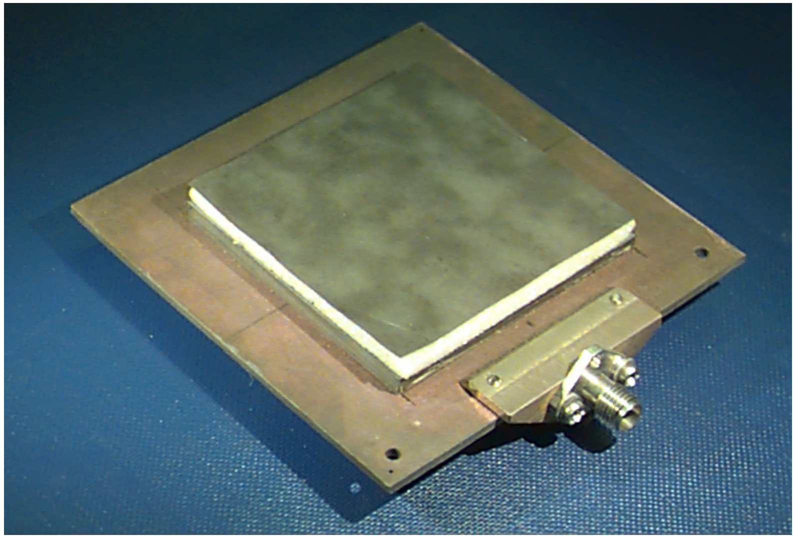
ECE 6345

Spring 2015

Prof. David R. Jackson
ECE Dept.



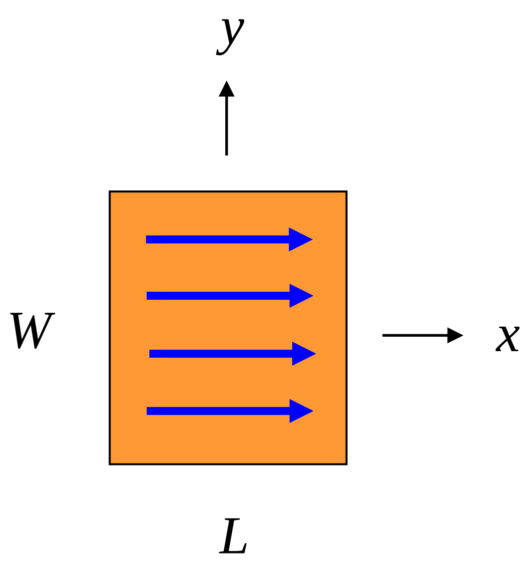
Notes 20

Overview

In this set of notes we apply the SDI method to investigate the fields produced by a patch current.

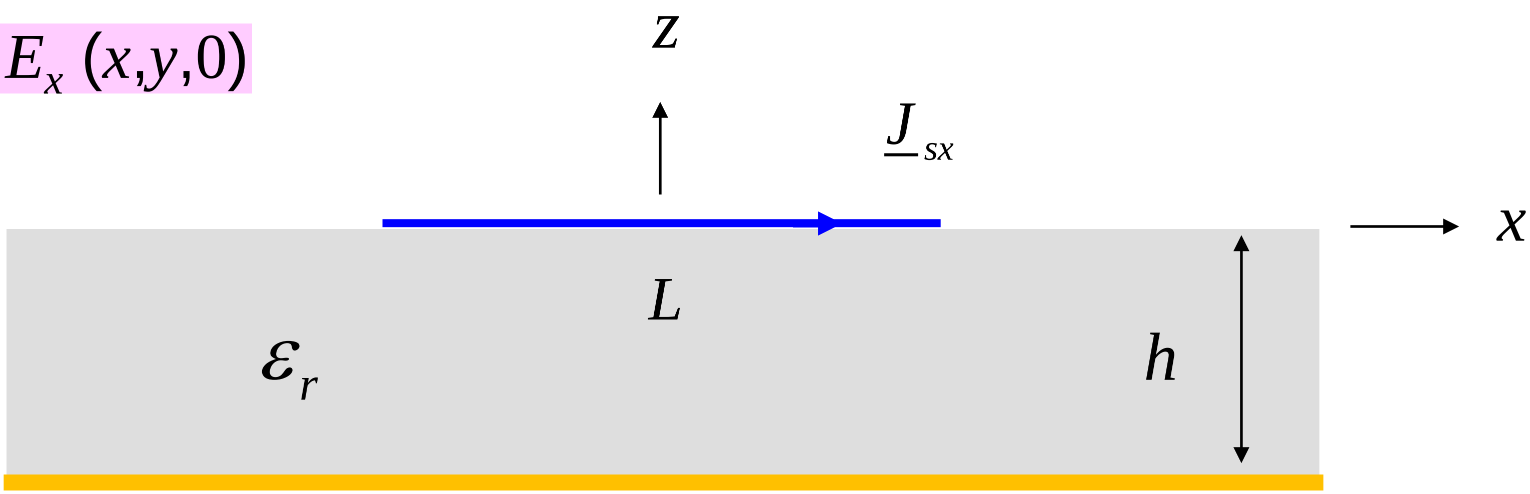
- We calculate the field due to a rectangular patch on top of a substrate.
- We examine the pole and branch point singularities in the complex plane.
- We examine the path of integration in the complex plane.

Patch Fields



$$\underline{J}_{sx}^{10} = \underline{\hat{x}} \cos\left(\frac{\pi x}{L}\right)$$

Find $E_x\,(x,y,0)$



Patch Fields (cont.)

From Notes 19 we have

$$E_x = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{G}_{xx}(k_x, k_y, z)\, \tilde{J}_{sx}(k_x, k_y)\, e^{-j(k_x x + k_y y)}\, dk_x\, dk_y$$

where

$$\tilde{G}_{xx} = -\left(\frac{k_x}{k_t}\right)^2 V_i^{TM}(z) - \left(\frac{k_y}{k_t}\right)^2 V_i^{TE}(z)$$

For the patch current, we have:

$$\tilde{J}_{sx}(k_x, k_y) = \left(\frac{\pi}{2} LW\right) \operatorname{sinc}\left(k_y \frac{W}{2}\right) \left[\frac{\cos\left(k_x \frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(k_x \frac{L}{2}\right)^2} \right]$$

Patch Fields (cont.)

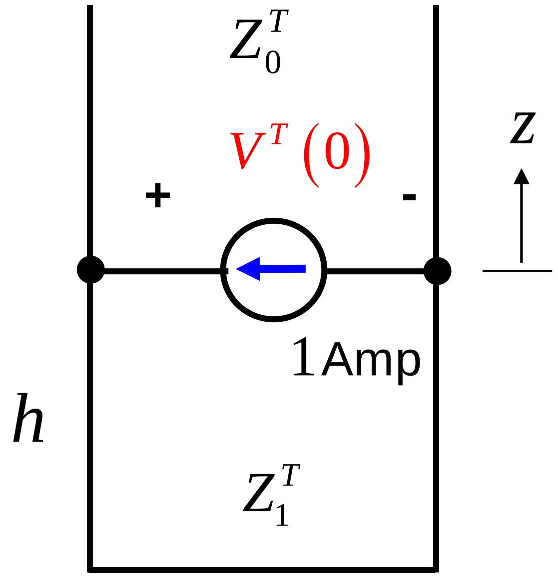
From the TEN we have

$$V_i^T(0) = (1)\, Z_{in}^T(0)$$

$$= \frac{1}{Y_{in}^T(0)}$$

$$= \frac{1}{Y_0^T - jY_1^T \cot\left(k_{z1}h\right)}$$

T denotes TM or TE



(TM$_z$ or TE$_z$)

$$Y_0^{TM} = \frac{\omega\varepsilon_0}{k_{z0}}$$

$$Y_1^{TM} = \frac{\omega\varepsilon_1}{k_{z1}}$$

$$Y_0^{TE} = \frac{k_{z0}}{\omega\mu_0}$$

$$Y_1^{TE} = \frac{k_{z1}}{\omega\mu_1}$$

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2}$$

$$k_{z1} = \left(k_1^2 - k_t^2\right)^{1/2}$$

$$k_t^2 = \left(k_x^2 + k_y^2\right)^{1/2}$$

Patch Fields (cont.)

