ECE 6345

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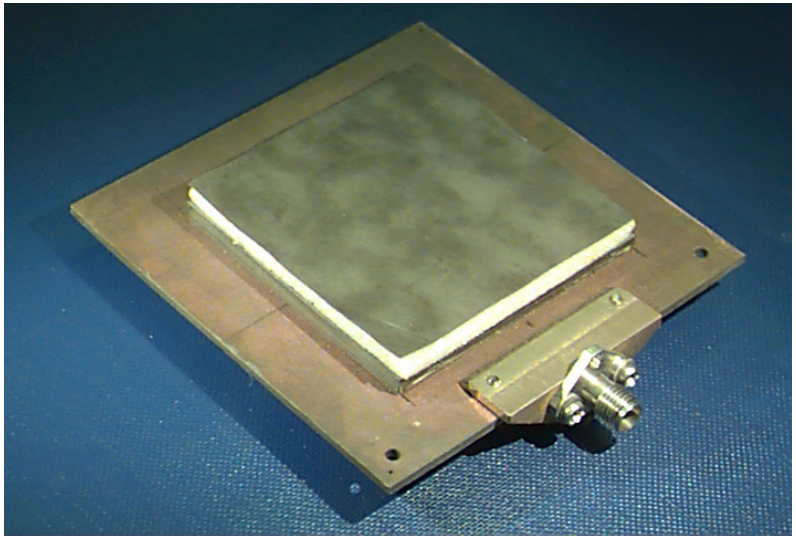
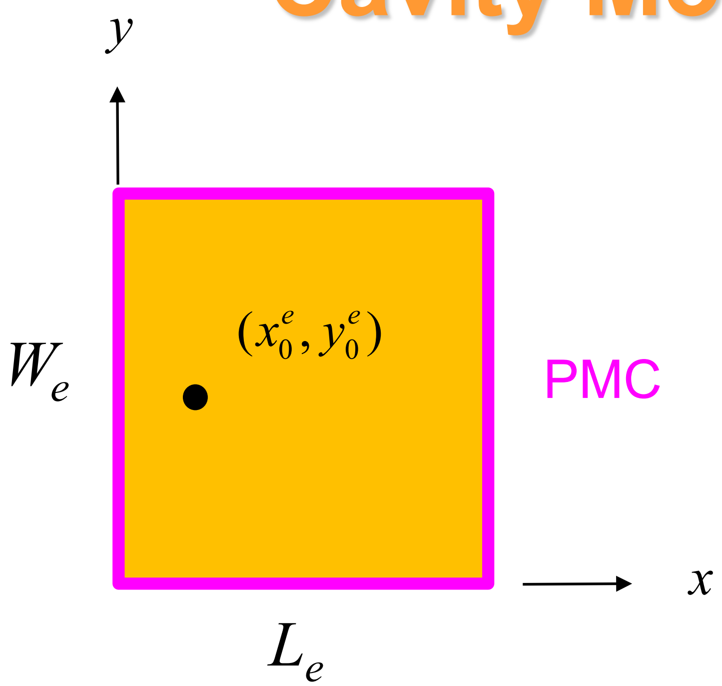


Notes 28

Overview

In this set of notes we use the cavity model and the method of eigenfunction expansion to solve for the input impedance of the rectangular patch antenna.

Cavity Model



$$x_0^e = x_0 + \Delta L$$

$$y_0^e = y_0 + \Delta W$$

The coordinates (x_0, y_0) are measured from the corner of the physical patch.

Let $\quad k_e = k_0 \sqrt{\varepsilon_{rl}^{eff}}$

Assume no z variation (the probe current is constant in the z direction.)

$$\varepsilon_{rl}^{eff} = \varepsilon_r' \left(1 - j l_{eff}\right)$$

Note:

ΔL is from Hammerstad's formula
ΔW is from Wheeler's formula

$$l_{eff} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{sp}} + \frac{1}{Q_{sw}}$$

Cavity Model (cont.)

We first derive the Helmholtz equation for E_z.

$$\nabla \times \underline{H} = \underline{J}^{i} + j\omega\varepsilon_l^{eff}\,\underline{E}$$

$$\nabla \times \underline{E} = -j\omega\mu\underline{H}$$

Substituting Faradays law into Ampere's law, we have

$$-\frac{1}{j\omega\mu}\nabla \times \left(\nabla \times \underline{E}\right) = \underline{J}^{i} + j\omega\varepsilon_l^{eff}\,\underline{E}$$

$$\nabla \times \left(\nabla \times \underline{E}\right) = -j\omega\mu\underline{J}^{i} + k_e^2\underline{E}$$

$$\nabla\left(\nabla \cdot \underline{E}\right) - \nabla^2\underline{E} = -j\omega\mu\underline{J}^{i} + k_e^2\underline{E}$$

$$\nabla^2\underline{E} + k_e^2\underline{E} = j\omega\mu\underline{J}^{i}$$

Cavity Model (cont.)

Hence

$$\nabla^2 E_z + k_e^2 E_z = j\omega\mu J_z^i$$

Denote

$$\psi(x,y) = E_z(x,y)$$

Then

$$\nabla^2 \psi + k_e^2 \psi = f(x,y)$$

where

$$f(x,y) = j\omega\mu J_z^i(x,y)$$

Eigenfunction Expansion

Introduce "eigenfunctions"

$$\psi_{mn}(x, y)$$

$$\nabla^2 \psi_{mn}(x, y) = -\lambda_{mn}^2 \psi_{mn}(x, y)$$

$$\left. \frac{\partial \psi_{mn}}{\partial n} = 0 \right|_C \qquad -\lambda_{mn}^2 = \text{eigenvalue}$$

For rectangular patch we have, from separation of variables,

$$\psi_{mn}(x, y) = \cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$\lambda_{mn}^2 = \left[\left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2\right]$$

Cavity Model (cont.)

