ECE 6345

Spring 2015

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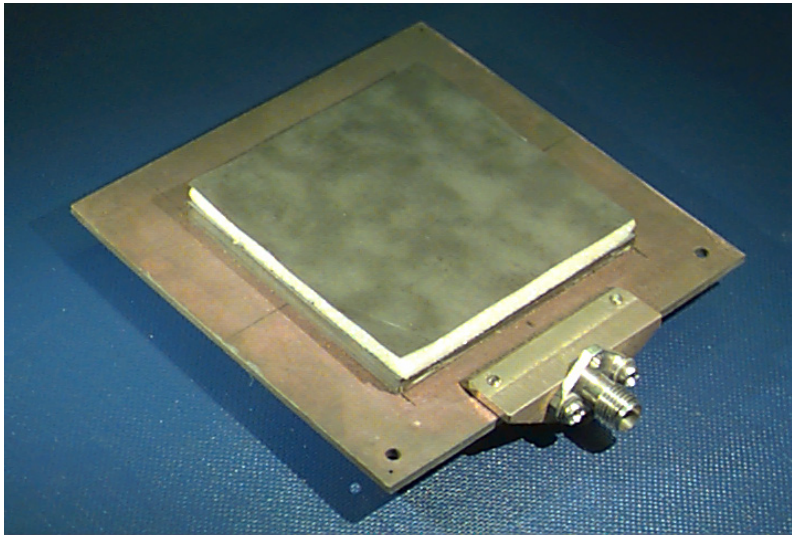
ECE Dept.



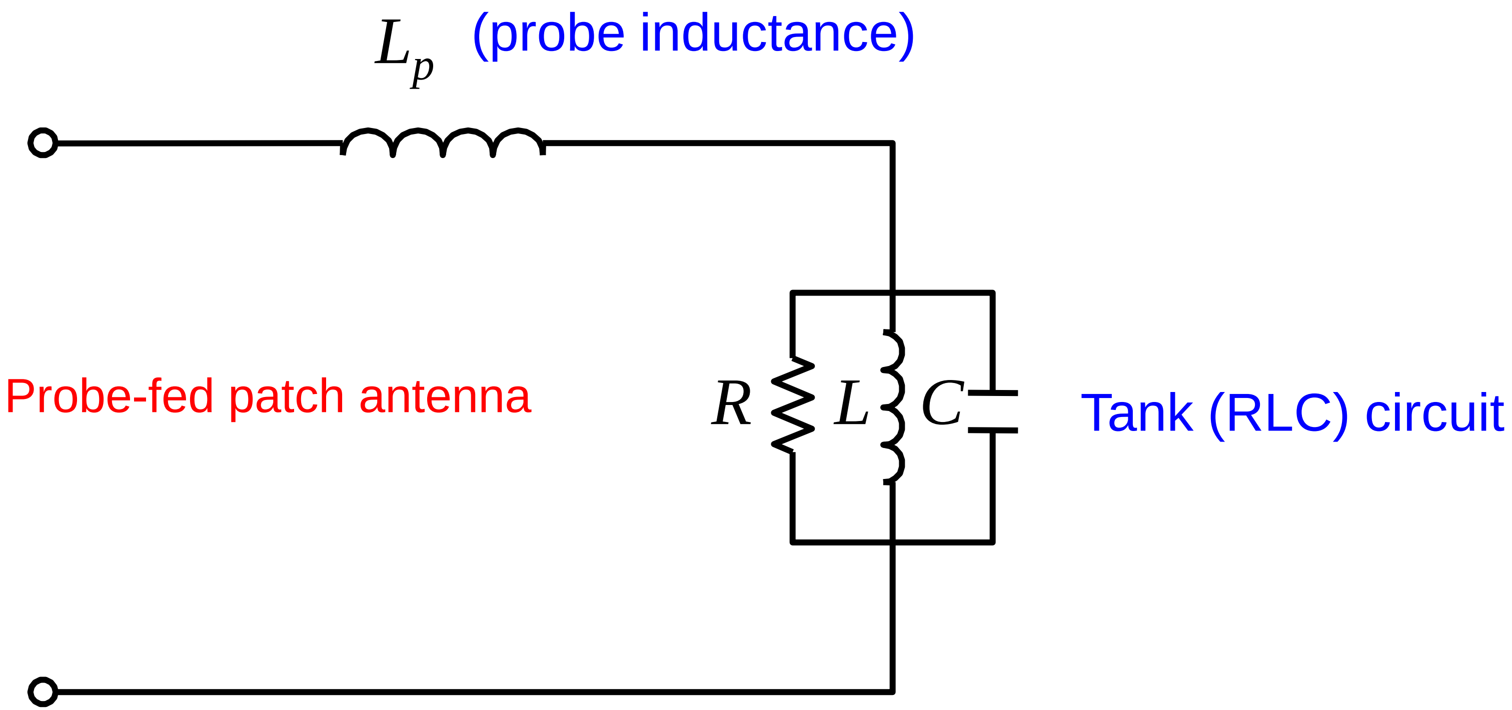
Notes 1

Overview

In this set of notes we discuss the CAD model of the microstrip antenna.