Define the denominator term as

$$D^{T}(k_t) = Y_{in}^{T}(0)$$

so that

$$D^{TM}(k_t) = Y_0^{TM} - jY_1^{TM}\cot\left(k_{z1}h\right)$$
$$D^{TE}(k_t) = Y_0^{TE} - jY_1^{TE}\cot\left(k_{z1}h\right)$$

Patch Fields (cont.)

We then have

$$\tilde{G}_{xx} = -\left[\left(\frac{k_x}{k_t}\right)^2 \frac{1}{D^{TM}} + \left(\frac{k_y}{k_t}\right)^2 \frac{1}{D^{TE}}\right]$$

The final form of the electric field at the interface is then

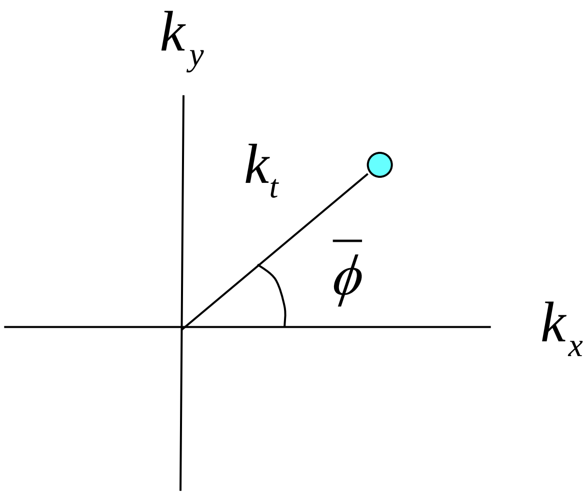
$$E_x(x,y,0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} -\left[\left(\frac{k_x}{k_t}\right)^2 \frac{1}{D^{TM}} + \left(\frac{k_y}{k_t}\right)^2 \frac{1}{D^{TE}}\right] \cdot \tilde{J}_{sx}(k_x,k_y)\, e^{-j(k_x x + k_y y)}\, dk_x\, dk_y$$

Polar Coordinates

Use the following change of variables:

$$dk_x\, dk_y = k_t\, dk_t\, d\overline{\phi}$$

(k_t is also often called k_ρ)



$$\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty}(\ldots)e^{-j\left(k_x x+k_y y\right)}dk_x\, dk_y = \int_{0}^{2\pi}\int_{0}^{+\infty}(\ldots)e^{-j\left(k_x x+k_y y\right)}k_t\, dk_t\, d\overline{\phi}$$

$$= 4\int_{0}^{\pi/2}\int_{0}^{+\infty}(\ldots)\cos\left(k_x x\right)\cos\left(k_y y\right)k_t\, dk_t\, d\overline{\phi}$$

$$\cos\overline{\phi} = \frac{k_x}{k_t}$$

$$\sin\overline{\phi} = \frac{k_y}{k_t}$$

Advantage: The poles and branch points are located at a fixed position in the complex k_t plane.

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2} \qquad k_{z1} = \left(k_1^2 - k_t^2\right)^{1/2}$$

Polar Coordinates (cont.)

Hence

$$E_x(x,y,0)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty}-\left[\left(\frac{k_x}{k_t}\right)^2\frac{1}{D^{TM}}+\left(\frac{k_y}{k_t}\right)^2\frac{1}{D^{TE}}\right]\tilde{J}_{sx}\left(k_x,k_y\right)e^{-j\left(k_x x+k_y y\right)}\,dk_x\,dk_y$$

$$E_x(x,y,0)=-\frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\infty}\tilde{J}_{sx}\left(k_t,\bar{\phi}\right)\left[\cos^2\bar{\phi}\frac{1}{D^{TM}(k_t)}+\sin^2\bar{\phi}\frac{1}{D^{TE}(k_t)}\right]\cdot\cos\left(k_x x\right)\cos\left(k_y y\right)k_t\,dk_t\,d\bar{\phi}$$

This is in the following general form:

$$E_x=\int_0^{\pi/2}\int_0^{\infty}F\left(k_t,\bar{\phi}\right)\,dk_t\,d\bar{\phi}$$

Poles

Poles occur when either of the following conditions are satisfied:

$$D^{TM}(k_t) = 0 \qquad \left(k_t = k_{tp}^{TM}\right)$$

$$D^{TE}(k_t) = 0 \qquad \left(k_t = k_{tp}^{TE}\right)$$

TM$_z$:

$$D^{TM} = 0$$

$$\Rightarrow \; Y_0^{TM} - jY_1^{TM} \cot\left(k_{z1}h\right) = 0$$

Poles (cont.)

This coincides with the well-known Transverse Resonance Equation (TRE) for determining the characteristic equation of a guided mode.

(e.g, TM_0 SW mode)

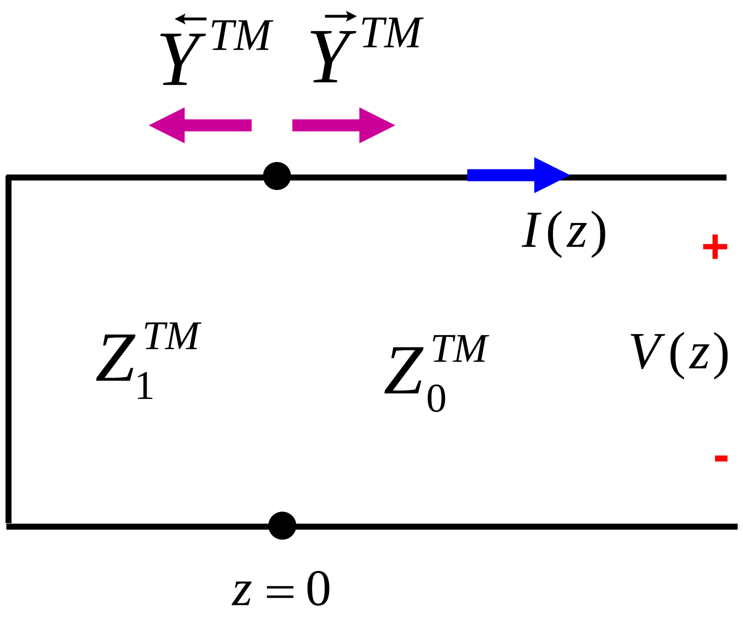
$$\overrightarrow{Y} = \frac{V(0^+)}{I(0^+)} \qquad \overleftarrow{Y} = \frac{V(0^-)}{-I(0^-)}$$

$$V(0^+) = V(0^-)$$
$$I(0^+) = I(0^-)$$

(Kirchhoff's laws)

Hence

$$\overleftarrow{Y} = -\overrightarrow{Y}$$



so that

$$-jY_1^{TM}\cot(k_{z1}h) = -Y_0^{TM}$$

TM_0 SW:

$$k_{z0} = \left(k_0^2 - \beta_{TM_0}^2\right)^{1/2}$$
$$k_{z1} = \left(k_1^2 - \beta_{TM_0}^2\right)^{1/2}$$

Poles (cont.)