Assume an "eigenfunction expansion"

$$\psi(x,y) = \sum_{m,n} A_{mn}\psi_{mn}(x,y)$$

This must satisfy $\nabla^2\psi + k_e^2\psi = f(x,y)$

Hence

$$\sum_{m,n} A_{mn}\nabla^2\psi_{mn} + k_e^2 \sum_{m,n} A_{mn}\psi_{mn} = f(x,y)$$

Using the properties of the eigenfunctions, we have

$$\sum_{m,n} A_{mn}\left(k_e^2 - \lambda_{mn}^2\right)\psi_{mn}(x,y) = f(x,y)$$

Cavity Model (cont.)

Multiply by $\psi_{m'n'}(x,y)$ and integrate.

Note that the eigenfunctions are *orthogonal*, so that

$$\int_S \psi_{mn}(x,y)\psi_{m'n'}(x,y)\,dS = 0 \qquad (m,n) \neq (m',n')$$

Denote

$$<\psi_{mn},\psi_{mn}> = \int_S \psi_{mn}^2(x,y)\,dS$$

Note: Here the bracket notation denote inner product, not reaction.

We then have

$$A_{mn}\left(k_e^2 - \lambda_{mn}^2\right) <\psi_{mn},\psi_{mn}> = < f,\psi_{mn} >$$

Cavity Model (cont.)

Hence, we have

$$A_{mn} = \frac{<f, \psi_{mn}>}{<\psi_{mn}, \psi_{mn}>}\left(\frac{1}{k_e^2 - \lambda_{mn}^2}\right)$$

For the patch we then have $f(x, y) = j\omega\mu J_z^i(x, y)$

$$A_{mn} = j\omega\mu\left(\frac{<J_z^i, \psi_{mn}>}{<\psi_{mn}, \psi_{mn}>}\right)\left(\frac{1}{k_e^2 - \lambda_{mn}^2}\right)$$

The field inside the patch cavity is then given by

$$E_z(x, y) = \sum_{m,n} A_{mn}\psi_{mn}(x, y)$$

Cavity Model (cont.)

To calculate the input impedance, we first calculate the complex power going into the patch as

$$P_{in} = -\frac{1}{2}\int_V E_z(x,y)J_z^{i*}\,dV$$

$$= -\frac{1}{2}h\int_S E_z(x,y)J_z^{i*}\,dS$$

$$= -\frac{1}{2}h\int_S \sum_{m,n} A_{mn}\psi_{mn}J_z^{i*}\,dS$$

Cavity Model (cont.)

or

$$P_{in} = -\frac{1}{2} h \sum_{m,n} A_{mn} < \psi_{mn}, J_z^{i*} >$$

$$= -\frac{1}{2} h \sum_{m,n} j\omega\mu \left(\frac{< \psi_{mn}, J_z^i >}{< \psi_{mn}, \psi_{mn} >} \right) \left(\frac{1}{k_e^2 - \lambda_{mn}^2} \right) < \psi_{mn}, J_z^{i*} >$$

$$= -\frac{1}{2} h \sum_{m,n} j\omega\mu \left(\frac{\left| < \psi_{mn}, J_z^i > \right|^2}{< \psi_{mn}, \psi_{mn} >} \right) \left(\frac{1}{k_e^2 - \lambda_{mn}^2} \right)$$

Also, $$P_{in} = \frac{1}{2} Z_{in} \left| I_{in} \right|^2$$

so $$Z_{in} = \frac{2 P_{in}}{\left| I_{in} \right|^2}$$

Cavity Model (cont.)

Hence we have

$$Z_{in} = -j\omega\mu h \frac{1}{\left|I_{in}\right|^2} \sum_{m,n} \left(\frac{\left| < \psi_{mn}, J_z^i > \right|^2}{< \psi_{mn}, \psi_{mn} >} \right) \left(\frac{1}{k_e^2 - \lambda_{mn}^2} \right)$$

where $\displaystyle \sum_{m,n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}$

Cavity Model (cont.)

Rectangular patch:

$$\psi_{mn} = \cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$\lambda_{mn}^2 = \left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2$$

$$k_e = k_0\sqrt{\varepsilon_{rl}^{eff}}$$

where $\varepsilon_{rl}^{eff} = \varepsilon_r'\left(1 - j\,l_{eff}\right)$

We need:

$$\langle \psi_{mn}, \psi_{mn} \rangle = \int_0^{L_e} \cos^2\left(\frac{m\pi x}{L_e}\right) dx \int_0^{W_e} \cos^2\left(\frac{n\pi y}{W_e}\right) dy$$