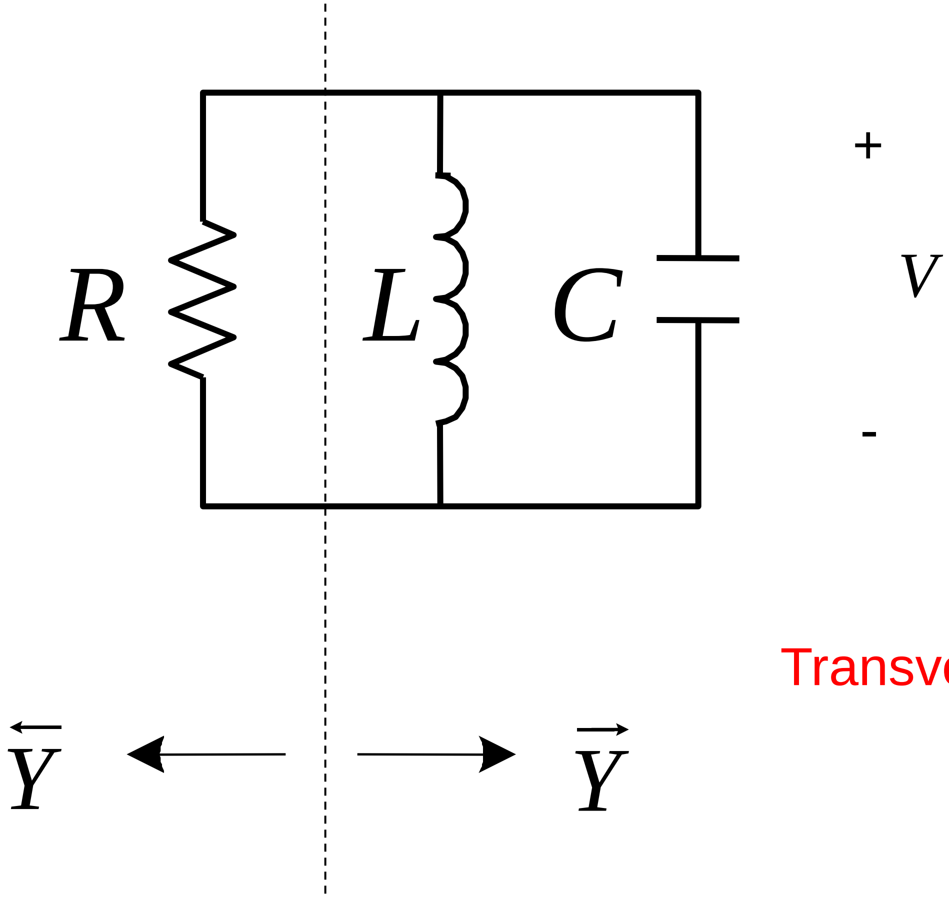
- Discuss complex resonance frequency
- Derive formula for Q
- Derive formula for input impedance
- Derive formula for impedance bandwidth

CAD Model of Microstrip Antennas



The circuit model is justified from the eigenfunction method in the cavity model, discussed later.

Tank Circuit: complex resonance frequency



$$G \equiv \frac{1}{R}$$

Transverse Resonance Equation (TRE):

$$\overleftarrow{Y} = -\overrightarrow{Y}$$

The complex resonance frequency is denoted as ω_0.

Resonance Frequency (cont.)

TRE:

$$G = -\left[j\omega_0 C + \frac{1}{j\omega_0 L} \right]$$

$$j\omega_0 LG = -1 + \omega_0^2 LC$$

$$\omega_0^2 \left(LC \right) + \omega_0 \left(-jLG \right) + \left(-1 \right) = 0$$

$$\omega_0 = \frac{jLG \pm \sqrt{-L^2G^2 + 4LC}}{2LC}$$

$$G \to 0, \quad \omega_0 \to \frac{1}{\sqrt{LC}}$$

so choose + sign

Resonance Frequency (cont.)

$$\omega_0 = j\frac{1}{2}\left(\frac{1}{RC}\right) + \frac{1}{\sqrt{LC}}\sqrt{1-\frac{1}{4}\frac{L}{R^2C}}$$

Denote: $\omega_0 = \omega_0' + j\omega_0''$

$$\omega_0' = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{1}{4}\frac{L}{R^2C}}$$

$$\omega_0'' = \frac{1}{2}\left(\frac{1}{RC}\right)$$

Resonance Frequency (cont.)

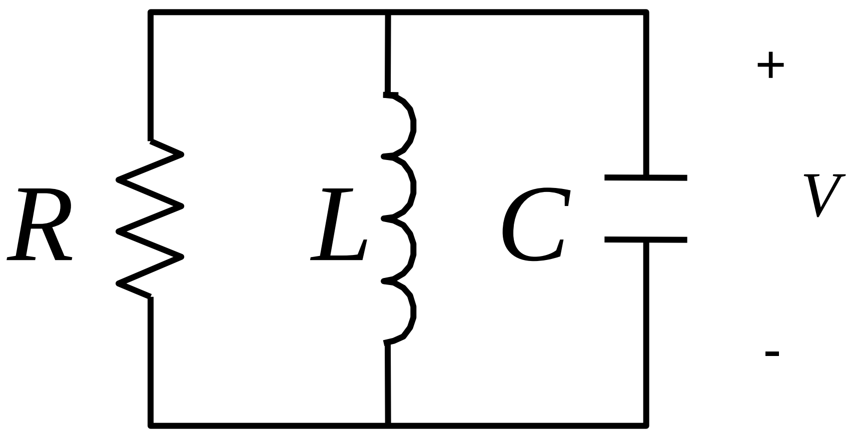
$$\omega_0' = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{1}{4}\frac{L}{R^2C}} \qquad \omega_0'' = \frac{1}{2}\left(\frac{1}{RC}\right)$$

Assume $R >> \sqrt{\frac{L}{C}}$ (a good resonator)

We then have:

$$\omega_0' \approx \frac{1}{\sqrt{LC}} \qquad \omega_0'' = \frac{1}{2}\left(\frac{1}{RC}\right)$$

Natural Response (no source)



The complex resonance frequency is ω_0.

