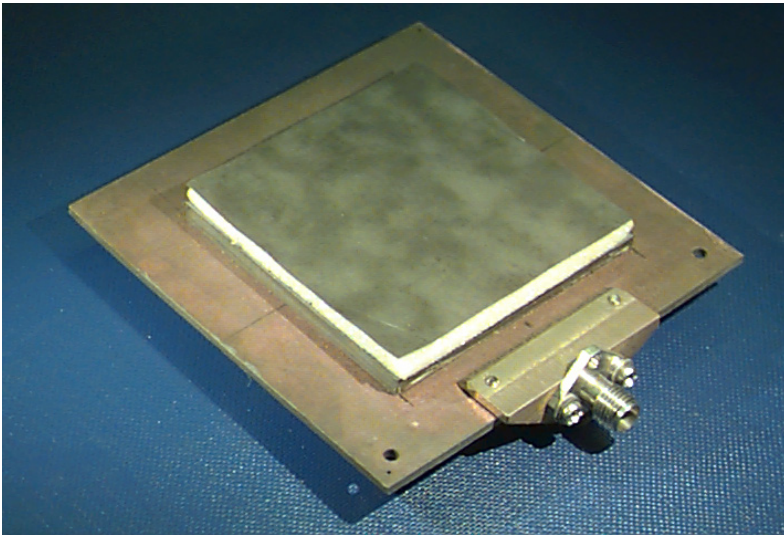


ECE 6345

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Notes 30

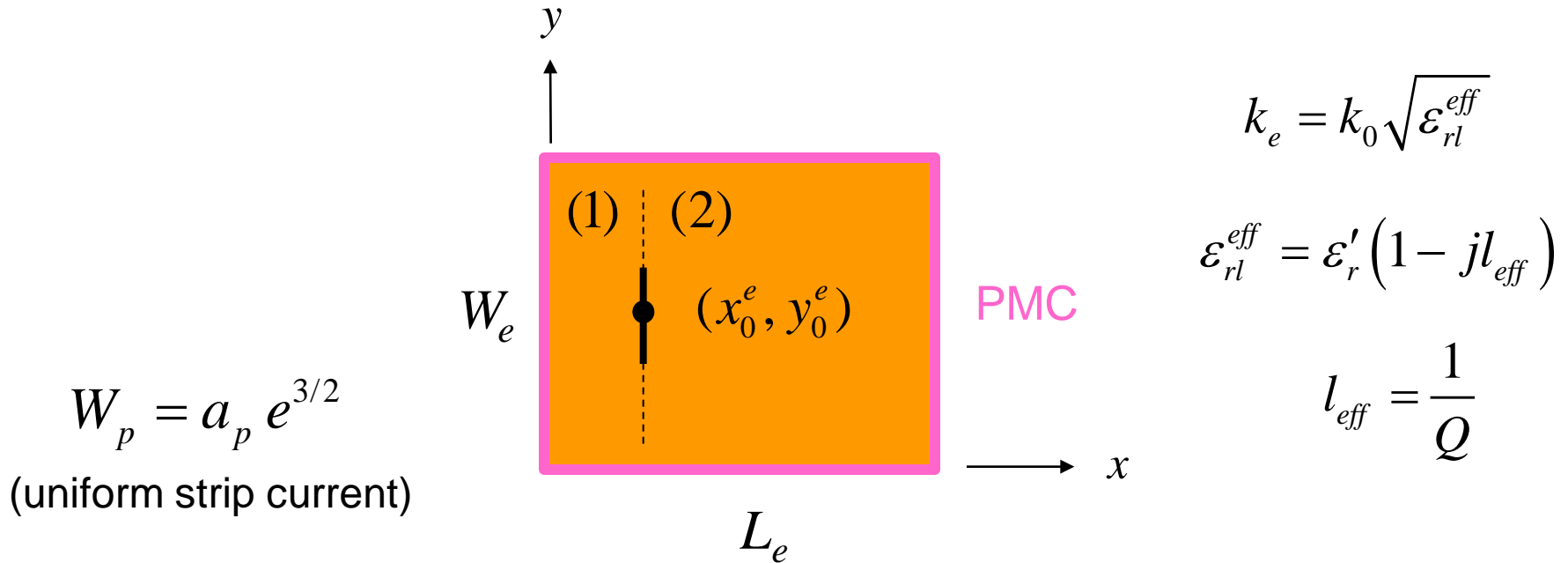
Overview

In this set of notes we develop the **mode matching method** for obtaining the input impedance in the cavity-model problem.

This is an alternative to the eigenfunction expansion method.

It is numerically convenient, requiring only a single sum instead of a double sum (as in the eigenfunction expansion method).

Mode Matching Method



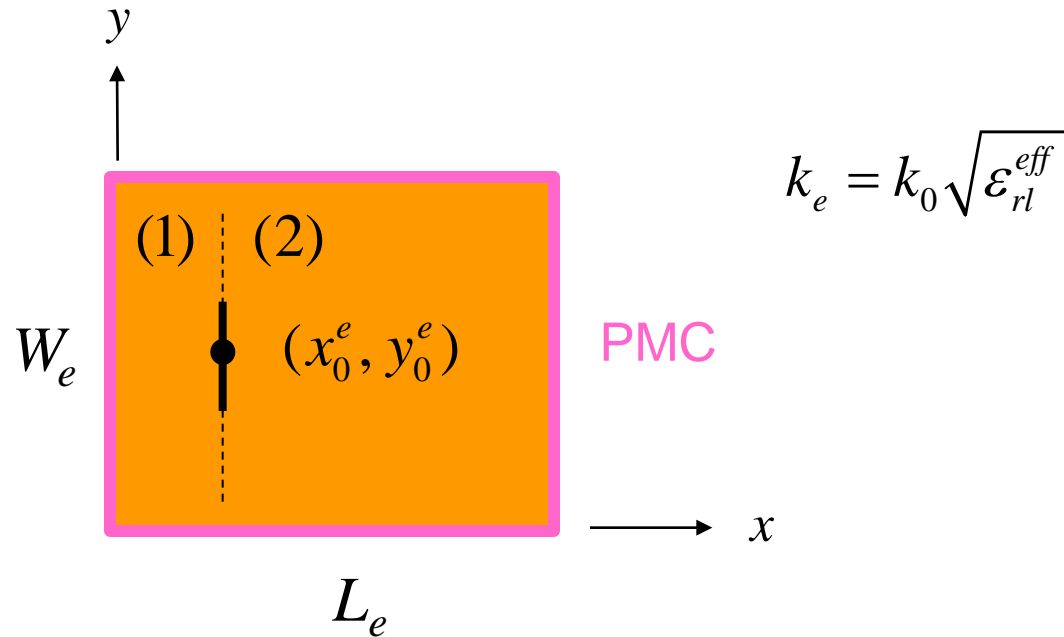
We view the cavity as a finite section of *waveguide*, of length L_e , having PMC side walls.

m^{th} waveguide mode (TM _{$m0$} mode):

$$E_z(x, y) = \cos\left(\frac{m\pi y}{W_e}\right) e^{-jk_{xm}x}$$

$$k_{xm} = \left(k_e^2 - \left(\frac{m\pi}{W_e}\right)^2\right)^{1/2}$$

Mode Matching Method



Applying B.C.s at the left and right walls, we have the following field representations:

$$(1) \quad E_{z1} = \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi y}{W_e}\right) \cos(k_{xm} x)$$

$$(2) \quad E_{z2} = \sum_{m=0}^{\infty} B_m \cos\left(\frac{m\pi y}{W_e}\right) \cos(k_{xm} [x - L_e])$$

Mode Matching (cont.)

