ECE 6345

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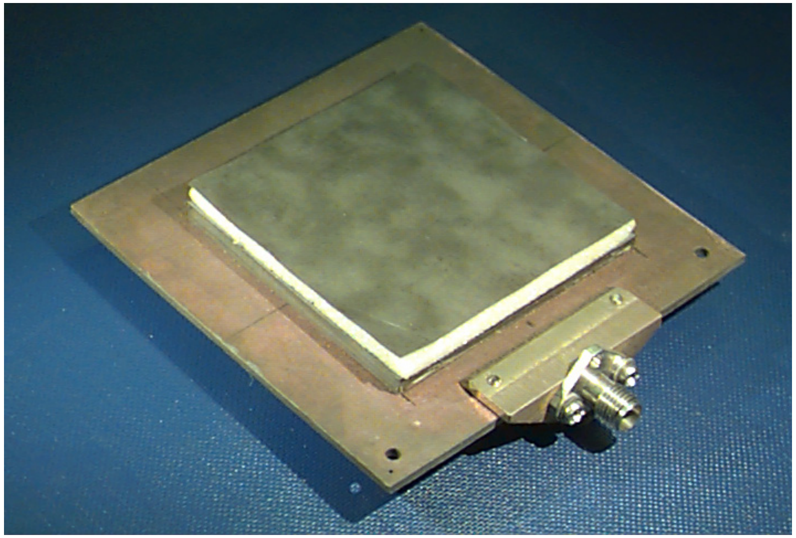


Notes 24

Overview

In this set of notes we derive the SDI formulation using a more mathematical, but general, approach (we directly Fourier transform Maxwell's equations).

This allows for all possible types of sources to be treated in one derivation.

General SDI Method

Start with Ampere's law: $\nabla \times \underline{H} = \underline{J}^{i} = j\omega\varepsilon \underline{E}$

$$\nabla = \nabla_t + \underline{\hat{z}} \frac{\partial}{\partial z}$$

where

$$\nabla_t = \underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y}$$

Assume a 2D spatial transform: $\tilde{\nabla}_t = \underline{\hat{x}}\left(-jk_x\right) + \underline{\hat{y}}\left(-jk_y\right)$

$$= -j\left(\underline{\hat{x}}\, k_x + \underline{\hat{y}}\, k_y\right)$$

$$= -j\underline{k}_t$$

$$= -jk_t \underline{\hat{u}}$$

General SDI Method (cont.)

Hence we have $\left(-jk_t\underline{\hat{u}}+\underline{\hat{z}}\dfrac{\partial}{\partial z}\right)\times\tilde{H}=\underline{\tilde{J}}^i+j\omega\varepsilon\underline{\tilde{E}}$

Next, represent the field as
$$\begin{aligned}\tilde{H}&=\underline{\hat{u}}\,\tilde{H}_u+\underline{\hat{v}}\,\tilde{H}_v+\underline{\hat{z}}\tilde{H}_z\\&=\underline{\hat{u}}\left(\tilde{H}\cdot\underline{\hat{u}}\right)+\underline{\hat{v}}\left(\tilde{H}\cdot\underline{\hat{v}}\right)++\underline{\hat{z}}\left(\tilde{H}\cdot\underline{\hat{z}}\right)\end{aligned}$$

Note that
$$\underline{\hat{u}}\times\underline{\hat{v}}=\underline{\hat{z}}$$
$$\underline{\hat{z}}\times\underline{\hat{u}}=\underline{\hat{v}}$$
$$\underline{\hat{z}}\times\underline{\hat{v}}=-\underline{\hat{u}}$$

Take the $\underline{\hat{z}},\underline{\hat{u}},\underline{\hat{v}}$ components of the transformed Ampere's equation

General SDI Method (cont.)

$$\underline{\hat{z}}) \; -jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z$$

$$\underline{\hat{u}}) \quad -\frac{\partial\tilde{H}_v}{\partial z} = \tilde{J}_u^i + j\omega\varepsilon\tilde{E}_u$$

$$\underline{\hat{v}}) \quad jk_t\tilde{H}_z + \frac{\partial\tilde{H}_u}{\partial z} = \tilde{J}_v^i + j\omega\varepsilon\tilde{E}_v$$