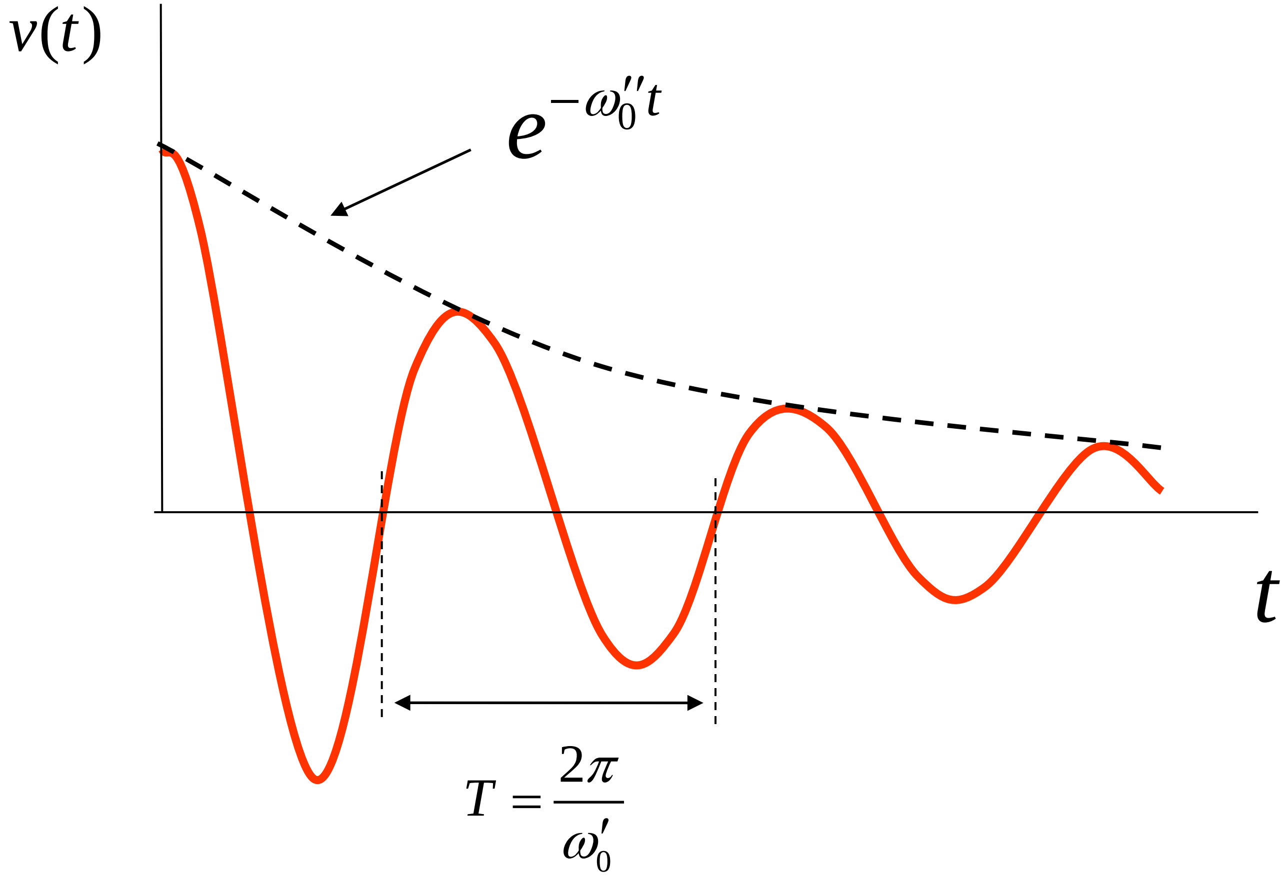
$$\omega_0 = \omega_0' + j\omega_0''$$

In the time domain: $v(t) = \mathrm{Re}\left(V\, e^{j\omega_0 t}\right)$ (Take V = 1)

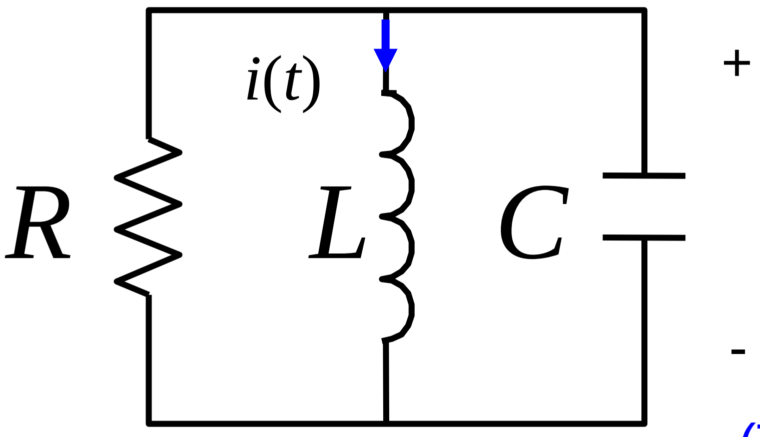
so $v(t) = \mathrm{Re}\left(e^{j\omega_0' t} e^{-\omega_0'' t}\right)$

Natural Response (cont.)

$$v(t) = e^{-\omega_0'' t} \cos\left(\omega_0' t\right)$$



Stored Energy



For the capacitor:

$$U_E\left(t\right)=\frac{1}{2}Cv^2(t)$$

$$=\frac{1}{2}Ce^{-2\omega_0''t}\cos^2\left(\omega_0't\right)$$

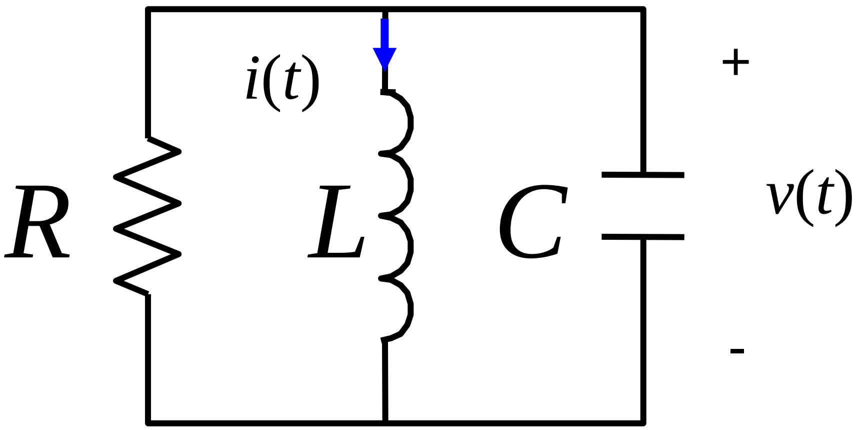
(Take $V = 1$)

For the inductor: $I=\dfrac{V}{j\omega_0 L}=\dfrac{1}{j\omega_0 L}\approx\dfrac{1}{j\omega_0' L}$