Comparison:

Poles in k_t plane

$$Y_0^{TM} - jY_1^{TM} \cot\left(k_{z1}h\right) = 0$$

$$Y_0^{TM} = \frac{\omega\varepsilon_0}{k_{z0}}$$

$$Y_1^{TM} = \frac{\omega\varepsilon_1}{k_{z1}}$$

$$k_{z0} = \left(k_0^2 - k_{tp}^2\right)^{1/2}$$

$$k_{z1} = \left(k_1^2 - k_{tp}^2\right)^{1/2}$$

TRE (surface-wave mode)

$$Y_0^{TM} - jY_1^{TM} \cot\left(k_{z1}h\right) = 0$$

$$Y^{TM} = \frac{\omega\varepsilon_0}{k_{z0}}$$

$$Y_1^{TM} = \frac{\omega\varepsilon_1}{k_{z1}}$$

$$k_{z0} = \left(k_0^2 - \beta_{TM_0}^2\right)^{1/2}$$

$$k_{z1} = \left(k_1^2 - \beta_{TM_0}^2\right)^{1/2}$$

(A similar comparison holds for the TE case)

Poles (cont.)

Hence, we have the conclusion that

$$k_{tp}^{TM} = \beta_{TM}$$

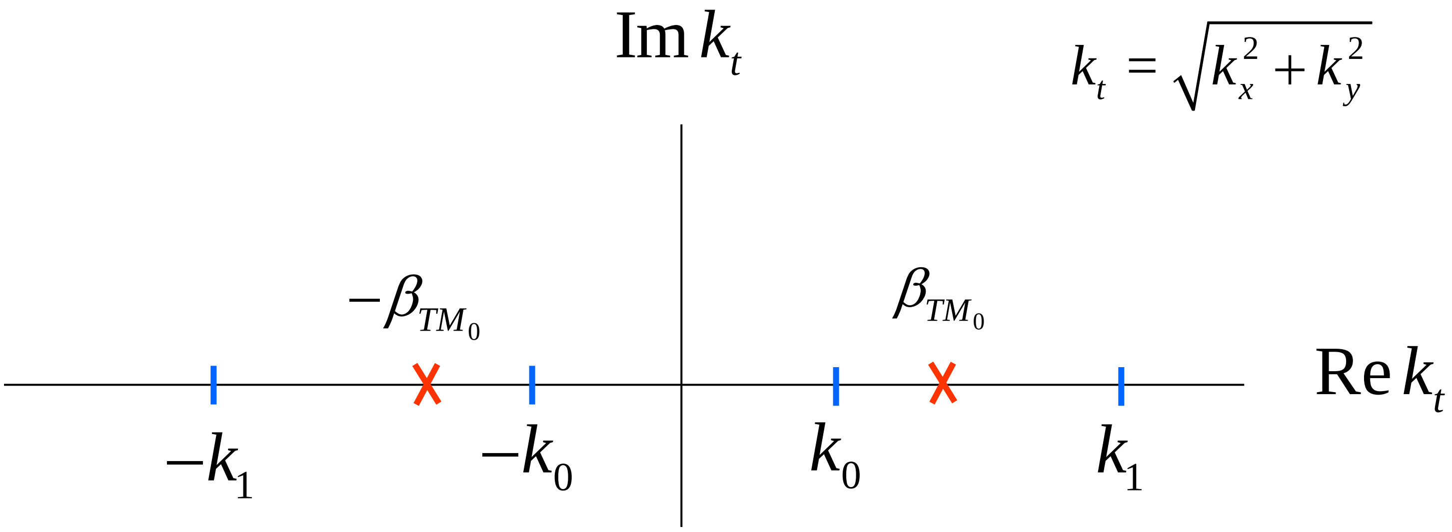
$$k_{tp}^{TE} = \beta_{TE}$$

That is, the poles are located at the wavenumbers of the guided modes (the surface-wave modes).

Note: In most practical cases, there is only a TM_0 surface-wave mode.

Poles (cont.)

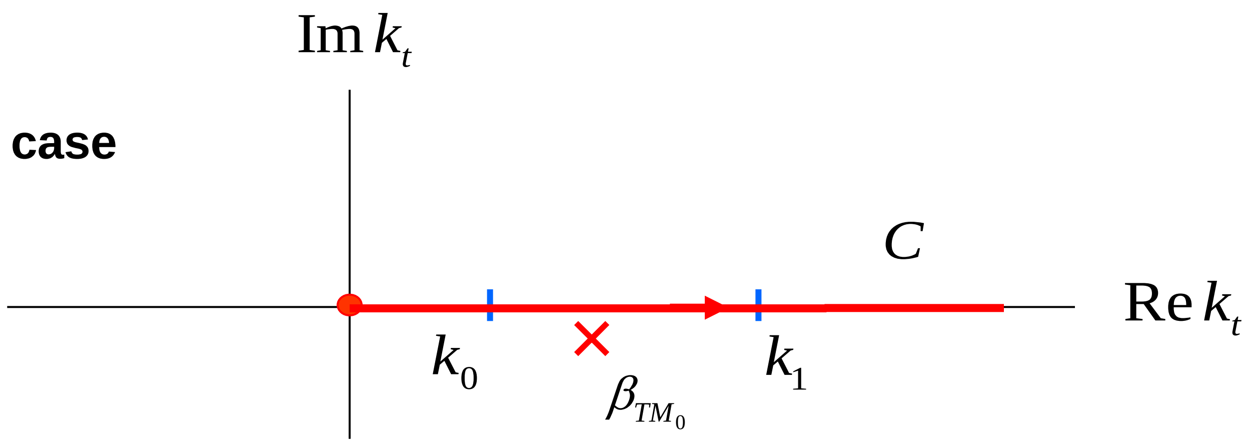
The complex plane thus has poles on the real axis at the wavenumbers of the surface waves.



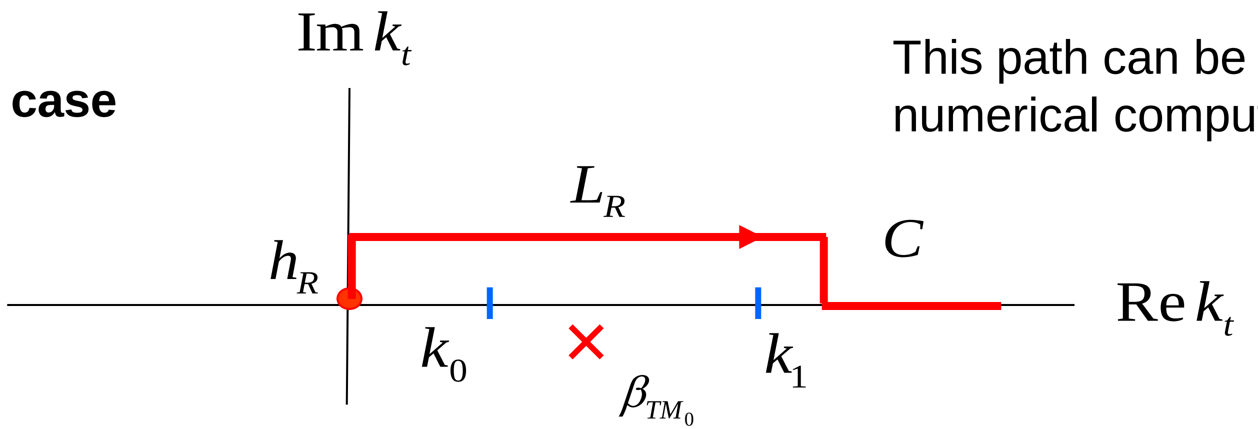
Path of Integration

The path avoids the poles by going above them.