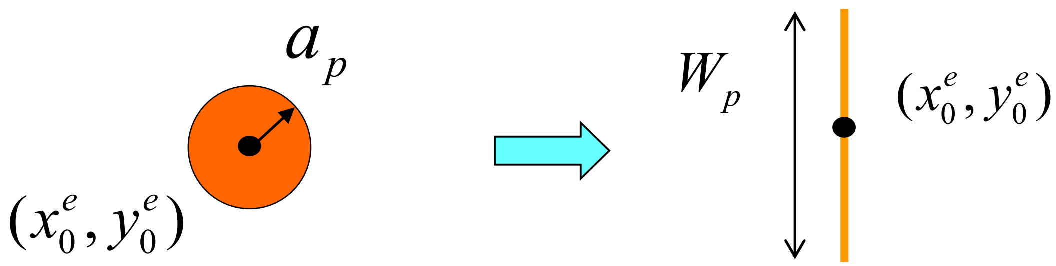
Cavity Model (cont.)

so

$$\left\langle \psi_{mn}, \psi_{mn} \right\rangle = \left(\frac{W_e}{2}\right)\left(\frac{L_e}{2}\right)\left(1+\delta_{m0}\right)\left(1+\delta_{n0}\right)$$

$$\delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

To calculate $\left\langle \psi_{mn}, J_z^i \right\rangle$, assume a strip model as shown below.



Maxwell Current

For a "Maxwell" strip current assumption, we have:

$$J_{sz} = \frac{I_{in}}{\pi\sqrt{\left(\frac{W_p}{2}\right)^2 - \left(y - y_0^e\right)^2}}, \qquad y \in \left(y_0^e - \frac{W_p}{2}, y_0^e + \frac{W_p}{2}\right)$$

$$W_p = 4a_p$$

Note: The total probe current is I_{in}.

Uniform Current

For a uniform strip current assumption, we have:

$$J_{sz} = \frac{I_{in}}{W_p}, \qquad y \in \left(y_0^e - \frac{W_p}{2}, y_0^e + \frac{W_p}{2} \right)$$

$$W_p = a_p \, e^{\frac{3}{2}} \doteq 4.482 \, a_p$$

Note: The total probe current is I_{in}.

Uniform Model

Assume uniform strip current model:

$$\left\langle \psi_{mn}, J_z^i \right\rangle = I_{in} \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \cos\left(\frac{m\pi x_0^e}{L_e}\right) \cos\left(\frac{n\pi y}{W_e}\right) \frac{1}{W_p} dy$$

Use

$$y = y_0^e + y'$$

$$= \frac{I_{in}}{W_p} \cos\left(\frac{m\pi x_0^e}{L_e}\right) \int_{-\frac{W_p}{2}}^{+\frac{W_p}{2}} \cos\left(\frac{n\pi}{W_e}\left[y_0^e + y'\right]\right) dy'$$

Integrates to zero

$$= \frac{I_{in}}{W_p} \cos\left(\frac{m\pi x_0^e}{L_e}\right) \int_{-\frac{W_p}{2}}^{+\frac{W_p}{2}} \cos\left(\frac{n\pi y_0^e}{W_e}\right) \cos\left(\frac{n\pi y'}{W_e}\right) - \sin\left(\frac{n\pi y_0^e}{W_e}\right) \sin\left(\frac{n\pi y'}{W_e}\right) dy'$$

$$= \frac{I_{in}}{W_p} \cos\left(\frac{m\pi x_0^e}{L_e}\right) \left[\cos\left(\frac{n\pi y_0^e}{W_e}\right) W_p \operatorname{sinc}\left(\frac{n\pi W_p}{2W_e}\right)\right]$$

Uniform Model (cont.)

Hence

$$\left\langle \psi_{mn}, J_z^i \right\rangle = I_{in} \cos\left(\frac{m\pi x_0^e}{L_e}\right) \cos\left(\frac{n\pi y_0^e}{W_e}\right) \operatorname{sinc}\left(\frac{n\pi W_p}{2W_e}\right)$$

Note: It is the $\operatorname{sinc}\left(\frac{n\pi W_p}{2W_e}\right)$ term that causes the series for Z_{in} to converge.

Note:
We cannot assume a probe of zero radius,
or else the series will not converge – the input reactance will be infinite.

Cavity Model (cont.)

Summary

$$Z_{in} = -j\omega\mu h \frac{1}{\left|I_{in}\right|^2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(\frac{\left| < \psi_{mn}, J_z^i > \right|^2}{< \psi_{mn}, \psi_{mn} >} \right) \left(\frac{1}{k_e^2 - \lambda_{mn}^2} \right)$$

where

$$\langle \psi_{mn}, \psi_{mn} \rangle = \left(\frac{W_e}{2} \right) \left(\frac{L_e}{2} \right) \left(1 + \delta_{m0}\right)\left(1 + \delta_{n0}\right)$$

$$\langle \psi_{mn}, J_z^i \rangle = I_{in} \cos\left(\frac{m\pi x_0^e}{L_e} \right) \cos\left(\frac{n\pi y_0^e}{W_e} \right) \operatorname{sinc}\left(\frac{n\pi W_p}{2W_e} \right)$$

$$W_p = a_p\, e^{\frac{3}{2}} \doteq 4.482\, a_p$$

$$\lambda_{mn} = \sqrt{\left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2} \qquad k_e = k_0 \sqrt{\varepsilon_{rl}^{eff}} \qquad \varepsilon_{rl}^{eff} = \varepsilon_r' \left(1 - j\, l_{eff}\right) \qquad l_{eff} = 1/Q$$

Probe Inductance

$$Z_{in} = -j\omega\mu h \frac{1}{\left|I_{in}\right|^2} \sum_{m,n} \left(\frac{\left| < \psi_{mn}, J_z^i > \right|^2}{< \psi_{mn}, \psi_{mn} >} \right) \left(\frac{1}{k_e^2 - \lambda_{mn}^2} \right)$$

Note that

$(1,0)$ = term that corresponds to the dominant patch mode current (impedance of RLC circuit).