Therefore,

$$i(t)=\mathrm{Re}\left(I\,e^{j\omega_0 t}\right)\approx\frac{1}{\omega_0' L}\mathrm{Re}\left(\frac{1}{j}e^{-\omega_0''t}e^{+j\omega_0't}\right)=\frac{1}{\omega_0' L}e^{-\omega_0''t}\sin\left(\omega_0't\right)$$

Stored Energy (cont.)



$$U_H(t) = \frac{1}{2} L i^2(t)$$

$$= \frac{1}{2} L \left(\frac{1}{\omega_0' L} \right)^2 e^{-2\omega_0'' t} \sin^2(\omega_0' t)$$

$$= \frac{1}{2} C e^{-2\omega_0'' t} \sin^2(\omega_0' t)$$

Note: $$< U_E(t) > = < U_H(t) > = \frac{1}{4} C e^{-2\omega_0'' t}$$

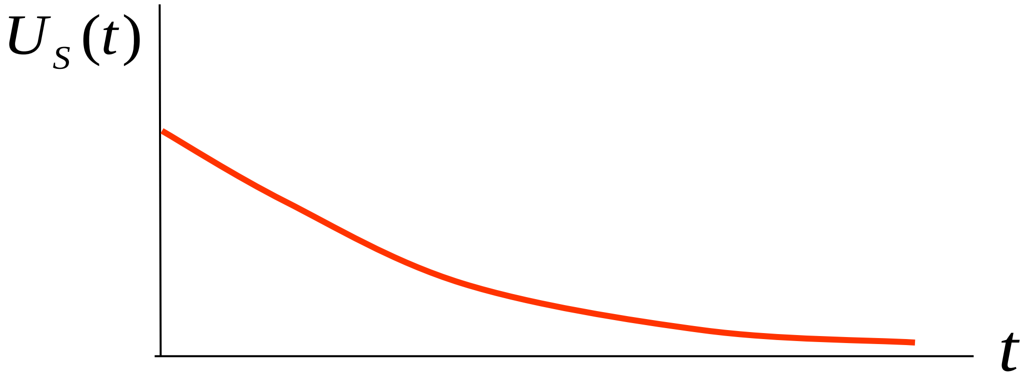
Also, note that $$U_S(t) = U_E(t) + U_H(t) = \frac{1}{2} C e^{-2\omega_0'' t}$$

Stored Energy (cont.)

$$U_S(t) = \left\langle U_S(t) \right\rangle = \left\langle U_E(t) \right\rangle + \left\langle U_H(t) \right\rangle = 2\left\langle U_E(t) \right\rangle$$

Hence

$$U_S(t) = \frac{1}{2} C\, e^{-2\omega_0'' t} = U_S(0)\, e^{-2\omega_0'' t}$$



Q of Cavity

$$Q \equiv 2\pi \left(\frac{U_S}{U_D^T} \right)$$

U_D^T = energy dissipated per cycle (period T)

$$Q \equiv \frac{2\pi}{T} \left(\frac{U_S}{U_D^T / T} \right)$$

or

$$Q \equiv \omega_0' \left(\frac{U_S}{P_D^{AVE}} \right)$$

P_D^{AVE} = average power dissipated

(this includes radiation loss)

Q of Cavity (cont.)

$$Q = \omega_0' \left(\frac{\frac{1}{2} C e^{-2\omega_0'' t}}{< G\, v(t)^2 >} \right) = \omega_0' \left(\frac{\frac{1}{2} C\, e^{-2\omega'' t}}{G < e^{-2\omega_0'' t} \cos^2\left(\omega_0' t\right) >} \right)$$

$$= \omega_0' \left(\frac{\frac{1}{2} C\, e^{-2\omega_0'' t}}{\frac{1}{2} G\, e^{-2\omega_0'' t}} \right)$$

$$= \frac{\omega_0' C}{G}$$

$$= \omega_0' R C$$

Note:

$$< e^{-2\omega_0'' t} \cos^2\left(\omega_0' t\right) >$$

$$\approx \quad e^{-2\omega_0'' t} < \cos^2\left(\omega_0' t\right) >$$

$$= e^{-2\omega_0'' t} \left(\frac{1}{2} \right)$$

Q of Cavity (cont.)

We then have

$$Q = \omega_0' RC = \frac{1}{\omega_0'}\left(\frac{R}{L}\right) = R\sqrt{\frac{C}{L}}$$

Recall that

$$\omega_0' \approx \frac{1}{\sqrt{LC}} \qquad \omega_0'' = \frac{1}{2}\left(\frac{1}{RC}\right)$$

Hence

$$\omega_0 \approx \omega_0'\left(1 + j\frac{1}{2}\left(\frac{1}{RC}\right)\sqrt{LC}\right)$$

Q of Cavity (cont.)

$$\omega \approx \omega_0' \left(1 + j\frac{1}{2}\left(\frac{1}{RC}\right)\sqrt{LC} \right)$$

$$= \omega_0' \left(1 + j\frac{1}{2}\left(\frac{1}{R}\right)\sqrt{\frac{L}{C}} \right)$$

Hence

$$\omega_0 \approx \omega_0' \left(1 + j\frac{1}{2Q} \right)$$

so

$$Q = \frac{1}{2}\frac{\omega_0'}{\omega_0''}$$

Q of Cavity (Cont.)