Lossy case



Lossy case

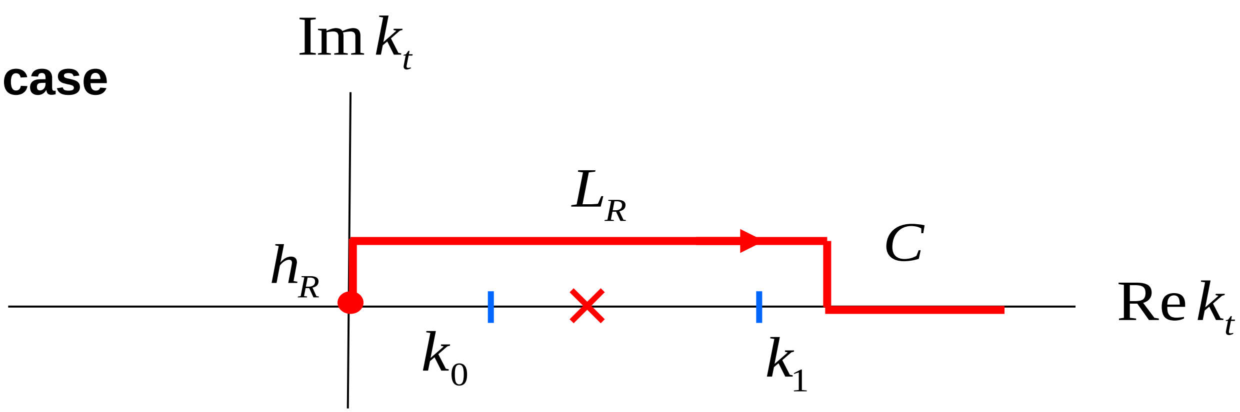


This path can be used for numerical computation.

Path of Integration (cont.)

The path avoids the poles by going above them.

Lossless case



$$h_R = 0.05\, k_0$$

$$L_R = k_1 (1.1)$$

(typical choices)

Practical note: If h_R is too small, we are too close to the pole. If h_R is too large, there is too much round-off error due to exponential growth in the sin and cos functions.

Branch Points

To explain why we have branch points, consider the TM function:

$$D^{TM}\left(k_t\right) = Y_0^{TM} - jY_1^{TM}\cot\left(k_{z1}h\right)$$

$$= \left(\frac{\omega\varepsilon_0}{k_{z0}}\right) - j\left(\frac{\omega\varepsilon_1}{k_{z1}}\right)\cot\left(k_{z1}h\right)$$

with

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2}$$

$$k_{z1} = \left(k_1^2 - k_t^2\right)^{1/2}$$

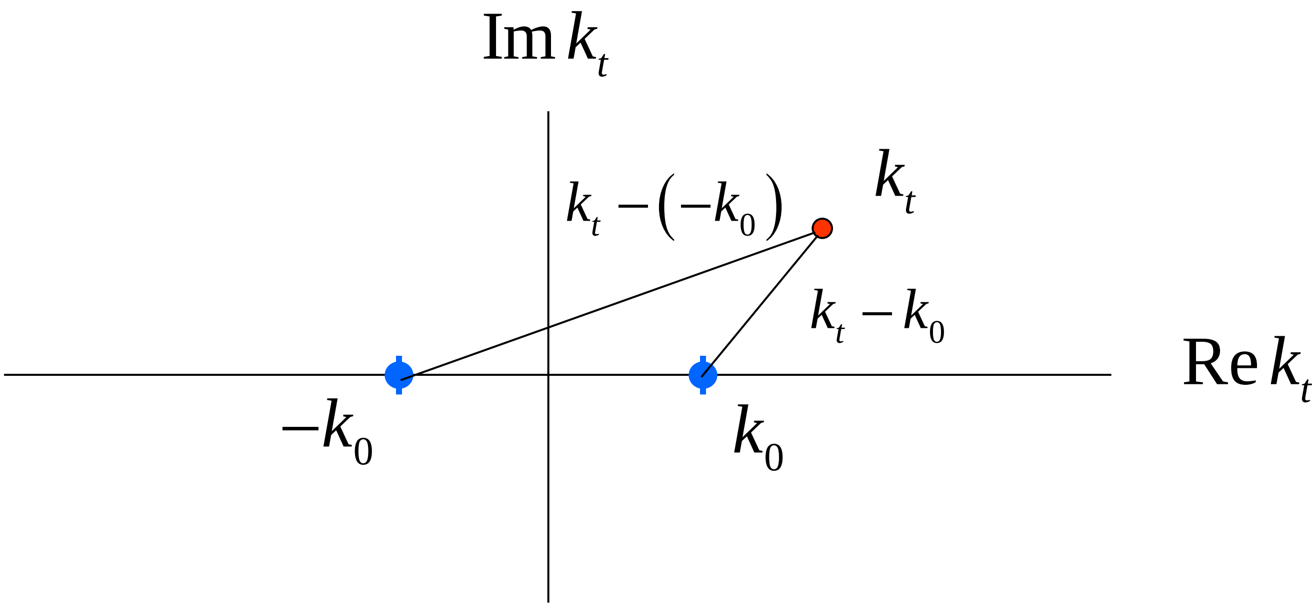
Note: There are no branch cuts for k_{z1}
(the function D^{TM} is an even function of k_{z1}).

If $k_{z0} \rightarrow -k_{z0}$ $\quad D^{TM}\left(k_t\right)$ changes (We need branch cuts for k_{z0}.)

Branch Points (cont.)

$$
\begin{aligned}
k_{z0} &= \left(k_0^2 - k_t^2\right)^{1/2} \\
&= \left(k_0 - k_t\right)^{1/2}\left(k_0 + k_t\right)^{1/2} \\
&= -j\left(k_t - k_0\right)^{1/2}\left(k_t + k_0\right)^{1/2} \\
&= -j\left(k_t - k_0\right)^{1/2}\left(k_t - \left(-k_0\right)\right)^{1/2}
\end{aligned}
$$

Note: The representation of the square root of –1 as –*j* is arbitrary here.

