ECE 6345

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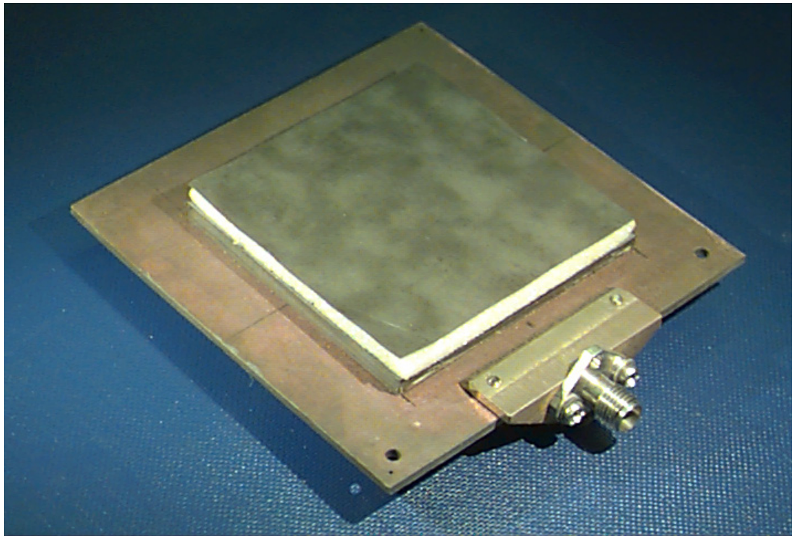


Notes 25

Overview

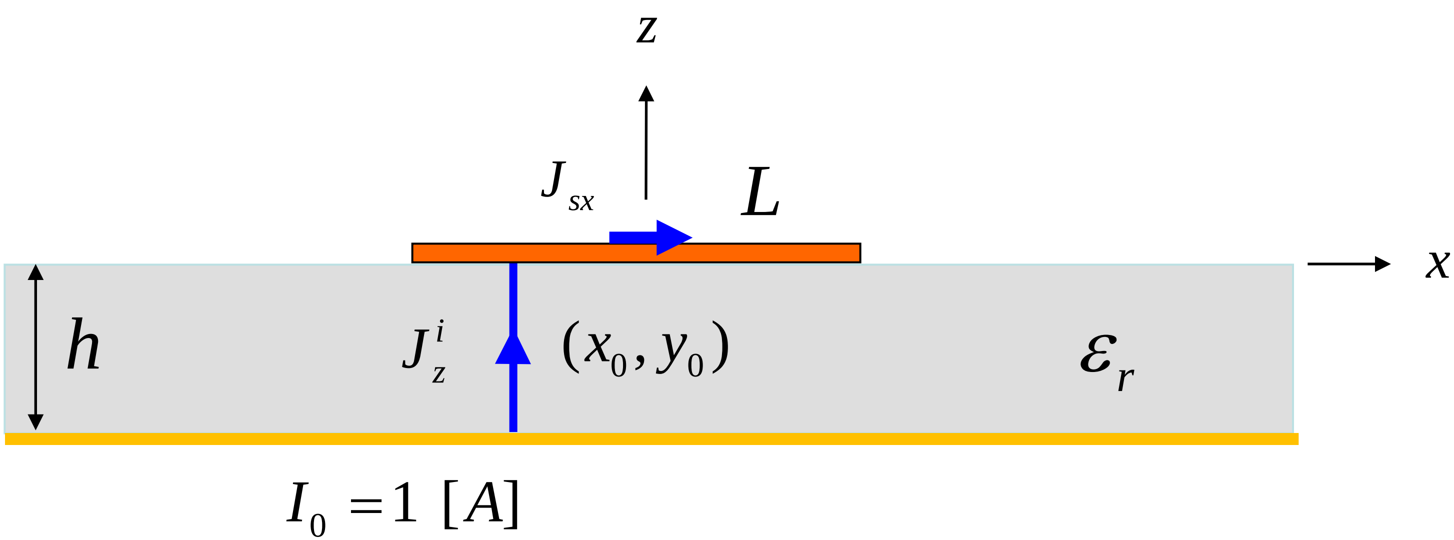
In this set of notes we use the spectral-domain method to find the input impedance of a rectangular patch antenna.

This method uses the exact spectral-domain Green's function, so all radiation physics, including surface-wave excitation, is automatically included (no need for an effective permittivity).

It does not account for the probe inductance (the way it is formulated here), so the CAD formula for probe inductance is added on at the end.

D. M. Pozar, "Input impedance and mutual coupling of rectangular microstrip antennas," *IEEE Trans. Antennas and Propagation*, vol. 30, pp. 1291-1196, Nov. 1982.

Spectral Domain Method



The probe is viewed as an impressed current.

Set $E_x = 0 \qquad (x, y) \in S$ S is the patch surface

$$E_x\left[J_{sx}\right] + E_x\left[J_z^i\right] = 0 \qquad (x, y) \in S$$

This is the "Electric Field Integral Equation (EFIE)"

Spectral Domain Method (cont.)

Let $J_{sx}(x,y) = A_x B_x(x,y)$

$$B_x(x,y) = \cos\left(\frac{\pi x}{L}\right)$$

The EFIE is then $A_x E_x[B_x] + E_x\left[J_z^i\right] = 0$

Pick a "testing" function $T(x,y)$:

$$\int_S T(x,y)\left\{A_x E_x[B_x] + E_x\left[J_z^i\right]\right\} dS = 0$$

$$A_x \int_S T(x,y) E_x[B_x]\, dS + \int_S T(x,y) E_x\left[J_z^i\right] dS = 0$$

Spectral Domain Method (cont.)

Galerkin's Method: $T(x,y) = B_x(x,y)$

(The testing function is the same as the basis function.)

Hence $$A_x \int_S B_x(x,y) E_x[B_x]\, dS + \int_S B_x(x,y) E_x\left[J_z^i\right] dS = 0$$

The solution for the unknown amplitude coefficient A_x is then

$$A_x = -\frac{\displaystyle\int_S B_x(x,y) E_x\left[J_z^i\right] dS}{\displaystyle\int_S B_x(x,y) E_x[B_x]\, dS} = -\frac{\left\langle J_z^i, B_x \right\rangle}{\left\langle B_x, B_x \right\rangle}$$

$$\left\langle J_z^i, B_x \right\rangle \equiv \int_S E_x\left[J_z^i\right] B_x(x,y)\, dS \qquad \left\langle B_x, B_x \right\rangle = \int_S E_x[B_x] B_x(x,y)\, dS$$

Spectral Domain Method (cont.)