We can thus write

$$v(t) = e^{-\left(\frac{\omega_0'}{2Q}\right)t} \cos\left(\omega_0' t\right)$$

$$U_S(t) = U_S(0)\, e^{-\left(\frac{\omega_0'}{Q}\right)t}$$

Input Impedance

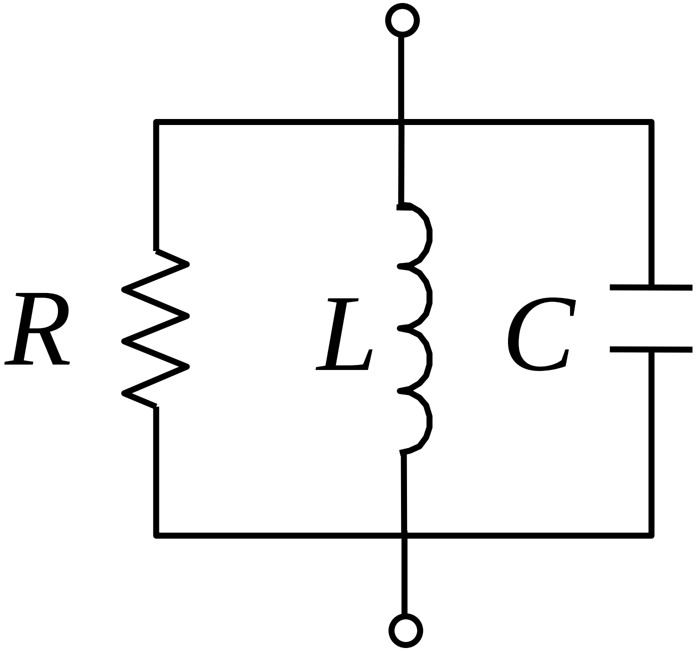
$$Y_{RLC} = G + j\omega C + \frac{1}{j\omega L}$$

$$Z_{RLC} = \frac{1}{G + j\omega C + \frac{1}{j\omega L}}$$

$$= \frac{R}{1 + j\omega RC + \frac{R}{j\omega L}}$$

$$= \frac{R}{1 + j\left(\omega RC - \frac{R}{\omega L}\right)}$$

The probe inductance is neglected here.



Input Impedance (cont.)

or

$$Z_{RLC} = \frac{R}{1 + j\left(\frac{\omega}{\omega_0'}\left(\omega_0' RC\right) - \frac{R}{\omega_0' L}\left(\frac{\omega_0'}{\omega}\right)\right)}$$

$$= \frac{R}{1 + j\left(\frac{\omega}{\omega_0'} Q - Q\left(\frac{\omega_0'}{\omega}\right)\right)}$$

Define $f_r \equiv \frac{f}{f_0} = \frac{\omega}{\omega_0'}$ where $f_0 \equiv \frac{\omega_0'}{2\pi}$

(real resonance frequency)

Then we have: $Z_{RCL} = \frac{R}{1 + jQ\left(f_r - \frac{1}{f_r}\right)}$

Input Impedance (cont.)

Define:

$$F \equiv f_r - \frac{1}{f_r}$$

$$F = \frac{1}{f_r}\left(f_r^{\,2} - 1\right)$$

$$= \frac{1}{f_r}\left(f_r - 1\right)\left(f_r + 1\right)$$

$$\approx 2\left(f_r - 1\right) \text{ for } f_r \approx 1$$

Hence, we have

$$Z_{RLC} = \frac{R}{1 + jQF} \approx \frac{R}{1 + j2Q\left(f_r - 1\right)}$$

Input Impedance (cont.)

$$Z_{RLC} = R\left[\frac{1}{1+jQF}\right]$$

Define: $$x \equiv QF = Q\left(f_r - \frac{1}{f_r}\right) \approx 2Q\left(f_r - 1\right)$$