Examine TM_z field: $\left(\tilde{E}_u, \tilde{H}_v, \tilde{E}_z\right)$ Ignore $\underline{\hat{v}}$ equation

$$-jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z \qquad (1)$$

$$-\frac{\partial\tilde{H}_v}{\partial z} = \tilde{J}_u^i + j\omega\varepsilon\tilde{E}_u \qquad (2)$$

TM$_z$ Fields

We wish to eliminate $\tilde{E}_z$. To do this, use Faraday's law:

$$\nabla \times \underline{E} = -\underline{M}^{i} - j\omega\mu\underline{H}$$

$$\left(-jk_t\underline{\hat{u}} + \underline{\hat{z}}\frac{\partial}{\partial z}\right) \times \underline{\tilde{E}} = -\underline{\tilde{M}}^{i} - j\omega\mu\underline{\tilde{H}}$$

Take the $\underline{\hat{v}}$ component of the transformed Faraday's Law:

$$jk_t\tilde{E}_z + \frac{\partial \underline{\tilde{E}}_u}{\partial z} = -\tilde{M}_v^{i} - j\omega\mu\tilde{H}_v \qquad (3)$$

TM$_z$ Fields (cont.)

Substitute $\tilde{E}_z$ from (1) into (3) to obtain

$$jk_t\left[\frac{1}{j\omega\varepsilon}\left(-\tilde{J}_z^i - jk_t\tilde{H}_v\right)\right]+\frac{\partial\tilde{E}_u}{\partial z}=-\tilde{M}_v^i - j\omega\mu\tilde{H}_v$$

Putting all the sources on the RHS:

$$\frac{\partial\tilde{E}_u}{\partial z}+\frac{k_t^2}{j\omega\varepsilon}\tilde{H}_v + j\omega\mu\tilde{H}_v = -\tilde{M}_v^i + \frac{k_t}{\omega\varepsilon}\tilde{J}_z^i$$

Note that
$$\frac{k_t^2}{j\omega\varepsilon}+j\omega\mu = \frac{1}{j\omega\varepsilon}\left(k_t^2-\omega^2\mu\varepsilon\right)$$
$$=\frac{1}{j\omega\varepsilon}\left(k_t^2-k^2\right)$$
$$=\frac{-1}{j\omega\varepsilon}k_z^2$$

TM$_z$ Fields (cont.)

Hence

$$\frac{\partial \tilde{E}_u}{\partial z} - \left(\frac{k_z^2}{j\omega\varepsilon} \right) \tilde{H}_v = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon} \right) \tilde{J}_z^i \qquad (4)$$

TM$_z$ Fields (cont.)

Equations (2) and (4) are rewritten as

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^i - j\omega\varepsilon \tilde{E}_u$$

$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i + \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v$$

TM$_z$ Fields (cont.)

Define:

$$V^{TM}(z) = \tilde{E}_u(k_x, k_y, z)$$

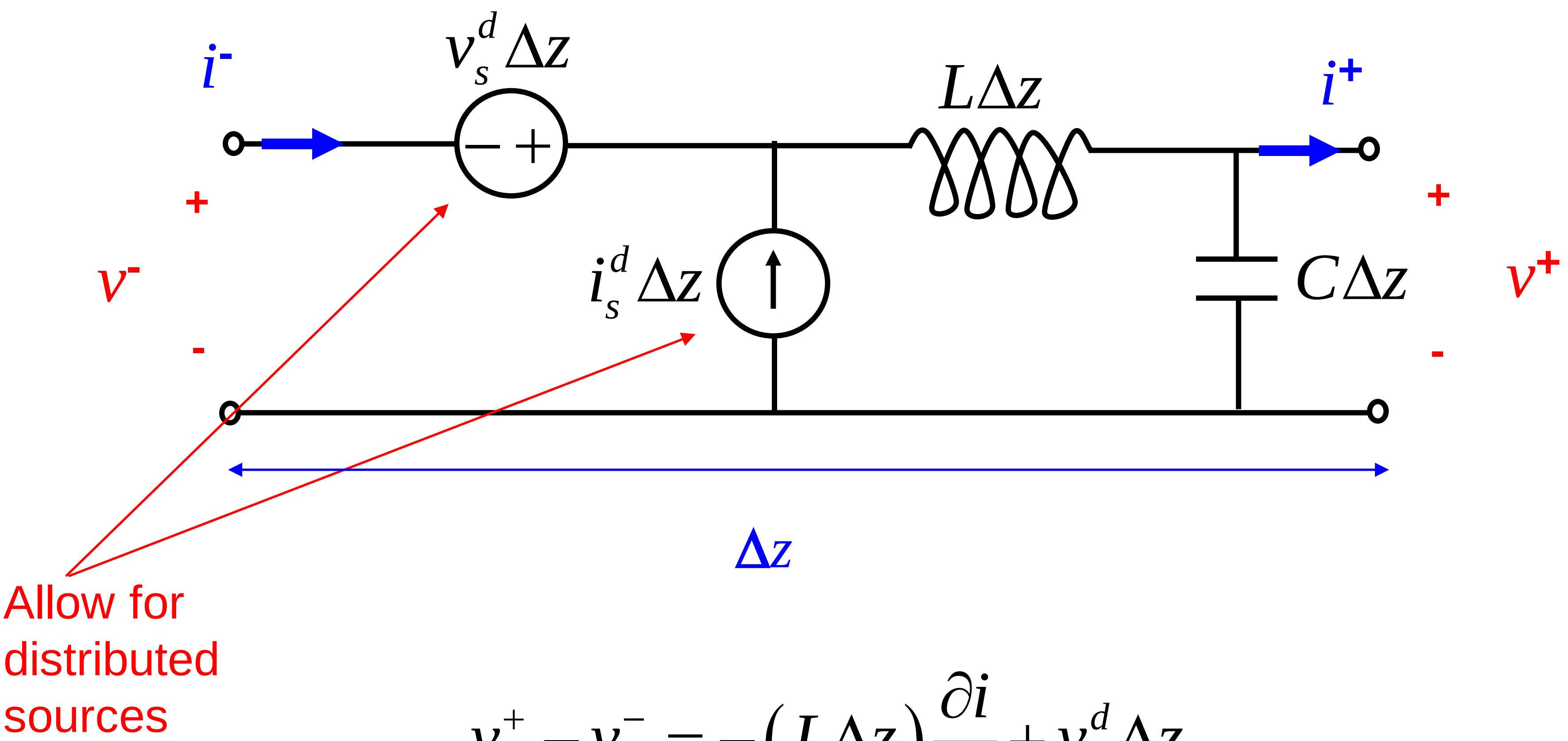
$$I^{TM}(z) = \tilde{H}_v(k_x, k_y, z)$$

We then have:

$$\frac{\partial I^{TM}}{\partial z} = -j\omega\varepsilon V^{TM} - \tilde{J}_u^i$$

$$\frac{\partial V^{TM}}{\partial z} = \left(\frac{k_z^2}{j\omega\varepsilon}\right) I^{TM} + \left[-\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i\right]$$

Telegrapher's Equations



$$v^+ - v^- = -\left(L\Delta z\right)\frac{\partial i}{\partial t} + v_s^d \Delta z$$

so $$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} + v_s^d$$

Telegrapher's Equations (cont.)

Hence, in the phasor domain,

$$\frac{\partial V}{\partial z} = -j\omega L I + V_s^d$$

Also, $$i^+ - i^- = -(C\Delta z)\frac{\partial v}{\partial t} + i_s^d \Delta z$$

so $$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} + i_s^d$$

Hence, in the phasor domain,

$$\frac{\partial I}{\partial z} = -j\omega C V + I_s^d$$

Telegrapher's Equations (cont.)

Compare field equations for TM$_z$ fields with TL equations:

$$\frac{\partial I^{TM}}{\partial z} = -j\omega(\varepsilon)V^{TM} + \left(-\tilde{J}_u^i\right)$$

$$\frac{\partial V^{TM}}{\partial z} = -j\omega\left(\frac{k_z^2}{\omega^2\varepsilon}\right)I^{TM} + \left[-\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i\right]$$

$$\frac{\partial I}{\partial z} = -j\omega CV + I_s^d$$

$$\frac{\partial V}{\partial z} = -j\omega LI + V_s^d$$

Telegrapher's Equations (cont.)