The input impedance is calculated as:

$$Z_{in} = \frac{2P_{in}}{\left|I_0\right|^2}$$

P_{in} = complex power coming from impressed probe current (in the presence of the patch).

$$= \frac{2}{\left|I_0\right|^2} \int_V -\frac{1}{2} E_z J_z^{i*} \, dV$$

V = volume of probe current

$$= -\int_V E_z J_z^i \, dV$$

(The probe current is real and equal to 1.0 [A].)

The total field comes from the patch and the probe:

$$E_z = E_z\left[J_z^i\right] + E_z\left[J_{sx}\right]$$

Spectral Domain Method (cont.)

Hence

$$Z_{in} = -\int_V J_z^i \, E_z\left[J_z^i\right] dV - \int_V J_z^i \, E_z\left[J_{sx}\right] dV$$

Define: $$Z_{probe} \equiv -\int_V J_z^i \, E_z\left[J_z^i\right] dV = -\left\langle J_z^i , J_z^i \right\rangle$$

Then we have

$$Z_{in} = Z_{probe} - \int_V J_z^i \, E_z\left[J_{sx}\right] dV$$

or

$$Z_{in} = Z_{probe} - A_x \int_V J_z^i \, E_z\left[B_x\right] dV = Z_{probe} - A_x \left\langle B_x , J_z^i \right\rangle$$

Spectral Domain Method (cont.)

$$Z_{in} = Z_{probe} - A_x \left\langle B_x, J_z^i \right\rangle$$

where

$$A_x = -\frac{\left\langle J_z^i, B_x \right\rangle}{\left\langle B_x, B_x \right\rangle}$$

We have from reciprocity that

$$\left\langle J_z^i, B_x \right\rangle = \left\langle B_x, J_z^i \right\rangle$$

Note: Z_{zx} is easier to calculate than Z_{xz}.

so that

$$Z_{in} = Z_{probe} + \frac{\left\langle B_x, J_z^i \right\rangle^2}{\left\langle B_x, B_x \right\rangle}$$

Spectral Domain Method (cont.)

Define:

$$Z_{xx} \equiv -\langle B_x, B_x \rangle$$

$$Z_{zx} \equiv -\langle B_x, J_z^i \rangle$$

Note: The minus sign is added to agree with typical MoM convention.

Note: The subscript notation on Z_{ij} follows the usual MoM convention.

We then have:

$$Z_{in} = Z_{probe} - \frac{Z_{zx}^2}{Z_{xx}}$$

Spectral Domain Method (cont.)

Note: The probe impedance may be approximately calculated by using a CAD formula:

$$Z_{probe} \approx jX_p$$

$$X_p = \eta_0 \mu_r \left(\frac{h}{\lambda_0} \right) \left[\ln \left(\frac{1}{a / \lambda_0} \right) - \gamma - \ln \pi - \ln \sqrt{\mu_r \varepsilon_r} \right]$$

$\gamma \doteq 0.57722$ (Euler's constant)

This result comes from a probe inside of an infinite parallel-plate waveguide.

Nlote: Calculating Z_{probe} exactly from the spectral-domain method can be done, but this would be a lot of work, and the improvement would be small.

Spectral Domain Method (cont.)

The next goal is to calculate the reactions Z_{xx} and Z_{xz} in closed form.

For the patch-patch reaction we have:

$$Z_{xx} = -\langle B_x, B_x \rangle = -\int_S E_x \left[B_x \right] B_x \, dS$$

From previous SDI theory, we have $\tilde{E}_x = \tilde{G}_{xx} \tilde{B}_x$

so $$E_x \left[B_x \right] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}_{xx} \tilde{B}_x \, e^{-j\left(k_x x + k_y y\right)} \, dk_x dk_y$$

Hence, integrating over the patch surface, we have

$$Z_{xx} = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} \tilde{G}_{xx}\left(k_x, k_y\right) \tilde{B}_x\left(k_x, k_y\right) \tilde{B}_x\left(-k_x, -k_y\right) dk_x dk_y$$

Spectral Domain Method (cont.)

Since the Fourier transform of the basis function (cosine function) is an even function of k_x and k_y, we can write:

$$Z_{xx} = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}_{xx}\left(k_x, k_y\right) \tilde{B}_x^2\left(k_x, k_y\right) dk_x dk_y$$

or

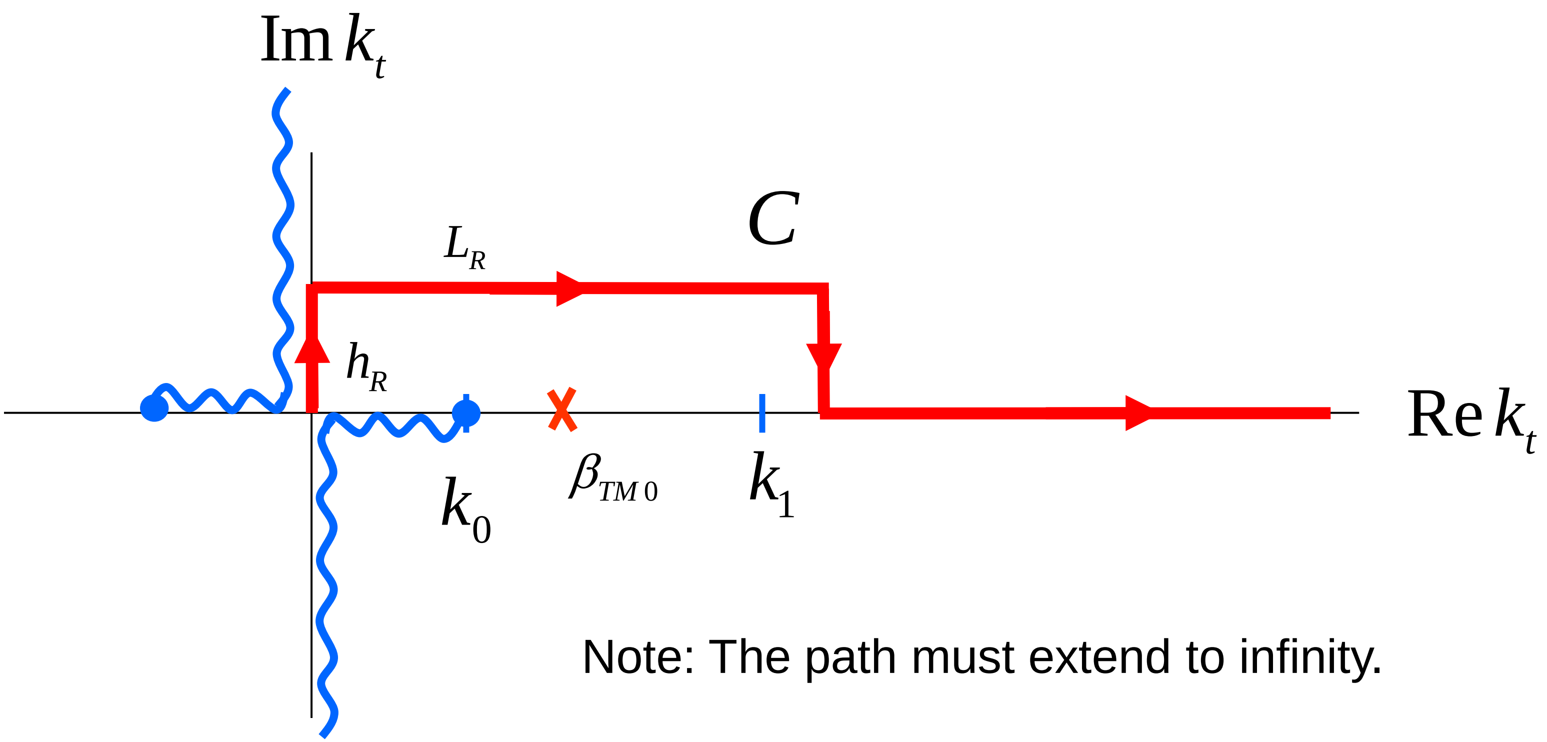
$$Z_{xx} = -\frac{1}{\pi^2} \int_{0}^{\infty} \int_{0}^{\infty} \tilde{G}_{xx}\left(k_x, k_y\right) \tilde{B}_x^2\left(k_x, k_y\right) dk_x dk_y$$