Hence

$$jX_p = -j\omega\mu h \frac{1}{\left|I_{in}\right|^2} \sum_{\substack{(m,n) \\ \neq (1,0)}} \left(\frac{\left| \left\langle \psi_{mn}, J_z^i \right\rangle \right|^2}{\left\langle \psi_{mn}, \psi_{mn} \right\rangle} \right) \left(\frac{1}{k_e^2 - \lambda_{mn}^2} \right)$$

RLC Model

We can write

$$Z_{in} = \sum_{m,n} Z_{in}^{m,n}$$

where

$$Z_{in}^{m,n} = -j\omega \left(\frac{P_{mn}}{k_e^2 - \lambda_{mn}^2} \right)$$

$$P_{mn} = \mu h \frac{1}{\left|I_{in}\right|^2} \frac{\left|\left\langle \psi_{mn}, J_z^i \right\rangle\right|^2}{\left\langle \psi_{mn}, \psi_{mn} \right\rangle}$$

(These coefficients are not a function of frequency or the current.)

RLC Model (cont.)

Eigenvalue equation: $\nabla^2 \psi_{mn} + \lambda_{mn}^2 \psi_{mn} = 0$

Assume an ideal *resonator* formed by a hypothetical lossless substrate ε_r'.

Fields allowed at *resonance frequencies*: $k = k_{mn} = k_0^{mn} \sqrt{\varepsilon_r'}$

Note: k is real here.

$k \equiv \omega \sqrt{\mu \varepsilon_0 \varepsilon_r'}$

k_{mn} = wavenumber of resonant patch mode (m,n) for a lossless substrate

Helmholtz equation: $\nabla^2 \psi_{mn} + k_{mn}^2 \psi_{mn} = 0$

Comparing, we have the conclusion that $\lambda_{mn} = k_{mn}$

This is the physical interpretation of the eigenvalues.

RLC Model (cont.)

We can then write

$$Z_{in}^{m,n} = -j\omega\left(\frac{P_{mn}}{k_e^2 - k_{mn}^2}\right)$$

$$= -j\omega\left(\frac{P_{mn}}{k^2\left(1 - jl_{eff}\right) - k_{mn}^2}\right)$$

$$= -j\omega\left(\frac{P_{mn}}{\left(k^2 - k_{mn}^2\right) - jk^2 l_{eff}}\right)$$

$$= \omega\frac{P_{mn}}{k^2 l_{eff} + j\left(k^2 - k_{mn}^2\right)}$$

Note :

$$k_e^2 = \omega^2 \mu \varepsilon_l^{eff}$$

$$= \omega^2 \mu \varepsilon' \left(1 - jl_{eff}\right)$$

$$= k^2\left(1 - jl_{eff}\right)$$

RLC Model (cont.)

or

$$Z_{in}^{m,n} = \left(\frac{P_{mn}}{k_{mn}^2 l_{eff}}\right)\left(\frac{\omega}{\dfrac{k^2}{k_{mn}^2} + j\left(\dfrac{1}{l_{eff}}\right)\left(\dfrac{k^2}{k_{mn}^2} - 1\right)}\right)$$

Next, use:

$$\frac{1}{l_{eff}} = Q$$

$$\omega = \omega_{mn}\left(\frac{\omega}{\omega_{mn}}\right) = \omega_{mn} f_{rmn}$$

$$\frac{k^2}{k_{mn}^2} = \frac{\omega^2}{\omega_{mn}^2} = f_{rmn}^2$$

Also, define

$$R_{mn} \equiv \left(\frac{P_{mn}}{k_{mn}^2 l_{eff}}\right)\omega_{mn}$$

$$\Rightarrow \left(\frac{P_{mn}}{k_{mn}^2 l_{eff}}\right)\omega = \left(\frac{P_{mn}}{k_{mn}^2 l_{eff}}\right)\omega_{mn}\left(\frac{\omega}{\omega_{mn}}\right) = R_{mn} f_{rmn}$$

RLC Model (cont.)

Then

$$Z_{in}^{m,n} = R_{mn}\left(\frac{f_{rmn}}{f_{rmn}^2 + jQ\left(f_{rmn}^2 - 1\right)}\right)$$

or

$$Z_{in}^{m,n} = \left(\frac{R_{mn}}{f_{rmn} + jQ\left(f_{rmn} - \dfrac{1}{f_{rmn}}\right)}\right)$$

RLC Model (cont.)

For $f_{rmn}^2 \approx 1$, we have

$$Z_{in}^{m,n} \approx \frac{R_{mn}}{1 + jQ\left(f_{rmn} - \dfrac{1}{f_{rmn}}\right)}$$

(RLC equation)

This justifies the RLC model near resonance.