We then make the following identifications:

$$C = \varepsilon$$

$$L = \frac{k_z^2}{\omega^2 \varepsilon}$$

Hence

$$k_z^{TL} = \omega\sqrt{LC} = \omega\sqrt{\frac{k_z^2}{\omega^2 \varepsilon}\varepsilon} = k_z$$

$$Z_0^{TL} = \sqrt{\frac{L}{C}} = \sqrt{\frac{k_z^2}{\omega^2 \varepsilon^2}} = \frac{k_z}{\omega\varepsilon}$$

or

$$k_z^{TL} = k_z$$

$$Z_0^{TL} = \frac{k_z}{\omega\varepsilon}$$

Sources: TM$_z$

For the sources we have, for the TM$_z$ case:

$$I_s^{d^{TM}} = -\tilde{J}_u^i$$

$$V_s^{d^{TM}} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i$$

Sources: TM_z (cont.)

Special case: *planar surface-current sources*

Assume $$\underline{J}\left(x,y,z\right)=\underline{J}_s\left(x,y\right)\delta(z)$$

$$\underline{M}\left(x,y,z\right)=\underline{M}_s\left(x,y\right)\delta(z)$$

$$I_s^{d^{TM}}=-\tilde{J}_{su}^{i}\delta\left(z\right)$$

$$V_s^{d^{TM}}=-\tilde{M}_{sv}^{i}\delta\left(z\right)$$

Then we have $$I_s^{TM}=-\tilde{J}_{su}^{i}$$

This is a lumped parallel current generator.

Similarly, $$V_s^{TM}=-\tilde{M}_{sv}^{i}$$

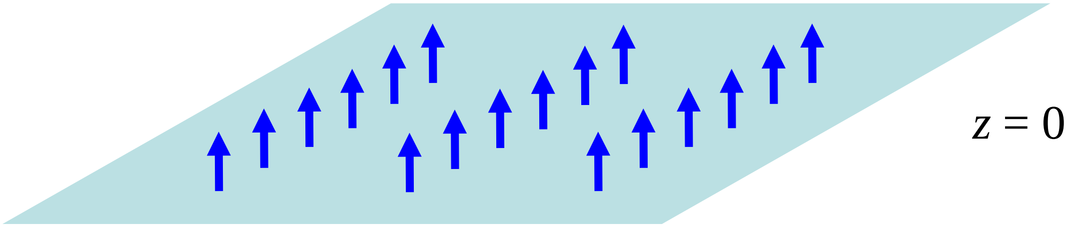
This is a lumped series voltage generator.

Sources: TM$_z$ (cont.)

Special case: *vertical planar electric current*

If $$J_z^i(x,y,z) = f(x,y)\,\delta(z)$$

Then we have $$V_s^{TM} = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{f}\left(k_x,k_y\right)$$



Example: $f(x,y) = \delta(x)\delta(y)$ (unit-amplitude vertical electric dipole)

$$\tilde{f}\left(k_x,k_y\right) = 1$$

TE$_z$ Fields

Use duality:

$$\underline{E} \to \underline{H}$$

$$\underline{H} \to -\underline{E}$$

$$\underline{J}^i \to \underline{M}^i$$

$$\underline{M}^i \to -\underline{J}^i$$

$$\varepsilon \rightleftarrows \mu$$

TM$_z$

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^i - j\omega\varepsilon \tilde{E}_u$$

$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i + \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v$$

TE$_z$

$$-\frac{\partial \tilde{E}_v}{\partial z} = -\tilde{M}_u^i - j\omega\mu \tilde{H}_u$$

$$\frac{\partial \tilde{H}_u}{\partial z} = +\tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i - \left(\frac{k_z^2}{j\omega\mu}\right)\tilde{E}_v$$

TE$_z$ (cont.)