$$\overline{Z}_{RLC} \equiv \frac{Z_{RLC}}{R}$$

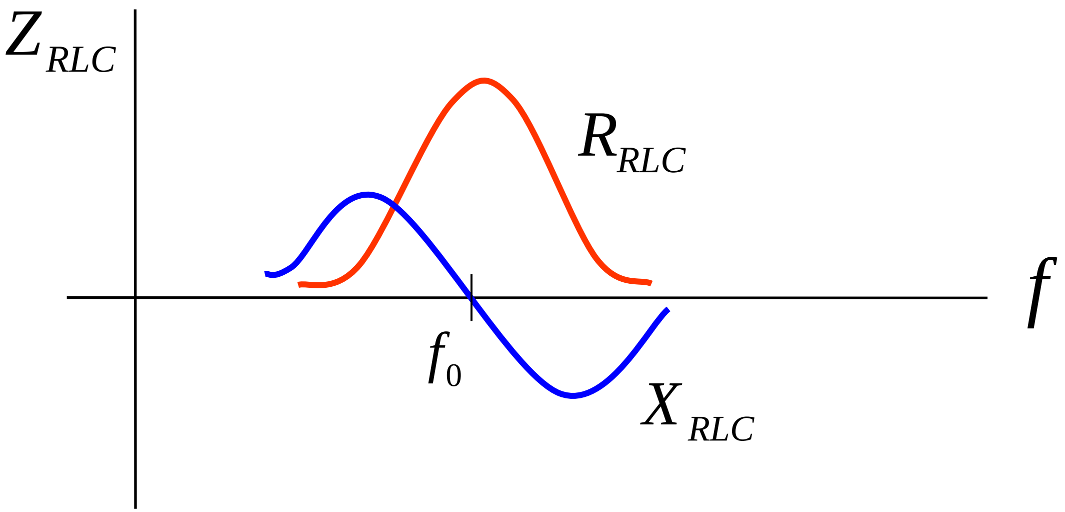
We then have

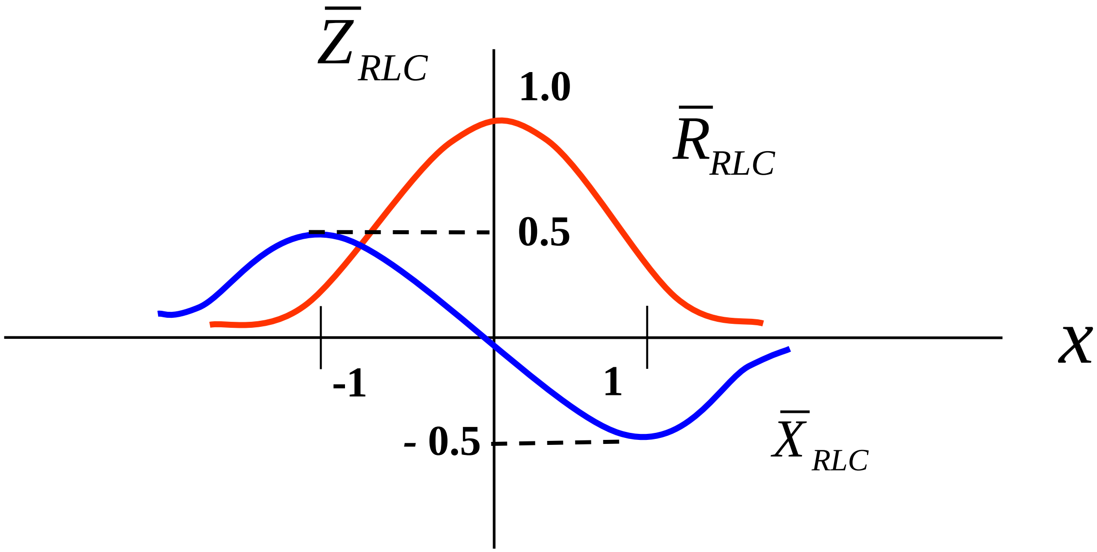
$$\overline{Z}_{RLC} = \frac{1}{1+jx}$$

$$\overline{R}_{RLC} = \frac{1}{1+x^2}$$

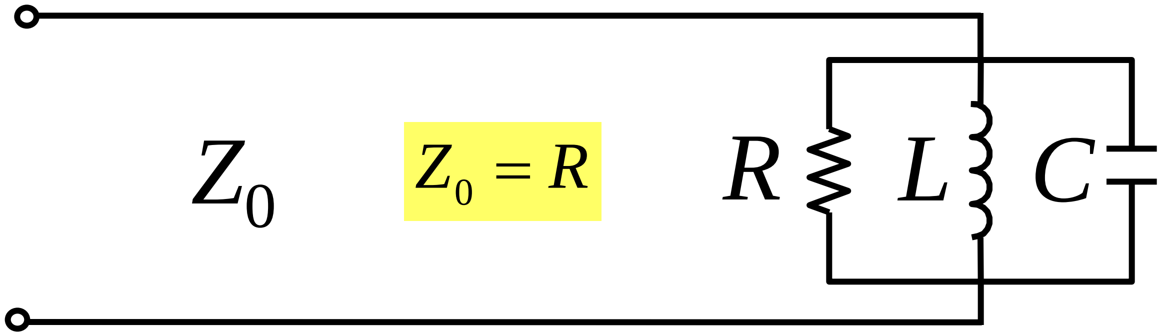
$$\overline{X}_{RLC} = \frac{-x}{1+x^2}$$

Input Impedance (cont.)





Reflection Coefficient



$$\Gamma = \frac{Z_{RLC} - Z_0}{Z_{RLC} + Z_0} = \frac{\bar{Z}_{RLC} - 1}{\bar{Z}_{RLC} + 1}$$

$$= \frac{1 - \bar{Y}_{RLC}}{1 + \bar{Y}_{RLC}} = \frac{1 - (1 + jx)}{1 + (1 + jx)}$$

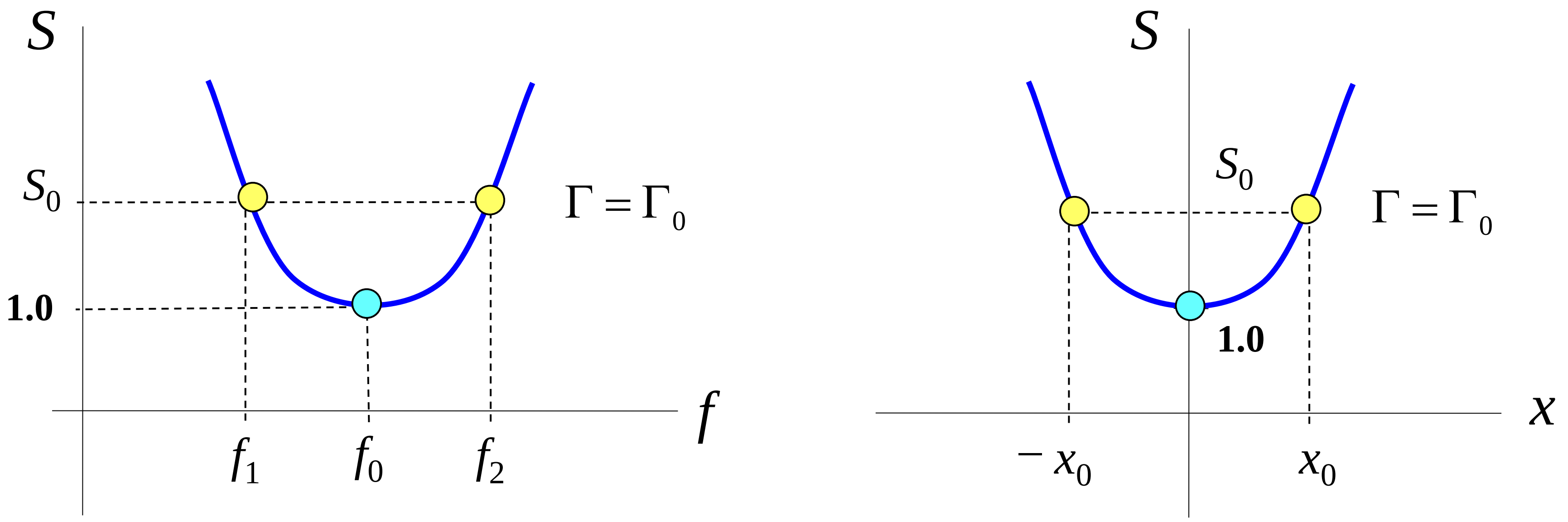
$$= \frac{-jx}{2 + jx}$$

Bandwidth

$$|\Gamma| = \frac{|x|}{\sqrt{4+x^2}} \qquad S = SWR = \frac{1+|\Gamma|}{1-|\Gamma|}$$

Bandwidth definition is based on SWR < S_0

The value S_0 is often chosen as 2.0.)



Bandwidth (cont.)