Note: $z = z' = 0$ in the spectral-domain Green's function here.

Spectral Domain Method (cont.)

Converting to polar coordinates, we have

$$Z_{xx} = -\frac{1}{\pi^2} \int_0^{\pi/2} \int_C \tilde{G}_{xx}\left(k_t, \bar{\phi}\right) \tilde{B}_x^2\left(k_t, \bar{\phi}\right) k_t \, dk_t \, d\bar{\phi}$$



Note: The path must extend to infinity.

Spectral Domain Method (cont.)

From previous calculations, we have:

$$\tilde{G}_{xx}\left(k_x, k_y, 0\right) \equiv -\left[\cos^2\overline{\phi}\,\frac{1}{D^{TM}} + \sin^2\overline{\phi}\,\frac{1}{D^{TE}}\right]$$

$$D^{TM}(k_t) = Y_0^{TM} - jY_1^{TM}\cot\left(k_{z1}h\right)$$
$$D^{TE}(k_t) = Y_0^{TE} - jY_1^{TE}\cot\left(k_{z1}h\right)$$

$$\tilde{B}_x\left(k_x, k_y\right) = \left(\frac{\pi}{2}LW\right)\mathrm{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\dfrac{L}{2}\right)}{\left(\dfrac{\pi}{2}\right)^2 - \left(k_x\dfrac{L}{2}\right)^2}\right]$$

Spectral Domain Method (cont.)

For the patch-probe reaction we have

$$Z_{zx} = -\int_V E_z[B_x](x,y,z)\, J_z^i \, dV$$
$$= -\int_{-h}^{0} E_z[B_x](x_0,y_0,z)\, dz$$

Note: Z_{zx} is the voltage drop at the feed location due to the current B_x.

$$\tilde{E}_z = \tilde{G}_{zx} \tilde{J}_{sx}$$

so

$$E_z(x_0,y_0,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{G}_{zx}(z)\, \tilde{B}_x\, e^{-j(k_x x_0 + k_y y_0)}\, dk_x dk_y$$

Spectral Domain Method (cont.)

To calculate $\tilde{G}_{zx}$ use:

$$E_z = \frac{1}{j\omega\varepsilon_1}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

so that

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon_1}\left(-jk_x\tilde{H}_y + jk_y\tilde{H}_x\right)$$

We need the transforms of the transverse magnetic field components.

Spectral Domain Method (cont.)

Using spectral-domain theory, we have

$$\begin{aligned}
\tilde{H}_x &= \tilde{H}_u \cos\bar{\phi} + \tilde{H}_v\left(-\sin\bar{\phi}\right) \\
&= I^{TE}\cos\bar{\phi} + I^{TM}\left(-\sin\bar{\phi}\right) \\
&= I_i^{TE}\left(\tilde{J}_{sv}\right)\cos\bar{\phi} + I_i^{TM}\left(-\tilde{J}_{su}\right)\left(-\sin\bar{\phi}\right) \\
&= I_i^{TE}\tilde{J}_{sx}\left(-\sin\bar{\phi}\right)\cos\bar{\phi} + I_i^{TM}\left(-\tilde{J}_{sx}\cos\bar{\phi}\right)\left(-\sin\bar{\phi}\right) \\
&= -I_i^{TE}\tilde{J}_{sx}\sin\bar{\phi}\cos\bar{\phi} + I_i^{TM}\tilde{J}_{sx}\cos\bar{\phi}\sin\bar{\phi}
\end{aligned}$$

$$V^{TM}(z) = \underline{\hat{u}} \cdot \underline{\tilde{E}}_t\left(k_x, k_y, z\right)$$

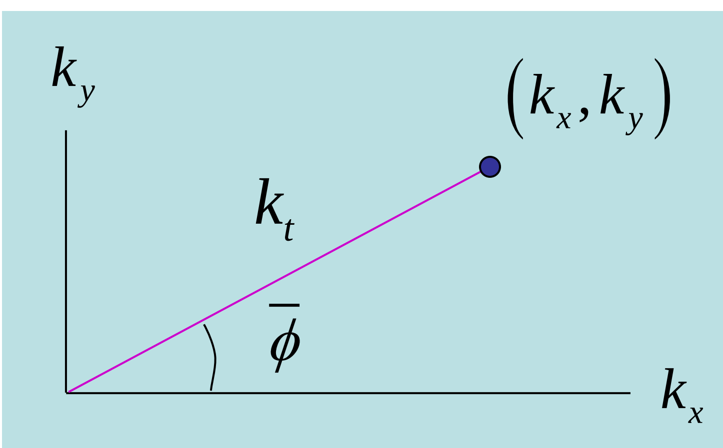
$$I^{TM}(z) = \underline{\hat{v}} \cdot \underline{\tilde{H}}_t\left(k_x, k_y, z\right)$$