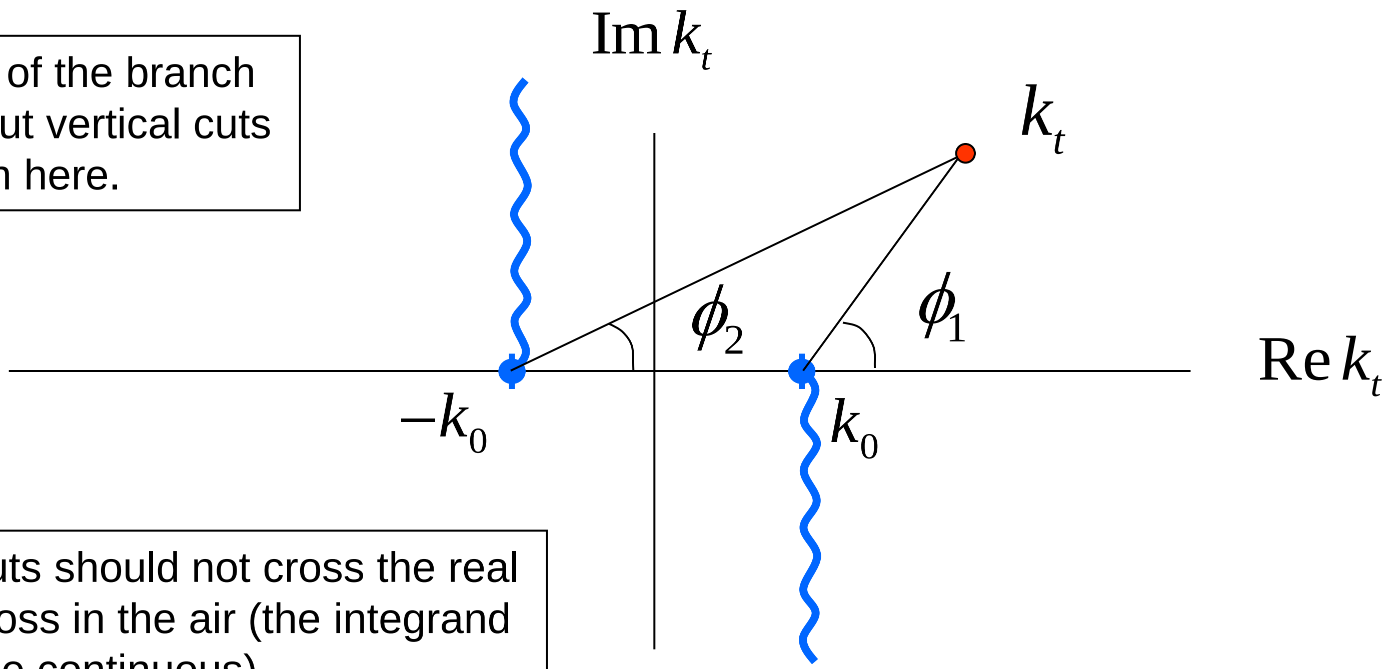


Branch Points (cont.)

$$k_{z0} = -j\left(k_t - k_0\right)^{1/2}\left(k_t - \left(-k_0\right)\right)^{1/2}$$
$$= -j\sqrt{\left|k_t - k_0\right|}\,\sqrt{\left|k_t - \left(-k_0\right)\right|}\; e^{j\phi_1/2}\, e^{j\phi_2/2}$$

Branch cuts are necessary to prevent the angles from changing by 2π:

Note: The shape of the branch cuts is arbitrary, but vertical cuts are shown here.



Note: The branch cuts should not cross the real axis when there is loss in the air (the integrand must be continuous).

Branch Points (cont.)

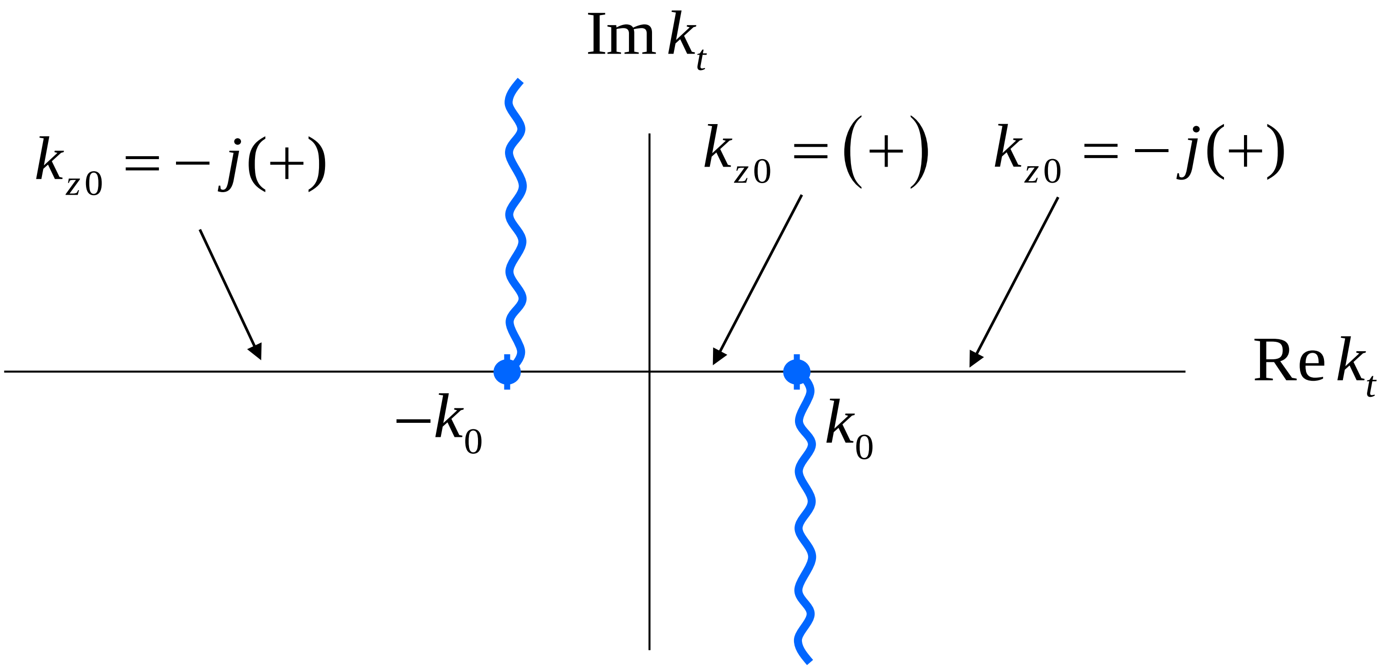
$$k_{z0} = -j\sqrt{\left|k_t - k_0\right|}\;\sqrt{\left|k_t - \left(-k_0\right)\right|}\; e^{j\phi_1/2}\, e^{j\phi_2/2}$$

We obtain the correct signs for k_{z0} if we choose the following branches:

$$-\pi/2 < \operatorname{Arg}\left(k_t - k_0\right) < 3\pi/2$$

$$-3\pi/2 < \operatorname{Arg}\left(k_t - \left(-k_0\right)\right) < \pi/2$$

The wave is then either decaying or outgoing in the air region when we are on the real axis.



Branch Points (cont.)

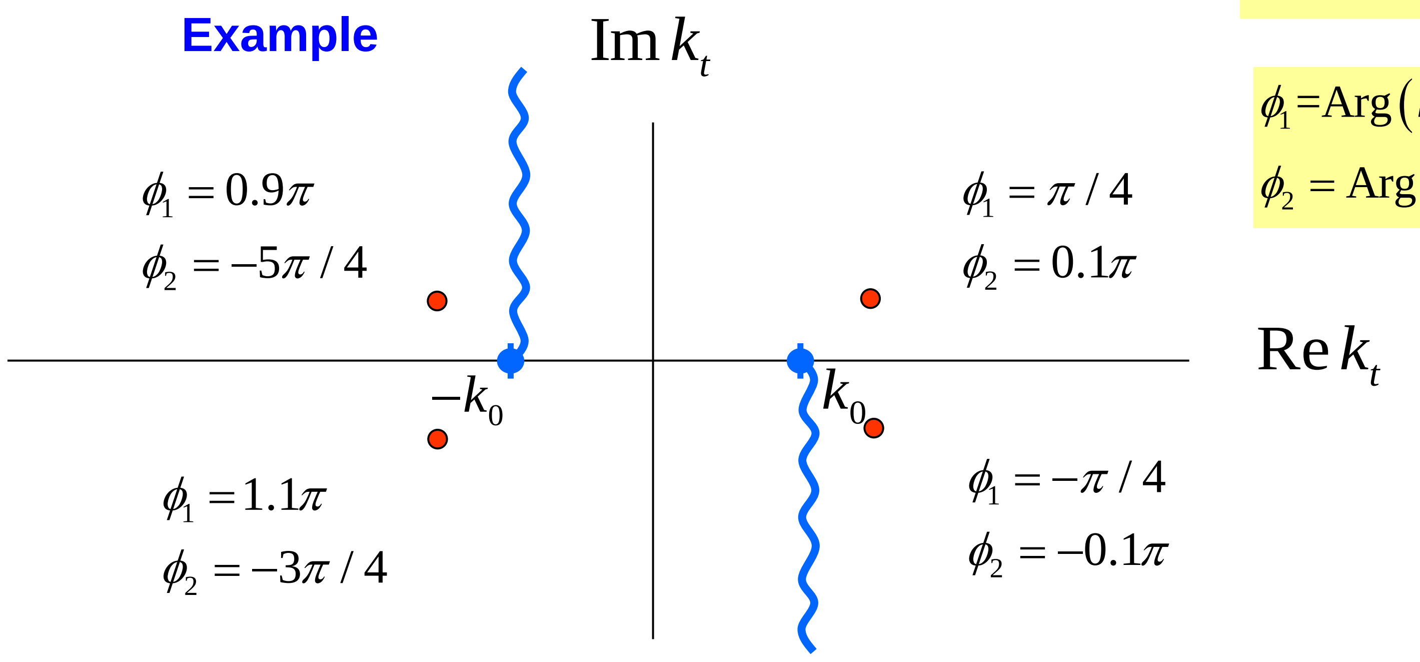
The wavenumber k_{z0} is then uniquely defined everywhere in the complex plane:

$$k_{z0} = -j\sqrt{|k_t - k_0|}\,\sqrt{|k_t - (-k_0)|}\; e^{j\phi_1/2}\, e^{j\phi_2/2}$$

$$-\pi/2 < \phi_1 < 3\pi/2$$
$$-3\pi/2 < \phi_2 < \pi/2$$

$$\phi_1 = \operatorname{Arg}(k_t - k_0)$$
$$\phi_2 = \operatorname{Arg}(k_t - (-k_0))$$

Example



Riemann Surface

- The Riemann surface is a pair of complex planes, connected by "ramps" (where the branch cuts used to be).
- The angles (and hence the function) change continuously over the surface.
- All possible values of the function are found on the surface.

Riemann Surface (cont.)

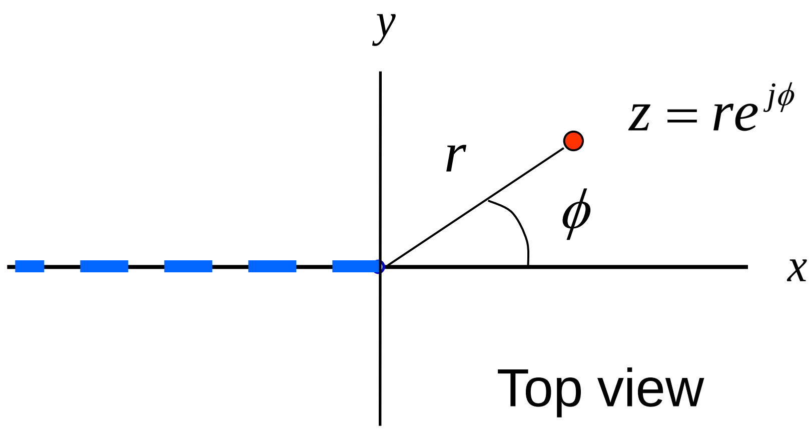
Riemann surface for $z^{1/2}$

Top sheet

$-\pi < \phi < \pi$

Bottom sheet

$\pi < \phi < 3\pi$



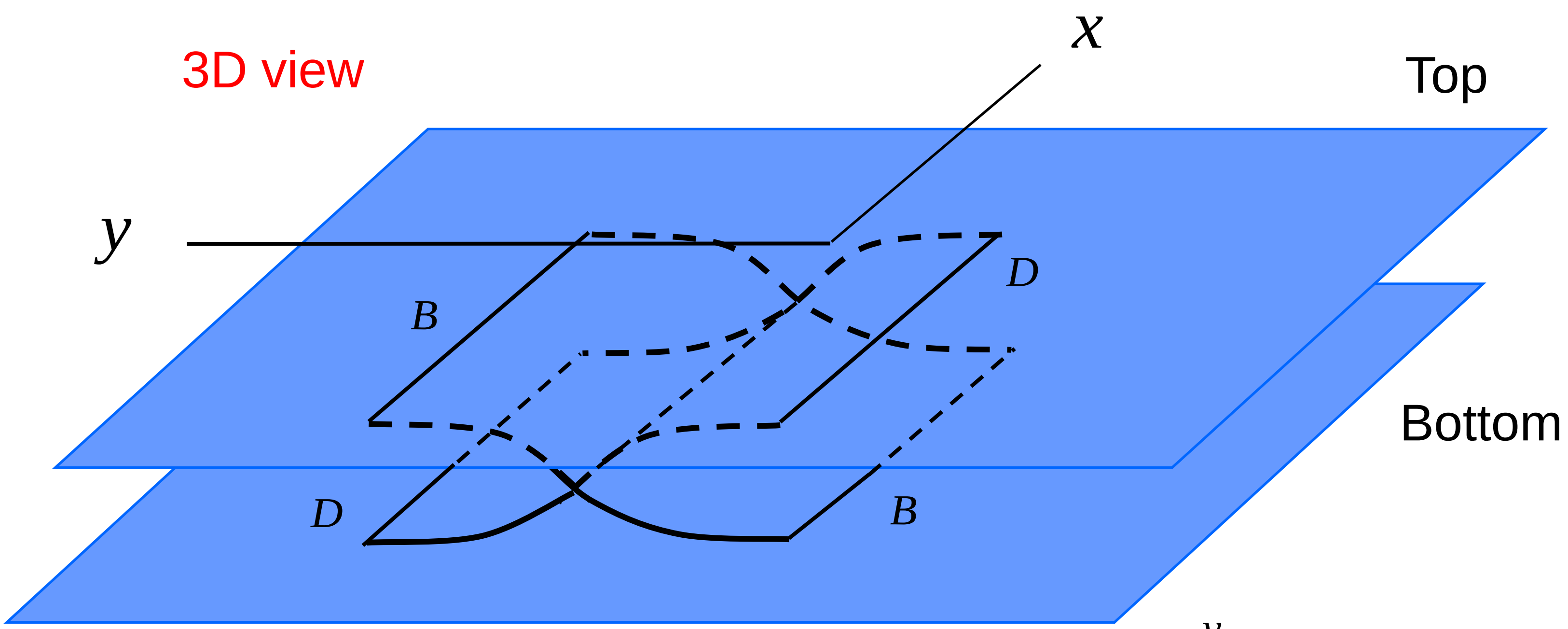
MATLAB : $-\pi < \phi \leq \pi$

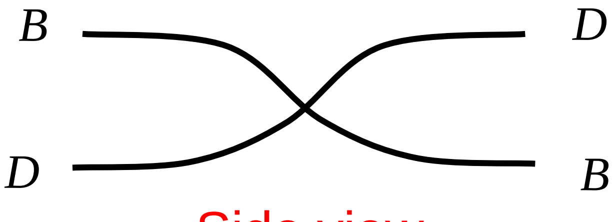
Note: A horizontal branch cut has been arbitrarily chosen.

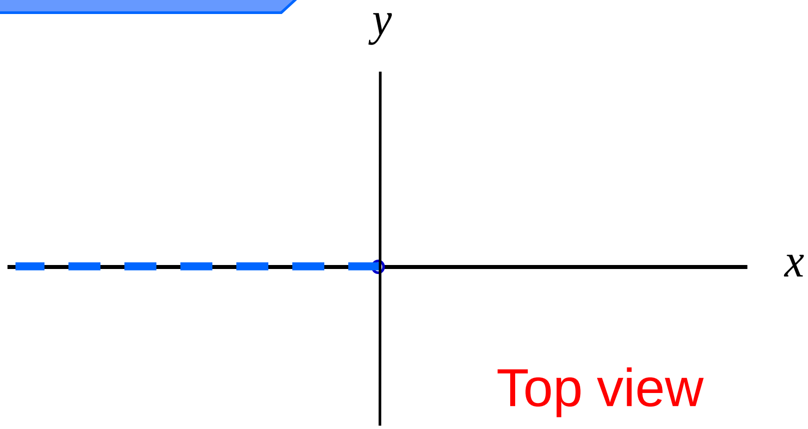
A ramp now exists where the branch cut used to be.