Define

$$V^{TE}(z) = -\tilde{E}_v(k_x, k_y, z)$$
$$I^{TE}(z) = \tilde{H}_u(k_x, k_y, z)$$

$$\frac{\partial V^{TE}}{\partial z} = -j\omega(\mu) I^{TE} + \left[-\tilde{M}_u^i\right]$$
$$\frac{\partial I^{TE}}{\partial z} = -j\omega\left(\frac{k_z^2}{\omega^2 \mu}\right) V^{TE} + \tilde{J}_v^i + \left[\left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i\right]$$

We then identify:

$$L = \mu$$
$$C = \frac{k_z^2}{\omega^2 \mu}$$

$\Rightarrow$

$$k_z^{TL} = k_z$$
$$Z_0^{TE} = \frac{\omega\mu}{k_z}$$

TE$_z$ (cont.)

For the sources, we have

$$V_s^{d\,TE} = -\tilde{M}_u^i$$

$$I_s^{d\,TE} = \tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i$$

Special case of horizontal surface currents:

$$V_s^{TE} = -\tilde{M}_{su}^i$$

$$I_s^{TE} = +\tilde{J}_{sv}^i$$

Special case of vertical planar currents: $\left(M_z^i = g(x,y)\,\delta(z)\right)$

$$I_s^{TE} = \left(\frac{k_t}{\omega\mu}\right)\tilde{g}$$

Summary

$$V^{TM} = \tilde{E}_u$$
$$I^{TM} = \tilde{H}_v$$
$$V^{TE} = -\tilde{E}_v$$
$$I^{TE} = \tilde{H}_u$$

$$I_s^{d^{TM}} = -\tilde{J}_u^i$$
$$V_s^{d^{TM}} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i$$

$$V_s^{d^{TE}} = -\tilde{M}_u^i$$
$$I_s^{d^{TE}} = \tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i$$

Special case of horizontal surface currents:

$$I_s^{TM} = -\tilde{J}_{su}^i$$
$$V_s^{TM} = -\tilde{M}_{sv}^i$$

$$I_s^{TE} = -\tilde{M}_{su}^i$$
$$V_s^{TE} = +\tilde{J}_{sv}^i$$

Summary (cont.)

Special case of vertical planar currents:

$$V_s^{TM} = \left(\frac{k_t}{\omega\varepsilon} \right) \tilde{f}$$

$$\left(J_z^i = f(x, y)\,\delta(z) \right)$$

$$I_s^{TE} = \left(\frac{k_t}{\omega\mu} \right) \tilde{g}$$

$$\left(M_z^i = g(x, y)\,\delta(z) \right)$$

Example

Calculate G_{zz}

To calculate $\tilde{G}_{zz}$ use: $\nabla \times \underline{H} = \underline{J} + j\omega\varepsilon \underline{E}$

We then have

$$E_z = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) - \frac{1}{j\omega\varepsilon} J_z$$

so that

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon_{obs}}\left(-jk_x\tilde{H}_y + jk_y\tilde{H}_x\right) - \frac{1}{j\omega\varepsilon_{obs}}\tilde{J}_z$$

Example (cont.)

We have

$$\tilde{H}_x = \tilde{H}_u \cos\bar{\phi} + \tilde{H}_v \left(-\sin\bar{\phi}\right)$$
$$= I^{TE} \cos\bar{\phi} + I^{TM} \left(-\sin\bar{\phi}\right)$$

$$\tilde{H}_y = \tilde{H}_u \sin\bar{\phi} + \tilde{H}_v \cos\bar{\phi}$$
$$= I^{TE} \sin\bar{\phi} + I^{TM} \cos\bar{\phi}$$

The TE part cancels when we substitute these expressions into the expression for the transform of E_z, so we have

$$\tilde{E}_z = \frac{-k_t}{\omega\varepsilon_{obs}}\left(\cos^2\bar{\phi} + \sin^2\bar{\phi}\right) I^{TM} - \frac{1}{j\omega\varepsilon_{obs}}\tilde{J}_z$$

Example (cont.)

or

$$\tilde{E}_z = \frac{-k_t}{\omega \varepsilon_{obs}} I^{TM} - \frac{1}{j\omega \varepsilon_{obs}} \tilde{J}_z$$

From the strength of the TM voltage generator due to a unit-amplitude vertical dipole at z', we then have

$$I^{TM} = \left(\frac{k_t}{\omega \varepsilon_{src}} \right) I_v^{TM}$$

We also have

$$J_z = J_z^i = f(x,y)\,\delta(z - z')$$

where

$$f(x,y) = \delta(x)\delta(y) \qquad \tilde{f}(k_x, k_y) = 1$$

Example (cont.)