(0,0) Mode

Note that for the $(0,0)$ mode $\omega_{00} = 0$ Recall : $\lambda_{mn} = k_{mn} = \sqrt{\left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2}$

$$Z_{in}^{m,n} = \omega \frac{P_{mn}}{k^2 l_{eff} + j\left(k^2 - \cancel{k_{mn}^2}\right)} \quad \Rightarrow \quad Z_{in}^{0,0} = \omega \frac{P_{00}}{j\left(k^2 - jk^2 l_{eff}\right)}$$

or $$Z_{in}^{0,0} \approx \frac{1}{j\omega\left(\frac{\mu\varepsilon_0\varepsilon_r'}{P_{00}}\right)}$$ $\left(\text{Assume } l_{eff} = l_{eff}^{0,0} \ll 1\right)$

Also, we have

$$P_{mn} = \mu h \frac{1}{\left|I_{in}\right|^2} \frac{\left|\left\langle \psi_{mn}, J_z^i \right\rangle\right|^2}{\left\langle \psi_{mn}, \psi_{mn} \right\rangle} \quad \Rightarrow \quad P_{00} = \mu h \frac{1}{\left|I_{in}\right|^2} \frac{\left|I_{in}\right|^2}{L_e W_e} = \frac{\mu h}{L_e W_e}$$

RLC Model (cont.)

Hence

$$Z_{in}^{0,0} \approx \frac{1}{j\omega\left(\dfrac{\mu\varepsilon_0\varepsilon_r'}{P_{00}}\right)} = \frac{1}{j\omega\left(\dfrac{\mu\varepsilon_0\varepsilon_r'}{\left(\dfrac{\mu h}{L_e W_e}\right)}\right)}$$

or

$$Z_{in}^{0,0} \approx \frac{1}{j\omega\left(\varepsilon_0\varepsilon_r' \dfrac{L_e W_e}{h}\right)} = \frac{1}{j\omega C}$$

As expected, the (0,0) mode acts as a parallel-plate capacitor.

RLC Model (cont.)

For any other *nonresonant* mode $(m,n) \neq (1,0)$ or $(0,0)$

$$f_{rmn} < 1$$

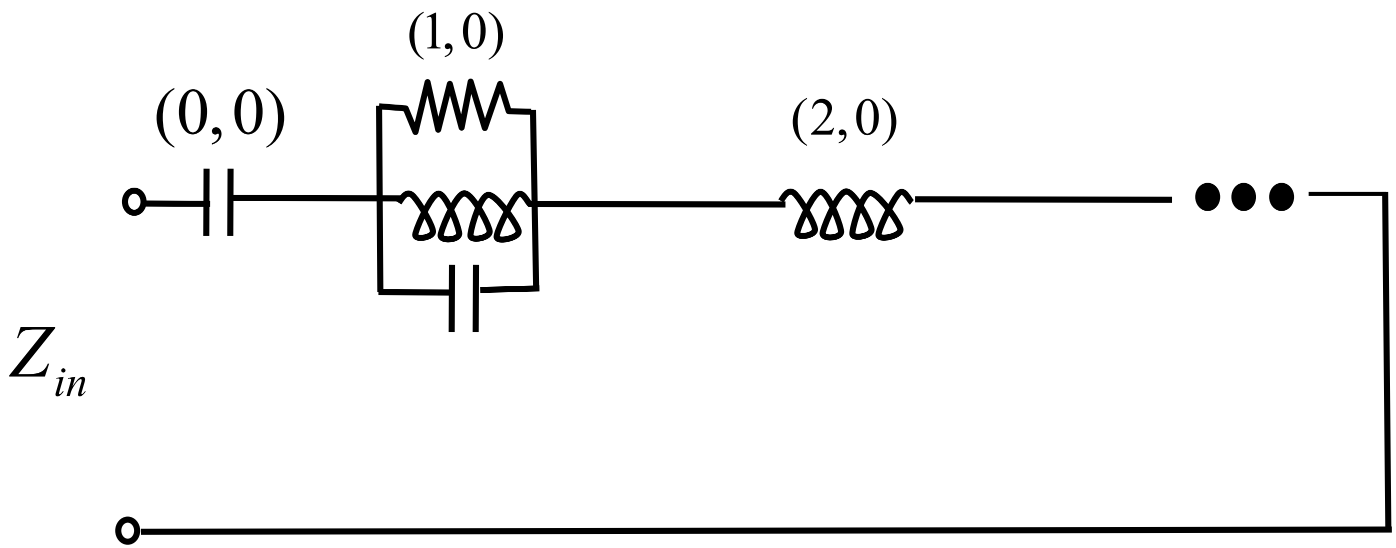
$$Z_{in}^{m,n} = \left(\frac{R_{mn}}{f_{rmn} + jQ\left(f_{rmn} - \frac{1}{f_{rmn}} \right)} \right)$$

$$Q >> 1$$

$$Z_{in}^{m,n} \approx \left(\frac{R_{mn}}{jQ\left(f_{rmn} - \frac{1}{f_{rmn}} \right)} \right) = jX_{mn} \qquad (X_{mn} > 0)$$

RLC Model (cont.)