Fractional bandwidth: $$BW = \frac{f_2 - f_1}{f_0} = f_{r2} - f_{r1}$$

Recall that $$x \equiv QF = Q\left(f_r - \frac{1}{f_r} \right)$$

We can solve for f_r in terms of x:

$$f_r - \frac{1}{f_r} = \frac{x}{Q} \quad \Rightarrow \quad f_r^{\,2} - f_r\left(\frac{x}{Q}\right) - 1 = 0$$

so

$$f_r = \frac{\dfrac{x}{Q} \pm \sqrt{\dfrac{x^2}{Q^2} + 4}}{2}$$

Bandwidth (cont.)

To determine correct sign, enforce that $x \to 0, \quad f_r \to 1$

(So choose the plus sign.)

Hence

$$f_r = \frac{\frac{x}{Q} + \sqrt{\frac{x^2}{Q^2} + 4}}{2}$$

Therefore

$$f_{r2} = \frac{\frac{x_0}{Q} + \sqrt{\frac{x_0^2}{Q^2} + 4}}{2}$$

$$f_{r1} = \frac{-\frac{x_0}{Q} + \sqrt{\frac{x_0^2}{Q^2} + 4}}{2}$$

Bandwidth (cont.)

Hence, $$BW = \frac{x_0}{Q}$$

Now we need to solve for x_0:

$$S_0 = \frac{1 + |\Gamma_0|}{1 - |\Gamma_0|}$$ so $$|\Gamma_0| = \frac{S_0 - 1}{S_0 + 1}$$

Also, $$|\Gamma| = \frac{|x|}{\sqrt{4 + x^2}} \Rightarrow |\Gamma_0| = \frac{|x_0|}{\sqrt{4 + x_0^{\;2}}} = \frac{x_0}{\sqrt{4 + x_0^{\;2}}}$$

Bandwidth (cont.)

Therefore
$$\frac{x_0}{\sqrt{4+x_0^2}} = \frac{S_0 - 1}{S_0 + 1}$$

so
$$\frac{x_0^2}{4+x_0^2} = \left(\frac{S_0 - 1}{S_0 + 1}\right)^2 \equiv A$$

Thus we have
$$4A + x_0^2 A = x_0^2$$

or
$$x_0^2\left(1 - A\right) = 4A$$

Bandwidth (cont.)

The solution is:

$$x_0 = 2\sqrt{\frac{A}{1-A}}$$

$$= 2\frac{\left(\dfrac{S_0-1}{S_0+1}\right)}{\sqrt{1-\left(\dfrac{S_0-1}{S_0+1}\right)^2}}$$

$$= 2\frac{S_0-1}{\sqrt{(S_0+1)^2-(S_0-1)^2}}$$

$$= 2\left(\frac{S_0-1}{\sqrt{4S_0}}\right)$$

Bandwidth (cont.)

Hence, $$x_0 = \frac{S_0 - 1}{\sqrt{S_0}}$$

We then have $$BW = \frac{1}{Q}\left(\frac{S_0 - 1}{\sqrt{S_0}}\right)$$

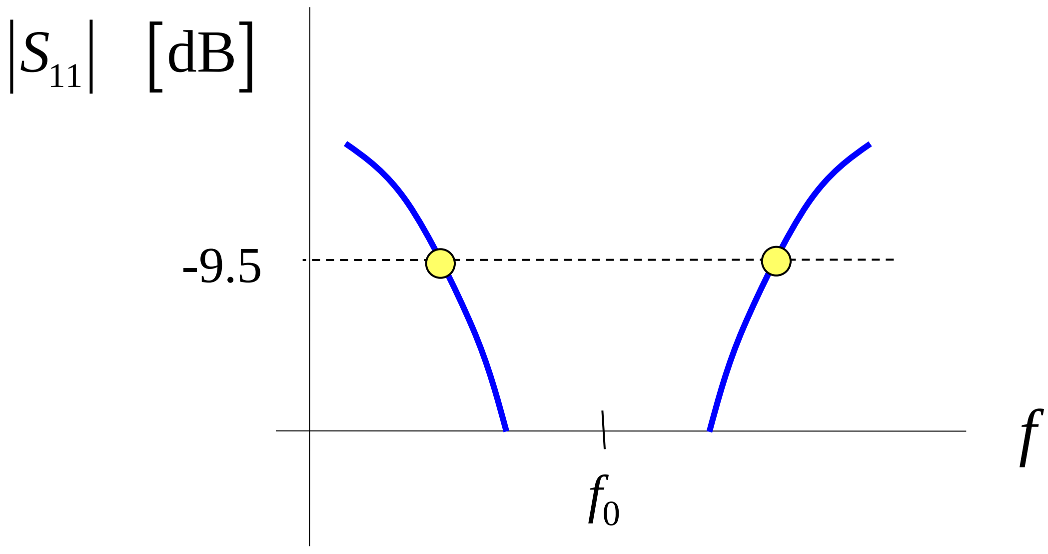
For $S_0 = 2$ we have: $$BW = \frac{1}{\sqrt{2}Q}$$

Bandwidth (cont.)

