ECE 6345

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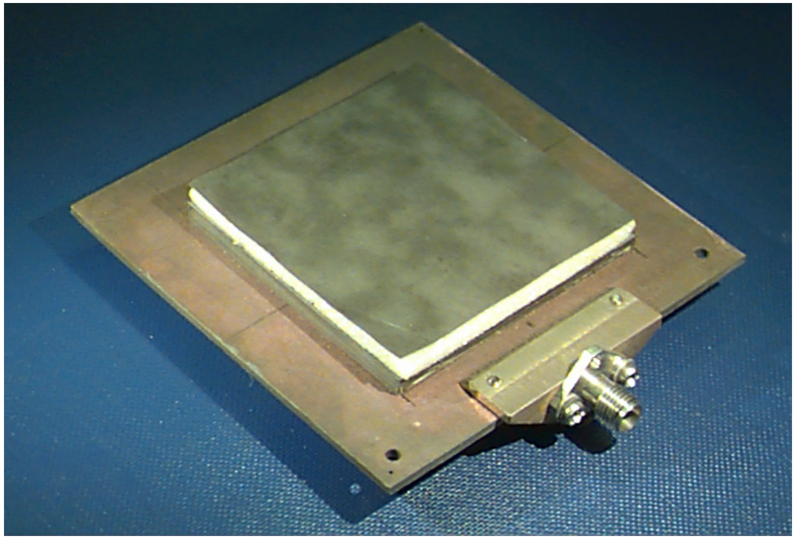
ECE Dept.