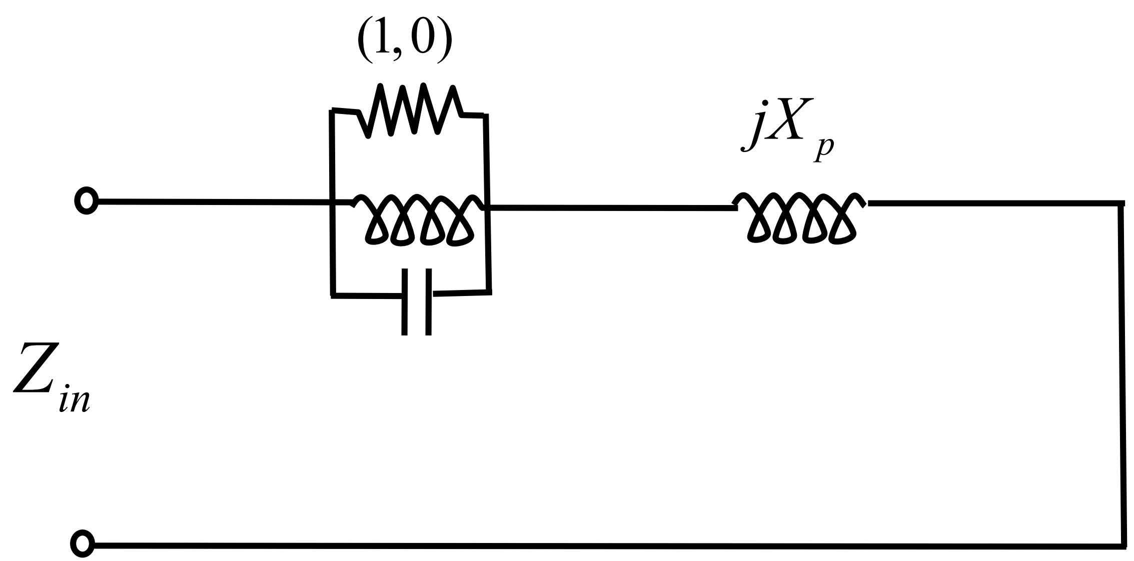
Circuit model:



Note:
This circuit model is accurate as long as we are near the resonance of the (1,0) circuit.

RLC Model (cont.)

Lumping all of the nonresonant circuits together, we have:

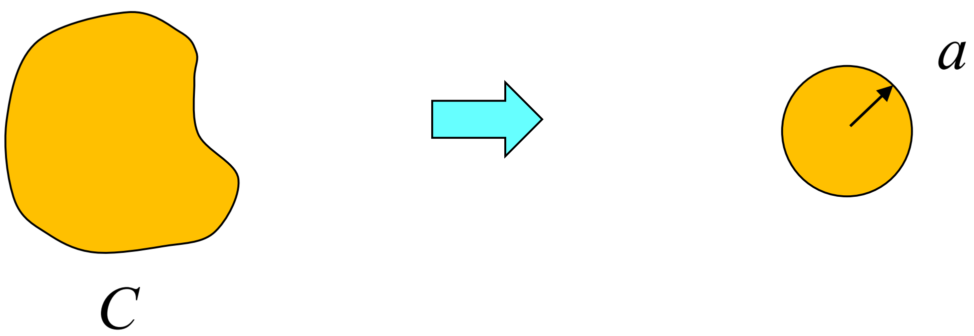


This gives us the CAD model for the patch.

Appendix

In this appendix we derive the equivalent radius approximation for a flat strip.

Start by considering a conductor of arbitrary cross section.



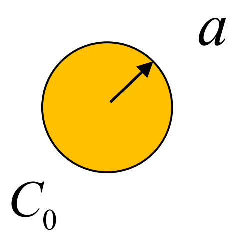
$$\text{Assume}: I_0 = 1\,\text{A}$$

We wish to find the effective radius a of the round wire that best models the object.

Appendix (cont.)

Approach: Equate complex power being radiated by the two objects.

Round wire:



ε, μ

Assume : $I_0 = 1\,\mathrm{A}$

From notes 4:

$$E_z = -\eta k \left(\frac{1}{4}\right) J_0(ka) H_0^{(2)}(k\rho), \quad \rho \ge a$$

$$P_{rad} = -\frac{1}{2}\int_{C_0} E_z J_{sz}^* dl = -\frac{1}{2}\int_0^{2\pi} E_z J_{sz}^* \, a \, d\phi = -\frac{1}{2} E_z J_{sz}^* \, a\left(2\pi\right)$$

$$= -\frac{1}{2}\left(-\eta k \left(\frac{1}{4}\right) J_0(ka) H_0^{(2)}(ka)\right)\left(\frac{1}{2\pi a}\right)^* a\left(2\pi\right)$$

Appendix (cont.)

$$P_{rad} = -\frac{1}{2}\left(-\eta k\left(\frac{1}{4}\right)J_0(ka)H_0^{(2)}(ka)\right)\left(\frac{1}{2\pi a}\right)^* a(2\pi)$$

Use $\eta k = \omega\mu$

$$P_{rad} = \frac{1}{8}\omega\mu J_0(ka)H_0^{(2)}(ka)$$

Assume that the radius is small compared with a wavelength.

$$P_{rad} \approx -j\frac{1}{8}\omega\mu Y_0(ka)$$

Next, use

$$Y_0(x) \sim \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right)+\gamma\right], \quad \gamma = 0.5772156$$

Appendix (cont.)

We then have

$$P_{rad} \approx -j\frac{1}{8}\omega\mu\left(\frac{2}{\pi}\left[\ln\left(\frac{ka}{2}\right)+\gamma\right]\right)$$

or

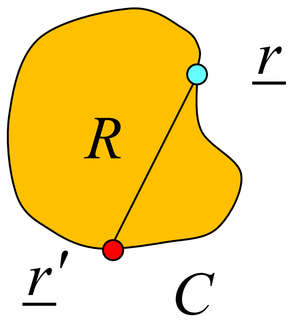
$$P_{rad} \approx -j\frac{1}{4\pi}\omega\mu\left[\ln\left(\frac{ka}{2}\right)+\gamma\right]$$

Next, we consider the arbitrary object.

Appendix (cont.)