Boundary conditions at interface:

at $x = x_0^e$:

$$\begin{cases} E_{z1} = E_{z2} \\ H_{y2} - H_{y1} = J_{sz}^i = \frac{I_{in}}{W_p}, \quad |y - y_0^e| < \frac{W_p}{2} \end{cases}$$

To calculate H_y we use

$$H_y = \frac{1}{j\omega\mu} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right)$$

Mode Matching (cont.)

From (1) we have

$$\sum_{m=0}^{\infty} A_m \cos(k_{xm} x_0^e) \cos\left(\frac{m\pi y}{W_e}\right) = \sum_{m=0}^{\infty} B_m \cos k_{xm} [x_0^e - L_e] \cos\left(\frac{m\pi y}{W_e}\right)$$

From orthogonality of the cosine functions, we then have

$$A_m \cos(k_{xm} x_0^e) = B_m \cos k_{xm} [x_0^e - L_e]$$

Mode Matching (cont.)

From (2) we have

$$H_{y2} - H_{y1} = J_{sz}^i = \frac{I_{in}}{W_p}, \quad |y - y_0^e| < \frac{W_p}{2}$$

which gives us

$$\sum_{m=0}^{\infty} \frac{1}{j\omega\mu} \left[B_m k_{xm} \left(-\sin k_{xm} (x_0^e - L_e) \right) - A_m k_{xm} \left(-\sin k_{xm} x_0^e \right) \right] \cos \left(\frac{m\pi y}{W_e} \right) = \frac{I_{in}}{W_p}$$

$$|y - y_0^e| < \frac{W_p}{2}$$

Mode Matching (cont.)

Using orthogonality of the cosine functions, we have

$$\begin{aligned} & -\frac{1}{j\omega\mu}(k_{xm}) \left[B_m \sin k_{xm} (x_0^e - L_e) - A_m \sin k_{xm} x_0^e \right] \left(\frac{W_e}{2} [1 + \delta_{m0}] \right) \\ &= \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \cos\left(\frac{m\pi y}{W_e}\right) \frac{I_{in}}{W_p} dy \\ &= I_{in} \cos\left(\frac{m\pi y_0^e}{W_e}\right) \text{sinc}\left(\frac{m\pi W_p}{2W_e}\right) \end{aligned}$$

Mode Matching (cont.)

Hence, we have the following two equations:

$$B_m = A_m \left[\frac{\cos(k_{xm} x_0^e)}{\cos[k_{xm} (x_0^e - L_e)]} \right] \quad (\text{from (1)})$$

$$\begin{aligned} & B_m \sin k_{xm} (x_0^e - L_e) - A_m \sin k_{xm} x_0^e \\ &= I_{in} \cos\left(\frac{m\pi y_0^e}{W_e}\right) \text{sinc}\left(\frac{m\pi W_p}{2W_e}\right) \left(\frac{2}{W_e}\right) \left(\frac{1}{1 + \delta_{m0}}\right) \left(\frac{-j\omega\mu}{k_{xm}}\right) \end{aligned}$$

(from (2))

Mode Matching (cont.)

For the second equation, we have (substituting in B_m from the first equation)

$$\begin{aligned} LHS &= A_m \left\{ \left(\frac{\cos k_{xm} x_0^e}{\cos k_{xm} (x_0^e - L_e)} \right) \sin k_{xm} (x_0^e - L_e) - \sin k_{xm} x_0^e \right\} \\ &= A_m \sec k_{xm} (x_0^e - L_e) \left\{ \cos k_{xm} x_0^e \sin k_{xm} (x_0^e - L_e) - \sin k_{xm} x_0^e \cos k_{xm} (x_0^e - L_e) \right\} \\ &= A_m \sec k_{xm} (x_0^e - L_e) (-1) \sin k_{xm} \left[x_0^e - (x_0^e - L_e) \right] \\ &= -A_m \sec k_{xm} (x_0^e - L_e) \sin k_{xm} L_e \end{aligned}$$

Mode Matching (cont.)