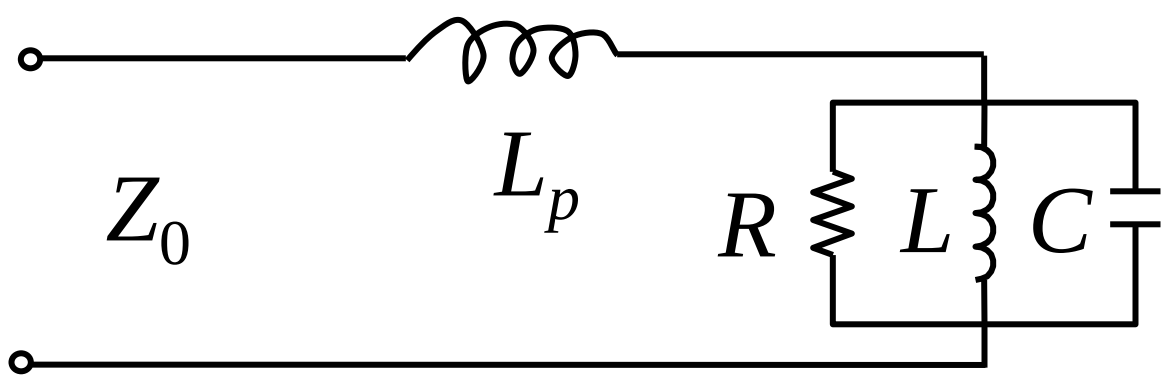
Note: $S_0 = SWR = 2.0$

$$|\Gamma| = |S_{11}| = \frac{1}{3}$$

$$20\log_{10}|S_{11}| = -9.5\ [\mathrm{dB}]$$



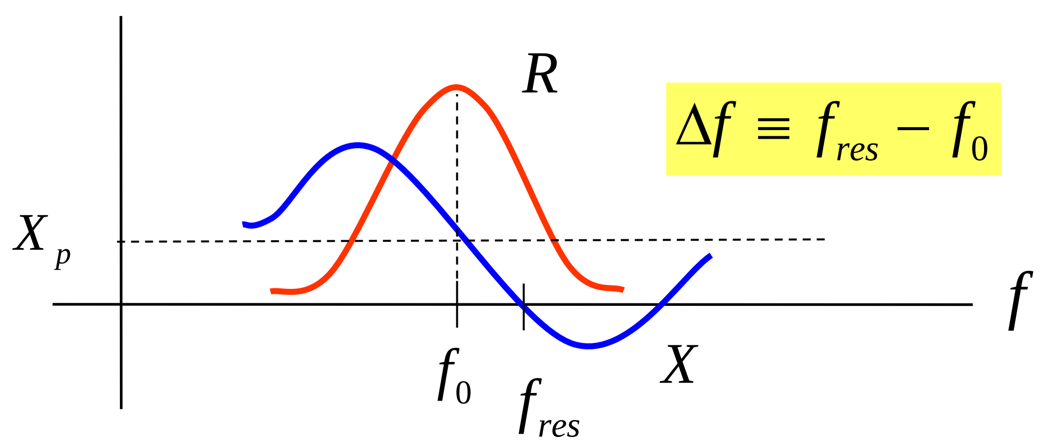
Complete Model



$$X_p = \omega L_p \approx \omega_0' L_p$$
$$Z_{in} = jX_p + Z_{RLC}$$

$$\frac{\Delta f}{f_0} \approx (BW)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{X_p}{R}\right)$$

(This will be derived in a HW problem.)



$$\Delta f \equiv f_{res} - f_0$$

Complete Model (cont.)

Define $\bar{X}_p \equiv X_p / R$

In terms of the normalized variable x, the resonance frequency f_{res} where the input impedance is purely real, corresponds to

$$x_{res} = \frac{1 - \sqrt{1 - 4\bar{X}_p^2}}{2\bar{X}_p}$$

(This will be derived in a HW problem.)

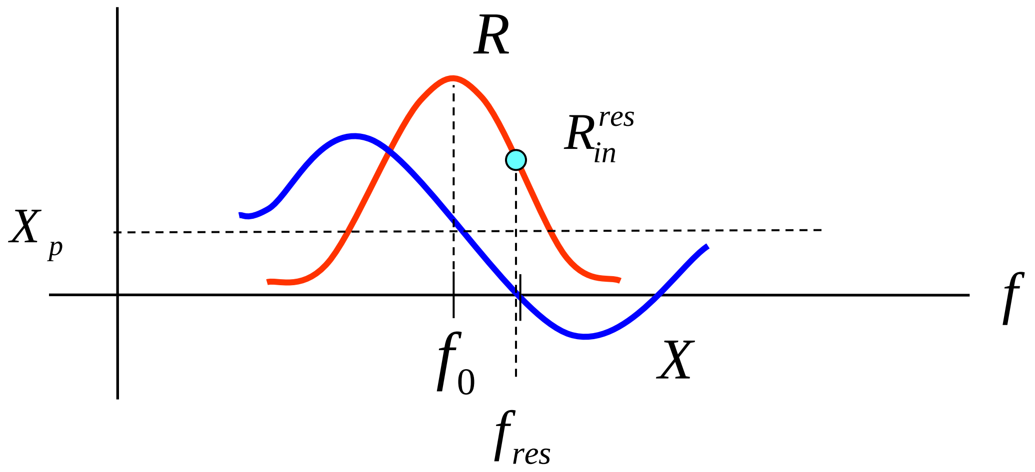
If $X_p << R$ then $x_{res} \approx \bar{X}_p$

(This follows from a binomial expansion of the square-root term in the numerator.)

Complete Model (cont.)

At the resonance frequency, the input resistance is then

$$R_{in}^{res} = \frac{R}{1 + x_{res}^2} \quad \Rightarrow \quad R_{in}^{res} \approx \frac{R}{1 + \left(\dfrac{X_p}{R} \right)^2}$$



Complete Model (cont.)

$$R_{in}^{res} \approx \frac{R}{1+\left(\dfrac{X_p}{R}\right)^2}$$

Note that the probe reactance changes the input resistance at resonance.

Given a specified value of the input resistance at resonance (e.g., $R_{in}{}^{res} = 50\ \Omega$), we wish to solve for the corresponding value of R.

Note that the CAD formula for resonant input resistance (in the short-course notes) gives us the value of R in terms of the feed location.

Complete Model (cont.)

To solve for R, use

$$R = R_{in}^{res} + R_{in}^{res}\left(\frac{X_p^2}{R^2}\right)$$

and solve iteratively:

$$R^{(i)} = R_{in}^{res} + R_{in}^{res}\left(\frac{X_p^2}{\left(R^{(i-1)}\right)^2}\right)$$

Zero iteration:

$$R = R_{in}^{res}$$

First iteration:

$$R = R_{in}^{res} + \left(\frac{X_p^2}{R_{in}^{res}}\right)$$

Second iteration:

$$R = R_{in}^{res} + R_{in}^{res}\left(\frac{X_p^2}{\left(R_{in}^{res} + \frac{X_p^2}{R_{in}^{res}}\right)^2}\right)$$