Arbitrary object:

ε, μ



Assume : $I_0 = 1\,\text{A}$

$$P_{rad} = -\frac{1}{2}\int_C J_{sz}^*(l)\,E_z(l)\,dl$$

where

$$E_z(l) = \int_C J_{sz}(l')\left[-\eta k\left(\frac{1}{4}\right)H_0^{(2)}(kR)\right]dl' \qquad R = \left|\underline{r} - \underline{r}'\right|$$

Appendix (cont.)

$$P_{rad} = -\frac{1}{2}\int_C J_{sz}^*(l)\int_C J_{sz}(l')\left[-\eta k\left(\frac{1}{4}\right)H_0^{(2)}(kR)\right]dl'\,dl$$

Denote $J_{sz}(l) = f(l)$ Assume : $I_0 = 1\,\mathrm{A}$

$$P_{rad} = \frac{1}{8}\omega\mu\int_C\int_C f(l')\,f^*(l)\,H_0^{(2)}(kR)\,dl'\,dl$$

so

$$P_{rad} \approx \frac{1}{8}\omega\mu\int_C\int_C f(l')\,f^*(l)\left(-j\frac{2}{\pi}\left[\ln\left(\frac{kR}{2}\right)+\gamma\right]\right)dl'\,dl$$

Appendix (cont.)

$$P_{rad} \approx \frac{1}{8}\omega\mu \int_C \int_C f(l') f^*(l) \left(-j\frac{2}{\pi}\left[\ln\left(\frac{kR}{2}\right) + \gamma \right] \right) dl' \, dl$$

Note that

$$\int_C f(l')\, dl' = 1 \quad \text{(1A on object)}$$

$$\int_C f(l)\, dl = 1 \quad \text{(1A on object)}$$

so that

$$P_{rad} \approx \frac{1}{8}\omega\mu \int_C \int_C f(l') f^*(l) \left(-j\frac{2}{\pi}\left[\ln\left(\frac{kR}{2}\right) \right] \right) dl' \, dl - j\gamma\left(\frac{1}{4\pi}\omega\mu\right)$$

Appendix (cont.)

Equate the two complex powers:

$$P_{rad} \approx -j\frac{1}{4\pi}\omega\mu\left[\ln\left(\frac{ka}{2}\right)+\gamma\right]$$

$$P_{rad} \approx -j\frac{1}{4\pi}\omega\mu\int_C\int_C f(l')\,f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl - j\gamma\left(\frac{1}{4\pi}\omega\mu\right)$$

$$-j\frac{1}{4\pi}\omega\mu\left[\ln\left(\frac{ka}{2}\right)+\gamma\right] = -j\frac{1}{4\pi}\omega\mu\int_C\int_C f(l')\,f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl - j\gamma\left(\frac{1}{4\pi}\omega\mu\right)$$

or

$$-j\frac{1}{4\pi}\omega\mu\ln\left(\frac{ka}{2}\right) = -j\frac{1}{4\pi}\omega\mu\int_C\int_C f(l')\,f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl$$

Appendix (cont.)

$$-j\frac{1}{4\pi}\omega\mu\ln\left(\frac{ka}{2}\right)=-j\frac{1}{4\pi}\omega\mu\int_C\int_C f(l')f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl$$

or

$$\ln\left(\frac{ka}{2}\right)=\int_C\int_C f(l')f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl$$

or

$$\ln(k)+\ln a-\ln 2=\int_C\int_C f(l')f^*(l)\left(\ln k+\ln R-\ln 2\right)dl'\,dl$$

or

$$\ln a=\int_C\int_C f(l')f^*(l)\ln R\,dl'\,dl$$

Appendix (cont.)

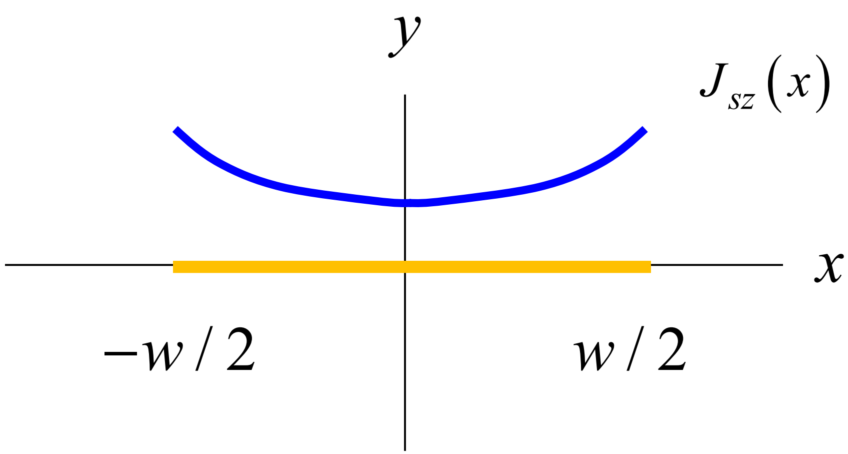
The general result (applicable to any arbitrary object) is thus

$$\ln a = \int_C \int_C f(l')\, f^*(l) \ln R(l,l')\, dl'\, dl$$

We next evaluate this for a flat strip.

Appendix (cont.)