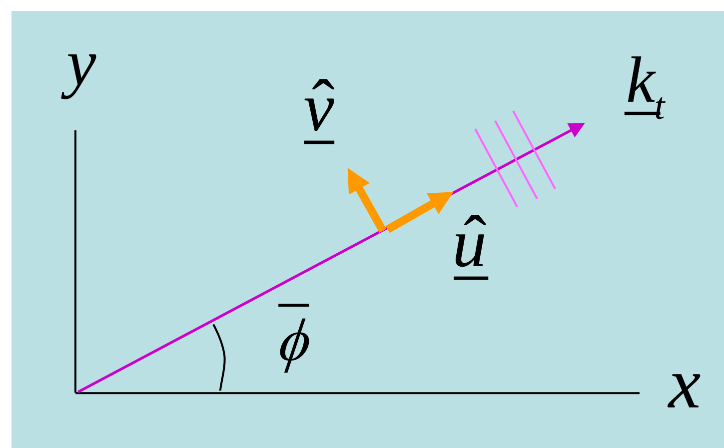
$$I_s^{TM}(z) = -\underline{\hat{u}} \cdot \underline{\tilde{J}}_s\left(k_x, k_y\right)$$

$$V^{TE}(z) = -\underline{\hat{v}} \cdot \underline{\tilde{E}}_t\left(k_x, k_y, z\right)$$

$$I^{TE}(z) = \underline{\hat{u}} \cdot \underline{\tilde{H}}_t\left(k_x, k_y, z\right)$$

$$I_s^{TE}(z) = \underline{\hat{v}} \cdot \underline{\tilde{J}}_s\left(k_x, k_y\right)$$



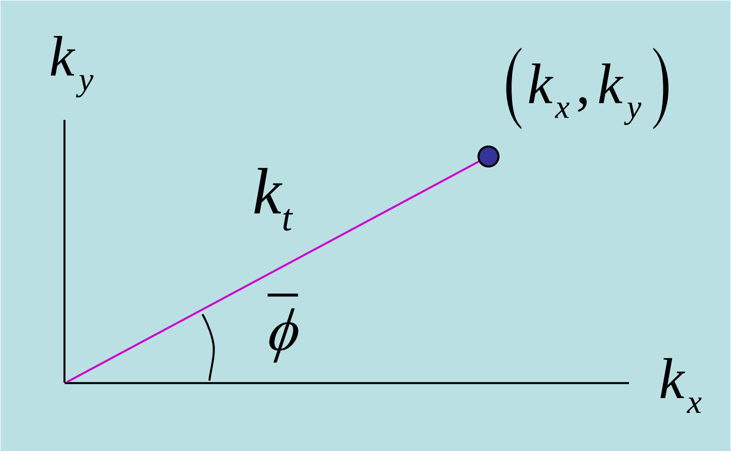


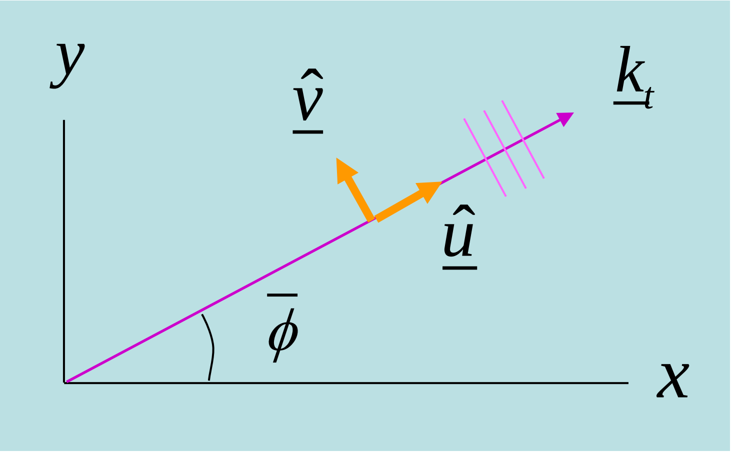
Spectral Domain Method (cont.)

We also have

$$\tilde{H}_y = \tilde{H}_u \sin\overline{\phi} + \tilde{H}_v \cos\overline{\phi}$$
$$= I^{TE} \sin\overline{\phi} + I^{TM} \cos\overline{\phi}$$
$$= I_i^{TE}\left(\tilde{J}_{sv}\right)\sin\overline{\phi} + I_i^{TM}\left(-\tilde{J}_{su}\right)\cos\overline{\phi}$$
$$= I_i^{TE}\left(\tilde{J}_{sx}\left(-\sin\overline{\phi}\right)\right)\sin\overline{\phi} + I_i^{TM}\left(-\tilde{J}_{sx}\cos\overline{\phi}\right)\cos\overline{\phi}$$
$$= -I_i^{TE}\sin^2\overline{\phi} - I_i^{TM}\tilde{J}_{sx}\cos^2\overline{\phi}$$

$$V^{TM}(z) = \underline{\hat{u}} \cdot \underline{\tilde{E}}_t\left(k_x, k_y, z\right)$$
$$I^{TM}(z) = \underline{\hat{v}} \cdot \underline{\tilde{H}}_t\left(k_x, k_y, z\right)$$
$$I_s^{TM}(z) = -\underline{\hat{u}} \cdot \underline{\tilde{J}}_s\left(k_x, k_y\right)$$
$$V^{TE}(z) = -\underline{\hat{v}} \cdot \underline{\tilde{E}}_t\left(k_x, k_y, z\right)$$
$$I^{TE}(z) = \underline{\hat{u}} \cdot \underline{\tilde{H}}_t\left(k_x, k_y, z\right)$$
$$I_s^{TE}(z) = \underline{\hat{v}} \cdot \underline{\tilde{J}}_s\left(k_x, k_y\right)$$





Spectral Domain Method (cont.)