Hence

$$A_m = -\cos k_{xm} (x_0^e - L_e) \csc(k_{xm} L_e) \\ \cdot I_{in} \left(\frac{2}{W_e} \right) \left(\frac{1}{1 + \delta_{mo}} \right) \left(\frac{-j\omega\mu}{k_{xm}} \right) \cos \left(\frac{m\pi y_0^e}{W_e} \right) \text{sinc} \left(\frac{m\pi W_p}{2W_e} \right)$$

Mode Matching (cont.)

We now calculate the input impedance, using complex power.

$$\begin{aligned} Z_{in} &= \frac{2P_{in}}{|I_{in}|^2} = \frac{2}{|I_{in}|^2} \int_V -\frac{1}{2} E_z J_z^{i*} dV = \frac{h}{|I_{in}|^2} \int_S -E_z J_{sz}^{i*} dS \\ &= -\frac{h}{|I_{in}|^2} \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} E_z J_{sz}^{i*} dy = -\frac{h}{|I_{in}|^2} \left(\frac{I_{in}^*}{W_p} \right) \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} E_z(x_0^e, y) dy \end{aligned}$$

Next, use (from the region 1 solution)

$$E_z = E_{z1} = \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi y}{W_e}\right) \cos(k_{xm} x)$$

Mode Matching (cont.)

Hence we have

$$\begin{aligned} Z_{in} &= -\frac{h}{W_p I_{in}} \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi y}{W_e}\right) \cos(k_{xm} x_0^e) dy \\ &= -\frac{h}{W_p I_{in}} \cos(k_{xm} x_0^e) \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi y}{W_e}\right) dy \end{aligned}$$

Mode Matching (cont.)

Note that

$$\frac{1}{W_p} \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \cos\left(\frac{m\pi y}{W_e}\right) dy = \cos\left(\frac{m\pi y_0^e}{W_e}\right) \text{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

Hence, we have

$$Z_{in} = -\frac{h}{I_{in}} \sum_{m=0}^{\infty} A_m \cos(k_{xm} x_0^e) \cos\left(\frac{m\pi y_0^e}{W_e}\right) \text{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

Summary

$$Z_{in} = -\frac{h}{I_{in}} \sum_{m=0}^{\infty} A_m \cos(k_{xm} x_0^e) \cos\left(\frac{m\pi y_0^e}{W_e}\right) \operatorname{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

with

$$A_m = -\cos k_{xm} (x_0^e - L_e) \operatorname{csc}(k_{xm} L_e) \\ \cdot I_{in} \left(\frac{2}{W_e}\right) \left(\frac{1}{1 + \delta_{mo}}\right) \left(\frac{-j\omega\mu}{k_{xm}}\right) \cos\left(\frac{m\pi y_0^e}{W_e}\right) \operatorname{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

Final Result

$$Z_{in} = -j\omega\mu h \left(\frac{2}{W_e} \right) \sum_{m=0}^{\infty} \left(\frac{1}{1 + \delta_{m0}} \right) \left(\frac{1}{k_{xm}} \right) \\ \cdot \cos(k_{xm} x_0^e) \cos k_{xm} (x_0^e - L_e) \csc(k_{xm} L_e) \\ \cdot \cos^2 \left(\frac{m\pi y_0^e}{W_e} \right) \operatorname{sinc}^2 \left(\frac{m\pi W_p}{2W_e} \right)$$

where

$$k_{xm} = \left(k_e^2 - \left(\frac{m\pi}{W_e} \right)^2 \right)^{1/2}$$

$$W_p = a_p e^{3/2}$$

$$k_e = k_0 \sqrt{\epsilon_{rl}^{eff}}$$

$$\epsilon_{rl}^{eff} = \epsilon_r' (1 - j l_{eff})$$

$$l_{eff} = \frac{1}{Q}$$