Hence, we have

$$\tilde{E}_z = \frac{-1}{\omega \varepsilon_{obs}} \left(\frac{k_t^2}{\omega \varepsilon_{src}} \right) I_v^{TM} - \frac{1}{j\omega \varepsilon_{obs}} \delta\left(z - z'\right)$$

Because of the delta function, we can replace the observation subscript with the source subscript, so that

$$\tilde{E}_z = \frac{-1}{\omega \varepsilon_{obs}} \left(\frac{k_t^2}{\omega \varepsilon_{src}} \right) I_v^{TM} - \frac{1}{j\omega \varepsilon_{src}} \delta\left(z - z'\right)$$

Note: If the vertical dipole is at an interface between two different materials, then the delta-function term is not well defined. In this case, we interpret the source as residing infinitesimally on one side of the interface (the source side)

Example (cont.)

In the space domain we have
(after taking the 2D inverse Fourier transform):

$$E_z = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-1}{\omega \varepsilon_{obs}} \left(\frac{k_t^2}{\omega \varepsilon_{src}} \right) I_v^{TM} e^{-j\left(k_x x + k_y y\right)} \, dk_x dk_y$$

$$- \frac{1}{j\omega \varepsilon_{src}} \delta(z - z') \big(\delta(x) \delta(y) \big)$$

where we have used

$$F^{-1}(1) = \delta(x) \delta(y)$$

Example (cont.)

Converting to polar coordinates, we have (for any function F):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F\left(k_t\right) e^{-j\left(k_x x+k_y y\right)}\, dk_x dk_y = \int_0^{\infty}\int_0^{2\pi} F\left(k_t\right)\, e^{-j\left(k_x x+k_y y\right)}\, k_t d\bar{\phi}\, dk_t$$

$$= \int_0^{\infty} F\left(k_t\right) k_t \int_0^{2\pi} e^{-j\left(k_t\rho\right)\left(\cos\bar{\phi}\cos\phi+\sin\bar{\phi}\sin\phi\right)}\, d\bar{\phi}\, dk_t$$

$$= \int_0^{\infty} F\left(k_t\right) k_t \int_0^{2\pi} e^{-j\left(k_t\rho\right)\cos\left(\bar{\phi}-\phi\right)}\, d\bar{\phi}\, dk_t$$

$$= \int_0^{\infty} F\left(k_t\right) k_t \int_0^{2\pi} e^{-j\left(k_t\rho\right)\cos\bar{\phi}}\, d\bar{\phi}\, dk_t$$

$$= \int_0^{\infty} F\left(k_t\right) k_t \left(2\pi J_0\left(k_t\rho\right)\right) dk_t$$

Example (cont.)

In the space domain we then have

$$E_z = \frac{1}{2\pi}\int_0^\infty \frac{-1}{\omega\varepsilon_{obs}}\left(\frac{k_t^3}{\omega\varepsilon_{src}}\right) I_v^{TM} J_0\left(k_t\rho\right) d\rho$$
$$-\frac{1}{j\omega\varepsilon_{src}}\delta\left(z-z'\right)\left(\delta\left(x\right)\delta\left(y\right)\right)$$

so that

$$G_{zz} = \frac{1}{2\pi}\int_0^\infty \frac{-1}{\omega\varepsilon_{obs}}\left(\frac{k_t^3}{\omega\varepsilon_{src}}\right) I_v^{TM} J_0\left(k_t\rho\right) d\rho$$
$$-\frac{1}{j\omega\varepsilon_{src}}\delta\left(z-z'\right)\left(\delta\left(x\right)\delta\left(y\right)\right)$$

Note: The integral does not converge when $z = z'$, and it must be interpreted in a limiting sense.