Recall:

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon_1}\left(-jk_x\tilde{H}_y + jk_y\tilde{H}_x\right)$$

We then have

$$-jk_x\tilde{H}_y + jk_y\tilde{H}_x = -jk_t\cos\overline{\phi}\begin{Bmatrix}-I_i^{TE}\tilde{J}_{sx}\sin^2\overline{\phi}\\ -I_i^{TM}\tilde{J}_{sx}\cos^2\overline{\phi}\end{Bmatrix}$$

$$+\,jk_t\sin\overline{\phi}\begin{Bmatrix}-I_i^{TE}\tilde{J}_{sx}\sin\overline{\phi}\cos\overline{\phi}\\ +I_i^{TM}\tilde{J}_{sx}\sin\overline{\phi}\cos\overline{\phi}\end{Bmatrix}$$

Note:

$$k_x = k_t\cos\overline{\phi}$$

$$k_y = k_t\sin\overline{\phi}$$

$$= jk_tI_i^{TM}\tilde{J}_{sx}\cos\overline{\phi}\left(\cos^2\overline{\phi} + \sin^2\overline{\phi}\right)$$

$$= jk_t\tilde{J}_{sx}\cos\overline{\phi}\,I_i^{TM}$$

Spectral Domain Method (cont.)

Hence we have

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon_1}\left(jk_t \tilde{J}_{sx} \cos\overline{\phi}\, I_i^{TM} \right)$$

Using $\tilde{E}_z = \tilde{G}_{zx}\tilde{J}_{sx}$

we then identify that

$$\tilde{G}_{zx}(z) = \frac{k_t}{\omega\varepsilon_1}\left(\cos\overline{\phi}\, I_i^{TM}(z)\right)$$

Spectral Domain Method (cont.)

From TL theory, we have the property that

$$I_i^{TM}(z) = I_i^{TM}(-h)\cos k_{z1}(z+h)$$

(The short circuit at $z = -h$ causes the current to have a zero derivative there.)

Hence

$$\tilde{G}_{zx}(z) = \frac{k_t}{\omega\varepsilon_1}\cos\bar{\phi}\, I_i^{TM}(-h)\cos k_{z1}(z+h)$$

Spectral Domain Method (cont.)

For the field due to the patch basis function, we then have

$$E_z\left(x_0, y_0, z\right) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{E}_z\left(z\right) e^{-j\left(k_x x_0 + k_y y_0\right)} \, dk_x dk_y$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{G}_{zx}\left(z\right) \tilde{B}_x \, e^{-j\left(k_x x_0 + k_y y_0\right)} \, dk_x dk_y$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\frac{k_t}{\omega \varepsilon_1} \cos\overline{\phi} \left(I_i^{TM}(-h) \right) \cos k_{z1}(z+h) \right] \tilde{B}_x \, e^{-j\left(k_x x_0 + k_y y_0\right)} dk_x dk_y$$

Recall that

$$Z_{zx} = -\int_{-h}^{0} E_z\left(x_0, y_0, z\right) \, dz$$

Note that

$$\int_{-h}^{0} \cos k_{z1}(z+h) dz = \int_{0}^{h} \cos k_{z1} z' dz' = h \, \mathrm{sinc}\left(k_{z1} h\right)$$

$$z' = z + h$$

Spectral Domain Method (cont.)

Hence we have

$$Z_{zx} = -\frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{k_t}{\omega\varepsilon_1} \left(I_i^{TM}(-h) \right) \tilde{B}_x \cos\bar{\phi} \, h \, \text{sinc}\left(k_{z1}h\right) e^{-j\left(k_x x_0 + k_y y_0\right)} dk_x dk_y$$

where

$$\cos\bar{\phi} = \frac{k_x}{k_t}$$

Spectral Domain Method (cont.)

The integrand is an even function of k_y and an odd function of k_x (due to the cosine term). Hence we use the following combinations to reduce the integration to one over the first quadrant:

Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4

$$
\begin{aligned}
& e^{-j(k_x x_0)} e^{-j(k_y y_0)} - e^{+j(k_x x_0)} e^{-j(k_y y_0)} - e^{+j(k_x x_0)} e^{+j(k_y y_0)} + e^{-j(k_x x_0)} e^{+j(k_y y_0)} \\
&= -2j \sin(k_x x_0) e^{-j(k_y y_0)} - 2j \sin(k_x x_0) e^{+j(k_y y_0)} \\
&= -2j \sin(k_x x_0) \left[e^{-j(k_y y_0)} + e^{+j(k_y y_0)} \right] \\
&= -2j \sin(k_x x_0) \left[2\cos(k_y y_0) \right] \\
&= -4j \sin(k_x x_0) \cos(k_y y_0)
\end{aligned}
$$