Strip model:

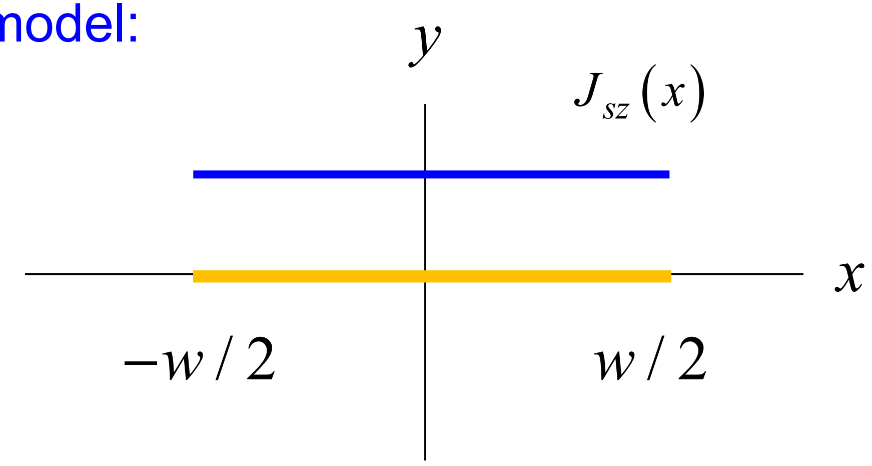


$$J_{sz}(x) = f(x)$$

$$\ln a = \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} f(x') f^*(x) \ln|x - x'| \, dx' dx$$

Appendix (cont.)

Uniform current model:



$$J_{sz}(x) = f(x) = 1/w$$

$$\ln a = \frac{1}{w^2} \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} \ln|x - x'| \, dx' dx$$

Appendix (cont.)

$$\ln a = \frac{1}{w^2} \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} \ln\left|x - x'\right| dx' dx$$

Use

$$s = x / w$$

$$t = x' / w$$

We then have

$$\ln a = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(w\left|s - t\right|\right) dt\, ds$$

Appendix (cont.)

$$\ln a = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \ln\left(w\left|s-t\right|\right) dt\, ds$$

Therefore, we have

$$\ln a = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \ln\left(\left|s-t\right|\right) dt\, ds + \ln w \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dt\, ds$$

or

$$\ln a = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \ln\left(\left|s-t\right|\right) dt\, ds + \ln w$$

Appendix (cont.)

$$\ln a = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(\left|s-t\right|\right) dt\, ds + \ln w$$

Define

$$I_2 \equiv \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(\left|s-t\right|\right) dt\, ds$$

We then have

$$\ln a = I_2 + \ln w$$

or

$$a = e^{I_2} w$$

or

$$w = e^{-I_2} a$$

Appendix (cont.)

We have

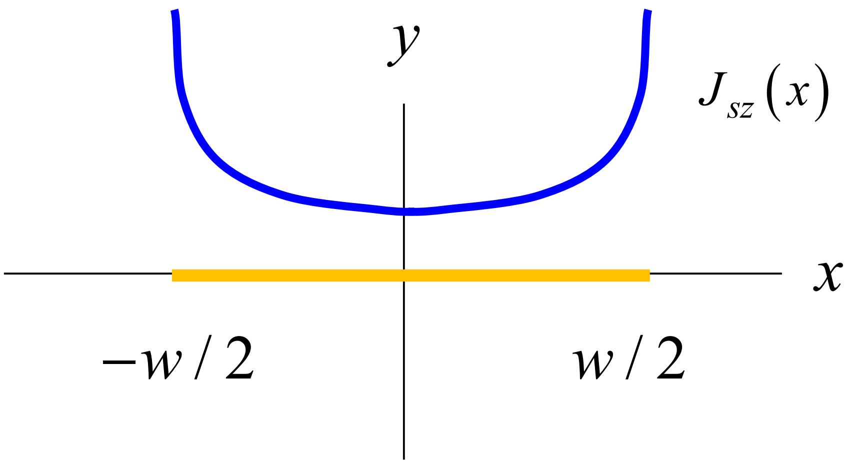
$$I_2 \equiv \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(\left|s-t\right|\right) dt\, ds = -\frac{3}{2}$$

We then have

$$w = e^{3/2} a$$

Appendix (cont.)

Maxwell current model:



$$J_{sz}(x) = \frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - x^2}}$$

(This corresponds to 1A.)

Appendix (cont.)

$$\ln a = \frac{1}{\pi^2} \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - x^2}} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - x'^2}} \ln|x - x'| dx' dx$$

Use

$$s = x / w$$

$$t = x' / w$$

$$\ln a = \frac{w^2}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - (ws)^2}} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - (wt)^2}} \ln\left(w|s - t|\right) ds dt$$

Appendix (cont.)

$$\ln a = \frac{w^2}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - (ws)^2}} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - (wt)^2}} \ln\left(w|s-t|\right) ds dt$$

$$\ln a = \frac{1}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} \ln\left(|s-t|\right) ds dt$$

$$+ \ln w \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1/\pi}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1/\pi}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} ds dt$$

This (separable) double integral equals 1.

Appendix (cont.)

$$\ln a = \ln w + \frac{1}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} \ln\left(|s-t|\right) ds\,dt$$

Define

$$I_2 \equiv \frac{1}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} \ln\left(|s-t|\right) ds\,dt$$

We then have

$$\ln a = I_2 + \ln w$$

or

$$a = e^{I_2} w$$

or

$$w = e^{-I_2} a$$

Appendix (cont.)

We have

$$I_2 \equiv \frac{1}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} \ln\left(|s-t|\right) ds dt = -\ln 4$$

We then have

$$w = e^{-(-\ln 4)} a$$

or

$$w = 4a$$