Riemann Surface (cont.)

Riemann surface for $z^{1/2}$







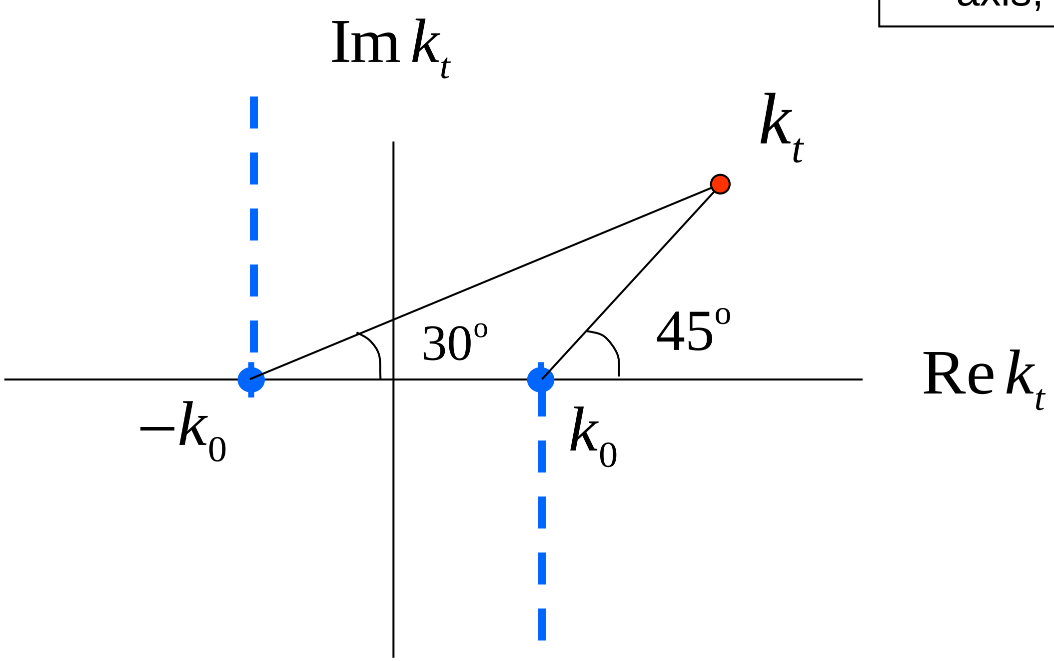
Riemann Surface

The Riemann surface can be constructed for the wavenumber function.

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2} = -j\left(k_t - k_0\right)^{1/2}\left(k_t - \left(-k_0\right)\right)^{1/2}$$

Example:

We go counter-clockwise around the branch point at k_0 starting on the top sheet on the real axis, and end up back where we started.



Top sheet

$$\phi_1 = \pi / 4$$
$$\phi_2 = \pi / 6$$

Bottom sheet

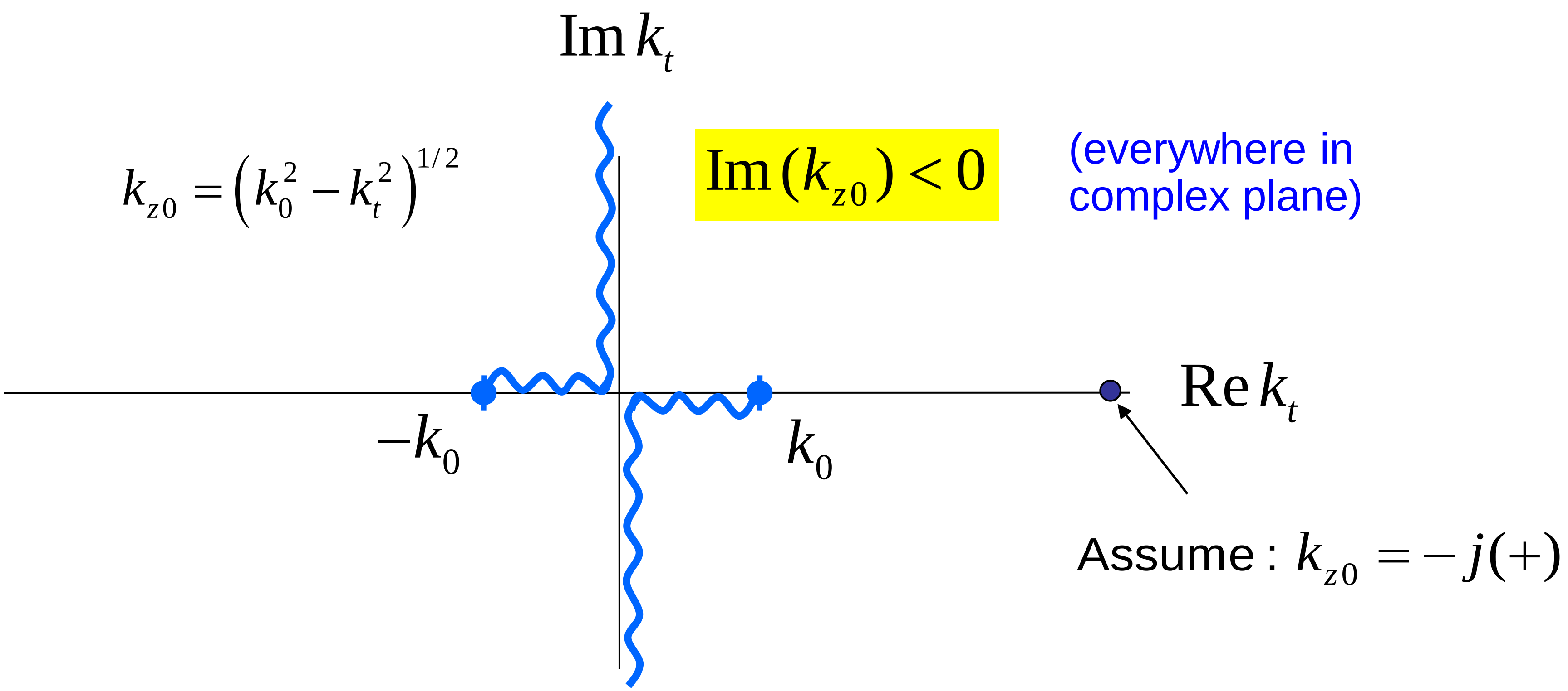
$$\phi_1 = \pi / 4 + 2\pi$$
$$\phi_2 = \pi / 6$$