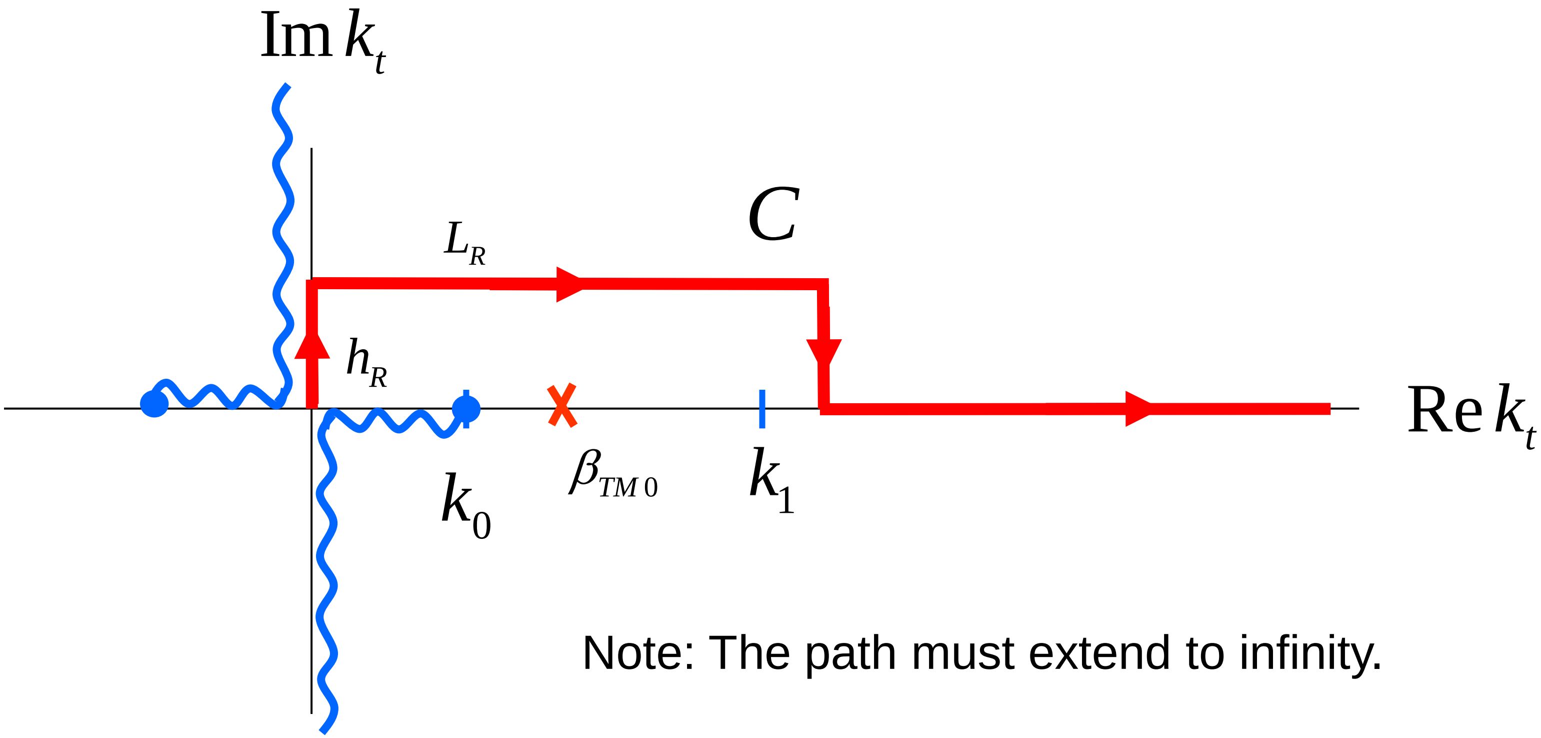
The result is then

$$
Z_{zx} = +\frac{j}{\pi^2} \left(\frac{h}{\omega \varepsilon_1} \right) \int_0^{\pi/2} \int_0^{\infty} \left\{ k_t \, I_i^{TM}(-h) \, \tilde{B}_x \cos\bar{\phi} \, \operatorname{sinc}(k_{z1} h) \right\} \cdot \sin(k_x x_0) \cos(k_y y_0) \, k_t \, dk_t \, d\bar{\phi}
$$

Spectral Domain Method (cont.)

The final result is then:

$$Z_{zx} = \frac{j}{\pi^2}\left(\frac{h}{\omega\varepsilon_1}\right)\int_0^{\pi/2}\int_C \left\{ k_t^2 \, I_i^{TM}(-h)\, \tilde{B}_x \cos\bar{\phi} \; \mathrm{sinc}\left(k_{z1}h\right)\right\}$$
$$\cdot \sin\left(k_x x_0\right)\cos\left(k_y y_0\right) dk_t \, d\bar{\phi}$$



Note: The path must extend to infinity.

Spectral Domain Method (cont.)

Note on material loss:

The spectral-domain method already accounts for radiation into space and into surface waves, and accounts for dielectric loss by using an complex permittivity.

In order to account for conductor loss, we can use

$$\tan\delta_{eff} = \frac{1}{Q_{loss}} = \frac{1}{Q_d} + \frac{1}{Q_c} \quad \Rightarrow \quad \varepsilon_r^{eff} = \varepsilon_r'\left(1 - j\tan\delta_{eff}\right)$$

where

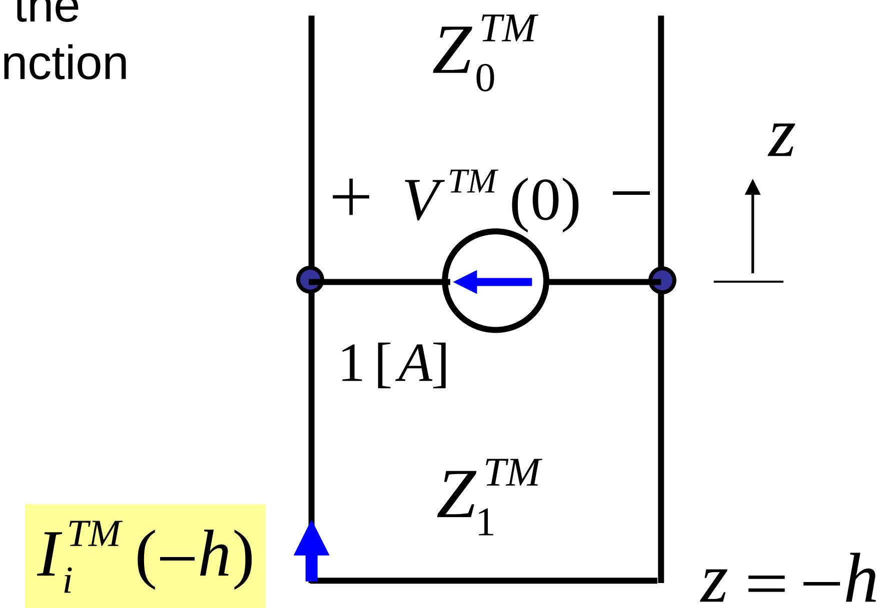
$$Q_d = \tan\delta \qquad Q_c = \left(\frac{\eta_0}{2}\right)\mu_r\left[\frac{(k_0 h)}{R_s^{ave}}\right]$$

It is also possible to account for conductor loss by using a impedance boundary condition on the patch, but using an effective loss tangent is a simpler approach (no need to modify the code – simply increase the loss tangent to account for conductor loss).

Spectral Domain Method (cont.)

We now calculate the needed current function

$$I_i^{TM}(-h)$$

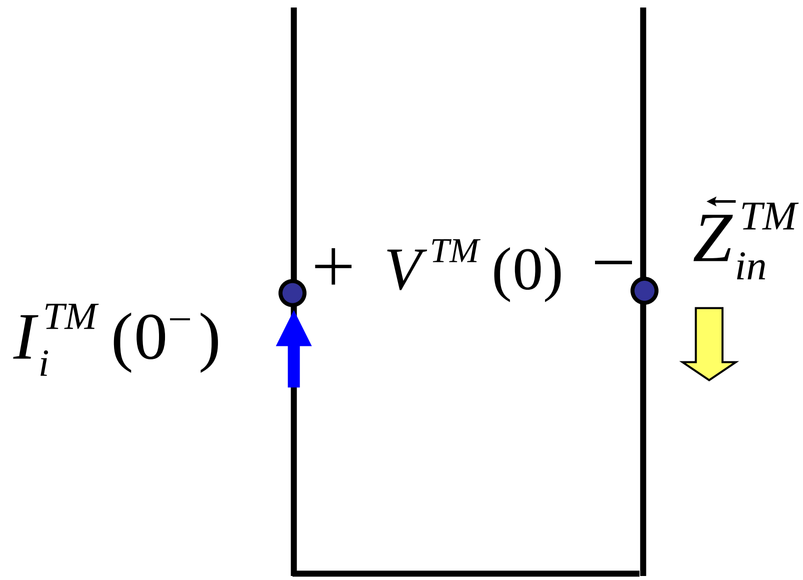


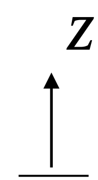
$$V_i^{TM}(0) = (1)Z_{in}^{TM}$$

$$= \frac{1}{Y_0^{TM} - jY_1^{TM}\cot\left(k_{z1}h\right)}$$

$$= \frac{1}{D^{TM}}$$

Spectral Domain Method (cont.)





From last slide:

$$V_i^{TM}(0) = \frac{1}{D^{TM}}$$

$$I_i^{TM}(0^-) = \frac{-V_i^{TM}(0)}{\overleftarrow{Z}_{in}^{TM}}$$

$$\overleftarrow{Z}_{in}^{TM} = jZ_1^{TM} \tan\left(k_{z1}h\right)$$

so

$$I_i^{TM}(0^-) = -\frac{1}{D^{TM}}\left[jZ_1^{TM} \tan\left(k_{z1}h\right)\right]^{-1}$$

Spectral Domain Method (cont.)

Also,

$$I_i^{TM}(z) = I_i^{TM}(-h)\cos\left(k_{z1}\left(z+h\right)\right) \quad -h \le z < 0$$

so

$$I_i^{TM}(0^-) = I_i^{TM}(-h)\cos\left(k_{z1}h\right)$$

And therefore

$$I_i^{TM}(-h) = I_i^{TM}(0^-)\sec\left(k_{z1}h\right)$$

Hence

$$= \left[-\frac{1}{D^{TM}} \left[jZ_1^{TM} \tan\left(k_{z1}h\right)\right]^{-1} \right] \sec\left(k_{z1}h\right)$$

$$I_i^{TM}(-h) = \left[-\frac{1}{Y_0^{TM} - jY_1^{TM}\cot\left(k_{z1}h\right)} \right] \left[\frac{1}{jZ_1^{TM}\tan\left(k_{z1}h\right)} \right] \sec\left(k_{z1}h\right)$$

Results

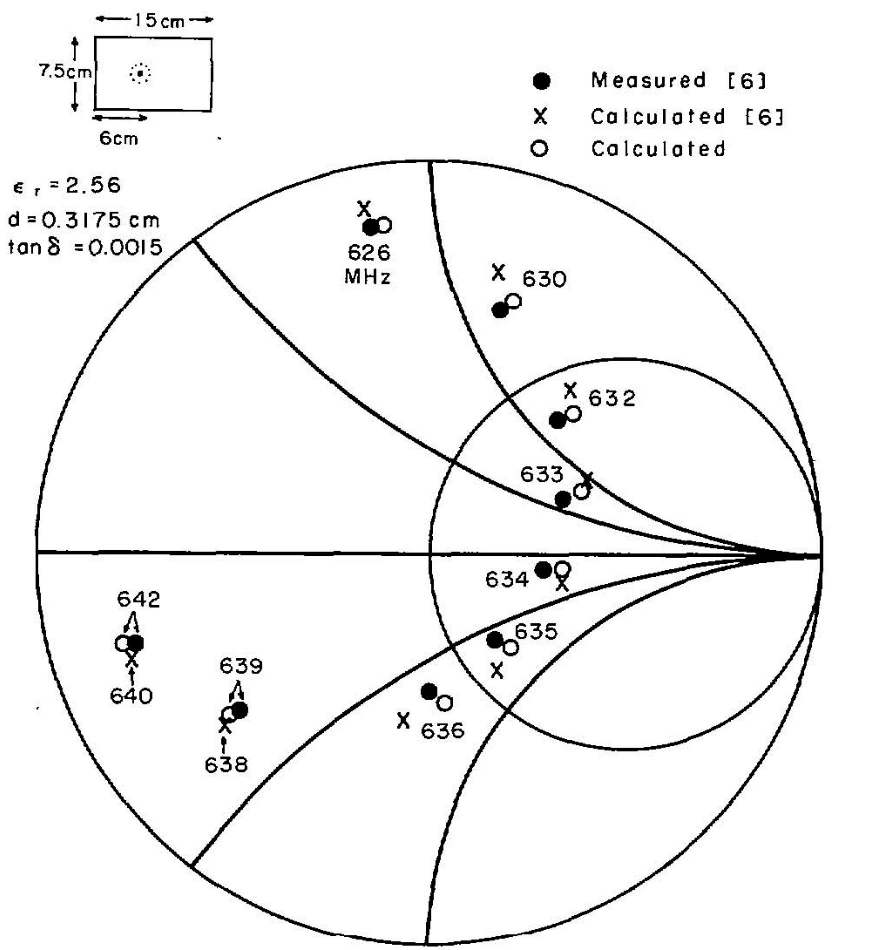


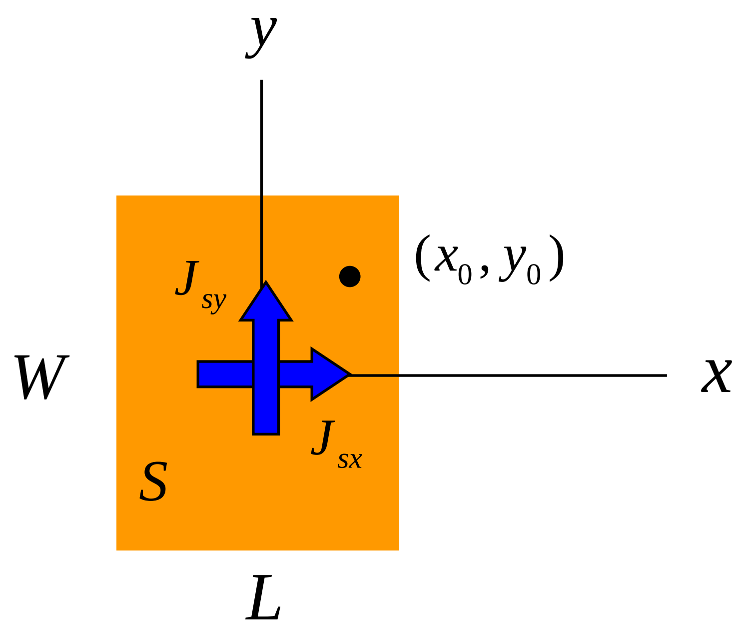
Fig. 5. Measured and calculated input impedance of a coax-fed microstrip antenna compared with calculations from [6].