Sommerfeld Branch Cuts

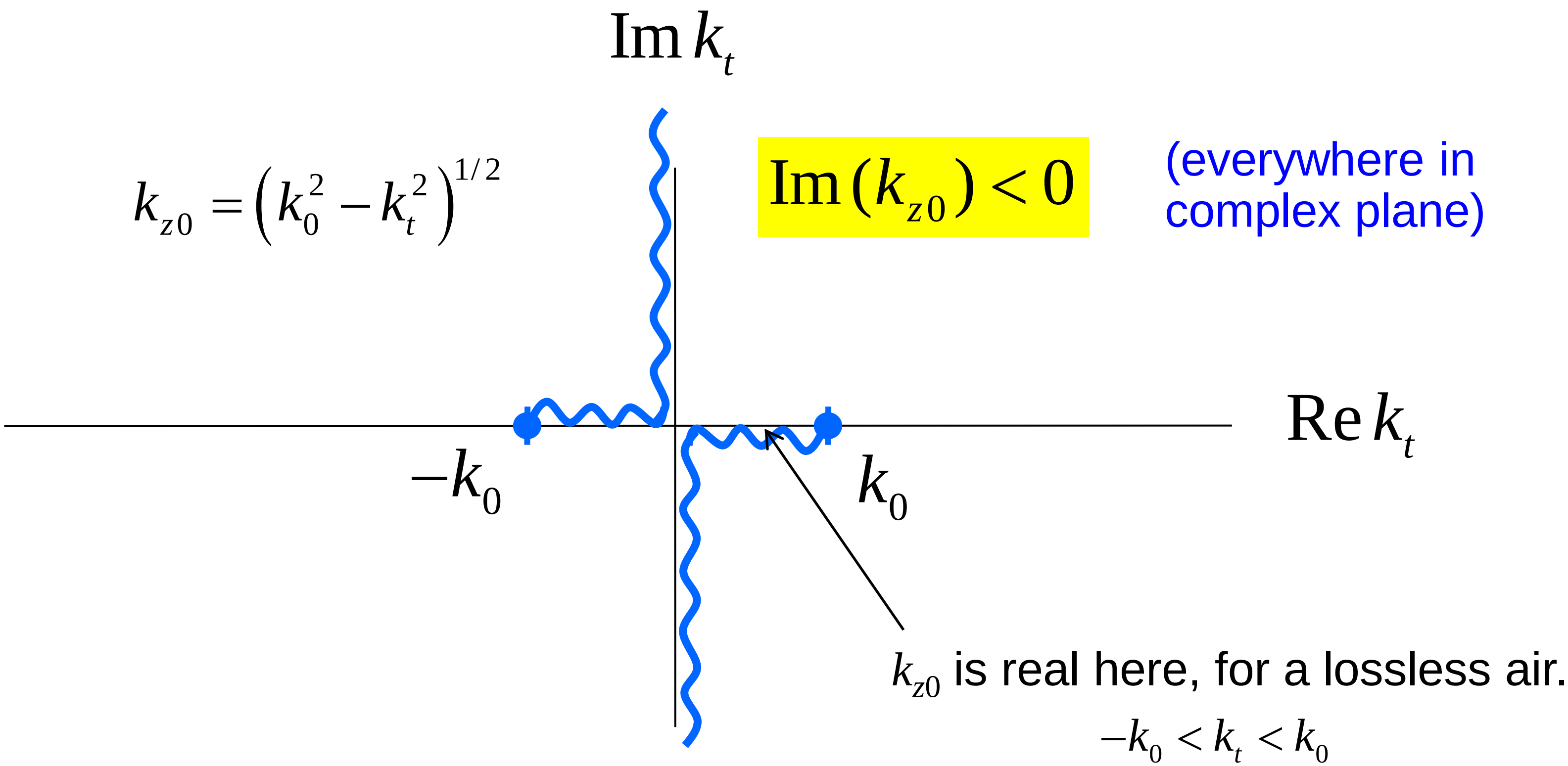
Sommerfeld branch cuts are a convenient choice for theoretical purposes (discussed more in ECE 6341):

$$\mathrm{Im}\,(k_{z0}) = 0 \quad \text{on branch cut}$$

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2}$$



Sommerfeld Branch Cuts (cont.)



Practical note:
If we give the air a small amount of loss, we can simply check to make sure that $\operatorname{Im}(k_{z0}) < 0$.

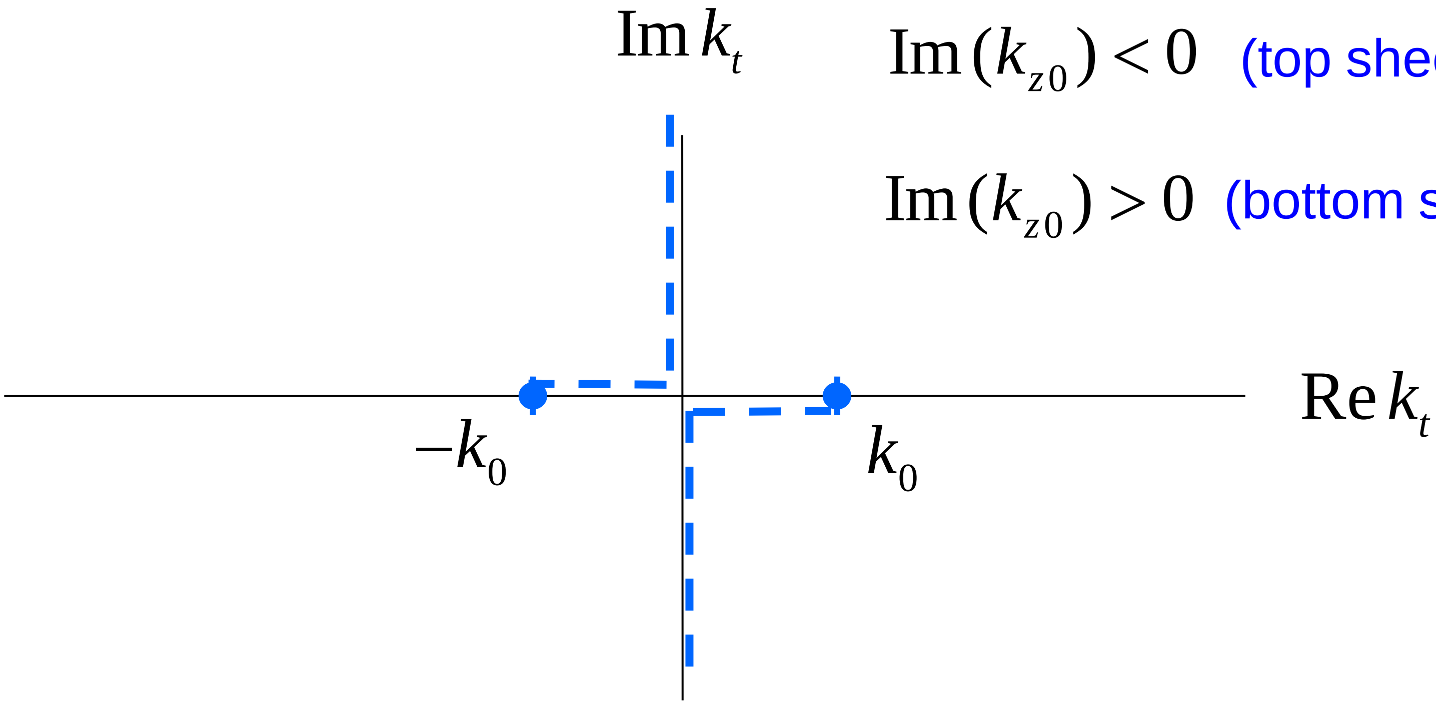
Note: The branch points move off of the axes for a lossy air.

Sommerfeld Branch Cuts (cont.)

The Riemann surface with Sommerfeld branch cuts:

$\mathrm{Im}(k_{z0}) < 0$ (top sheet)

$\mathrm{Im}(k_{z0}) > 0$ (bottom sheet)



Note: Surface wave poles must lie on the top sheet, and leaky-wave poles must lie on the bottom sheet.

Path of Integration

$$E_x = \int_0^{\pi/2} I\left(\bar{\phi}\right) d\bar{\phi}$$

where

$$I\left(\bar{\phi}\right) = \int_0^{\infty} F\left(k_t, \bar{\phi}\right)\, dk_t$$