D. M. Pozar, "Input impedance and mutual coupling of rectangular microstrip antennas," *IEEE Trans. Antennas Propagat.*, vol. AP-30. pp. 1191-1196, Nov. 1982.

[6] E. H. Newman and P. Tulyathan, "Analysis of microstrip antennas using moment methods," *IEEE Trans. Antennas Propagat.*, vol. AP-29. pp. 47-53, Jan. 1981.

Two Basis Functions

Note:
Using two basis functions is important for circular polarization or for dual-polarized patches.



$$\underline{J}_s(x,y) = A_x B_x(x,y) + A_y B_y(x,y)$$

EFIE:

$$E_x: \quad A_x E_x[B_x] + A_y E_x[B_y] + E_x[J_z^i] = 0 \ \text{ on } S$$

$$E_y: \quad A_x E_y[B_x] + A_y E_y[B_y] + E_y[J_z^i] = 0 \ \text{ on } S$$

Two Basis Functions (cont.)

Galerkin testing:

$$\int_S E_x B_x \, dS = 0$$

$$\int_S E_y B_y \, dS = 0$$

$$A_x \left\langle B_x , B_x \right\rangle + A_y \left\langle B_y , B_x \right\rangle = -\left\langle J_z^i , B_x \right\rangle$$

$$A_x \left\langle B_x , B_y \right\rangle + A_y \left\langle B_y , B_y \right\rangle = -\left\langle J_z^i , B_y \right\rangle$$

Define:

$$Z_{ij} \equiv -\left\langle B_j , B_i \right\rangle$$

$$Z_{zi} \equiv -\left\langle B_i , J_z^i \right\rangle$$

Two Basis Functions (cont.)

$$A_x Z_{xx} + A_y Z_{xy} = -Z_{xz}$$

$$A_x Z_{yx} + A_y Z_{yy} = -Z_{yz}$$

$$Z_{xy} = Z_{yx}$$

(reciprocity)

By symmetry, $Z_{xy} = Z_{yx} = 0$

This follows since

$$Z_{yx} = -\langle B_x, B_y \rangle = \int_S B_y(x,y) E_y\left[B_x(x,y)\right] dS$$

and

$$E_y\left[B_x\right](x,y) = \mathrm{Odd}(y)$$

$$B_y(x,y) = \mathrm{Even}(y)$$

(from symmetry)

Two Basis Functions (cont.)

Hence, the two testing equations reduce to:

$$A_x Z_{xx} = -Z_{xz}$$

$$A_y Z_{yy} = -Z_{yz}$$

The solution is:

$$A_x = -\frac{Z_{xz}}{Z_{xx}}$$

$$A_y = -\frac{Z_{yz}}{Z_{yy}}$$

Using reciprocity:

$$A_x = -\frac{Z_{zx}}{Z_{xx}}$$

$$A_y = -\frac{Z_{zy}}{Z_{yy}}$$

Two Basis Functions (cont.)

As before,

$$Z_{in} = Z_{probe} - \int_V J_z^i \, E_z\left[\underline{J}_s\right] dV \qquad \left(J_{sx} \rightarrow \underline{J}_s\right)$$

$$\begin{aligned} Z_{in} &= Z_{probe} - \left\langle \underline{J}_s, J_z^i \right\rangle \\ &= Z_{probe} - A_x \left\langle B_x, J_z^i \right\rangle - A_y \left\langle B_y, J_z^i \right\rangle \\ &= Z_{probe} - A_x Z_{zx} + A_y Z_{zy} \end{aligned}$$

Hence, using the previous results for A_x and A_y, we have

$$Z_{in} = Z_{probe} - \frac{Z_{zx}^2}{Z_{xx}} - \frac{Z_{zy}^2}{Z_{yy}}$$

Two Basis Functions (cont.)

From our derivation:

$$Z_{xx} = -\frac{1}{\pi^2} \int_0^{\pi/2} \int_C \tilde{G}_{xx}\left(k_t, \overline{\phi}\right) \tilde{B}_x^2\left(k_t, \overline{\phi}\right) k_t \, dk_t d\overline{\phi}$$

$$\tilde{G}_{xx} = -\frac{1}{k_t^2}\left[\frac{k_x^2}{D_m\left(k_t\right)} + \frac{k_y^2}{D_e\left(k_t\right)}\right]$$

Similarly,

$$Z_{yy} = -\frac{1}{\pi^2} \int_0^{\pi/2} \int_C \tilde{G}_{yy}\left(k_t, \overline{\phi}\right) \tilde{B}_y^2\left(k_t, \overline{\phi}\right) k_t \, dk_t d\overline{\phi}$$

$$\tilde{G}_{yy} = -\frac{1}{k_t^2}\left[\frac{k_y^2}{D_m\left(k_t\right)} + \frac{k_x^2}{D_e\left(k_t\right)}\right]$$

Two Basis Functions (cont.)

From our derivation:

$$Z_{zx} = \frac{j}{\pi^2}\left(\frac{h}{\omega\varepsilon_1}\right)\int_0^{\pi/2}\int_C \left\{ k_t^2 \, I_i^{TM}(-h)\,\tilde{B}_x \cos\bar{\phi}\,\operatorname{sinc}\left(k_{z1}h\right)\right\} \cdot \sin\left(k_x x_0\right)\cos\left(k_y y_0\right) dk_t \, d\bar{\phi}$$

Similarly,

$$Z_{zy} = \frac{j}{\pi^2}\left(\frac{h}{\omega\varepsilon_1}\right)\int_0^{\pi/2}\int_C \left\{ k_t^2 \, I_i^{TM}(-h)\,\tilde{B}_y \sin\bar{\phi}\,\operatorname{sinc}\left(k_{z1}h\right)\right\} \cdot \sin\left(k_y y_0\right)\cos\left(k_x x_0\right) dk_t \, d\bar{\phi}$$

Two Basis Functions (cont.)

The transforms of the basis functions are:

$$\tilde{B}_x\left(k_x,k_y\right)=\left(\frac{\pi}{2}LW\right)\operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

$$\tilde{B}_y\left(k_x,k_y\right)=\left(\frac{\pi}{2}WL\right)\operatorname{sinc}\left(k_x\frac{L}{2}\right)\left[\frac{\cos\left(k_y\frac{W}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_y\frac{W}{2}\right)^2}\right]$$