Notes 32

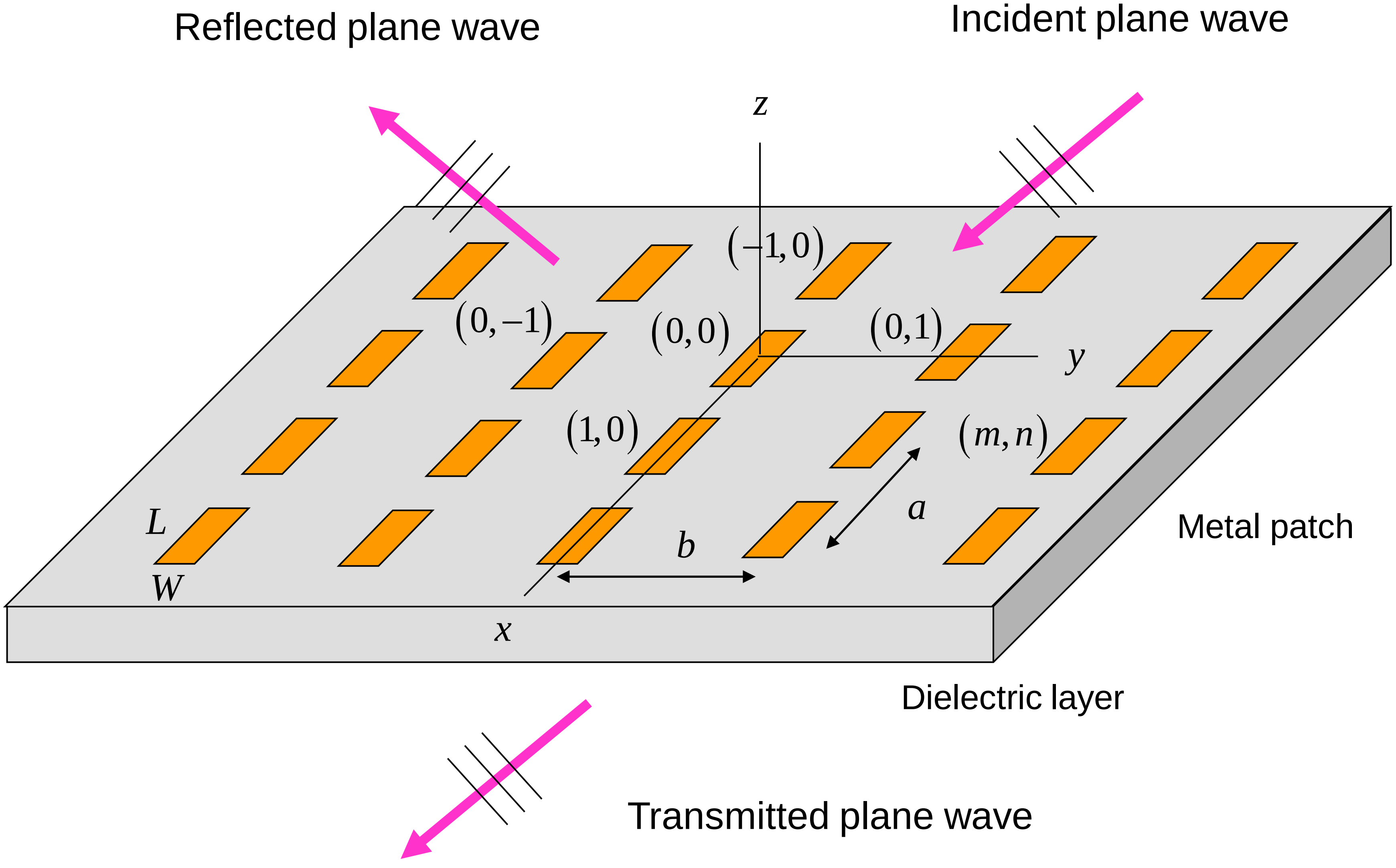
Overview

In this set of notes we extend the spectral-domain method to analyze infinite periodic structures.

Two typical examples of infinite periodic problems:

- Scattering from a frequency selective surface (FSS)
- Input impedance of a microstrip phased array

FSS Geometry



FSS Geometry (cont.)

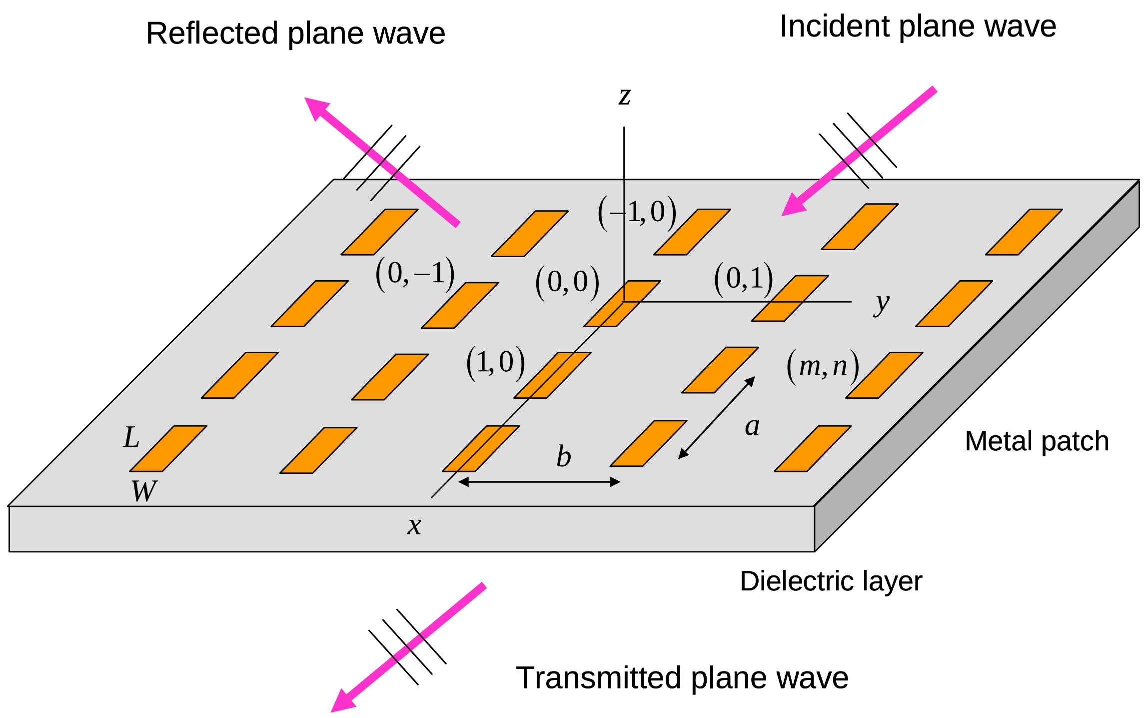
$$\psi^{inc} = A e^{-j\left(k_{x0}x + k_{y0}y\right)} e^{+jk_{z0}z}$$

$(\theta_0, \phi_0) =$ arrival angles

$$k_{x0} = -k_0 \sin\theta_0 \cos\phi_0$$

$$k_{y0} = -k_0 \sin\theta_0 \sin\phi_0$$

$$k_{z0} = k_0 \cos\theta_0$$

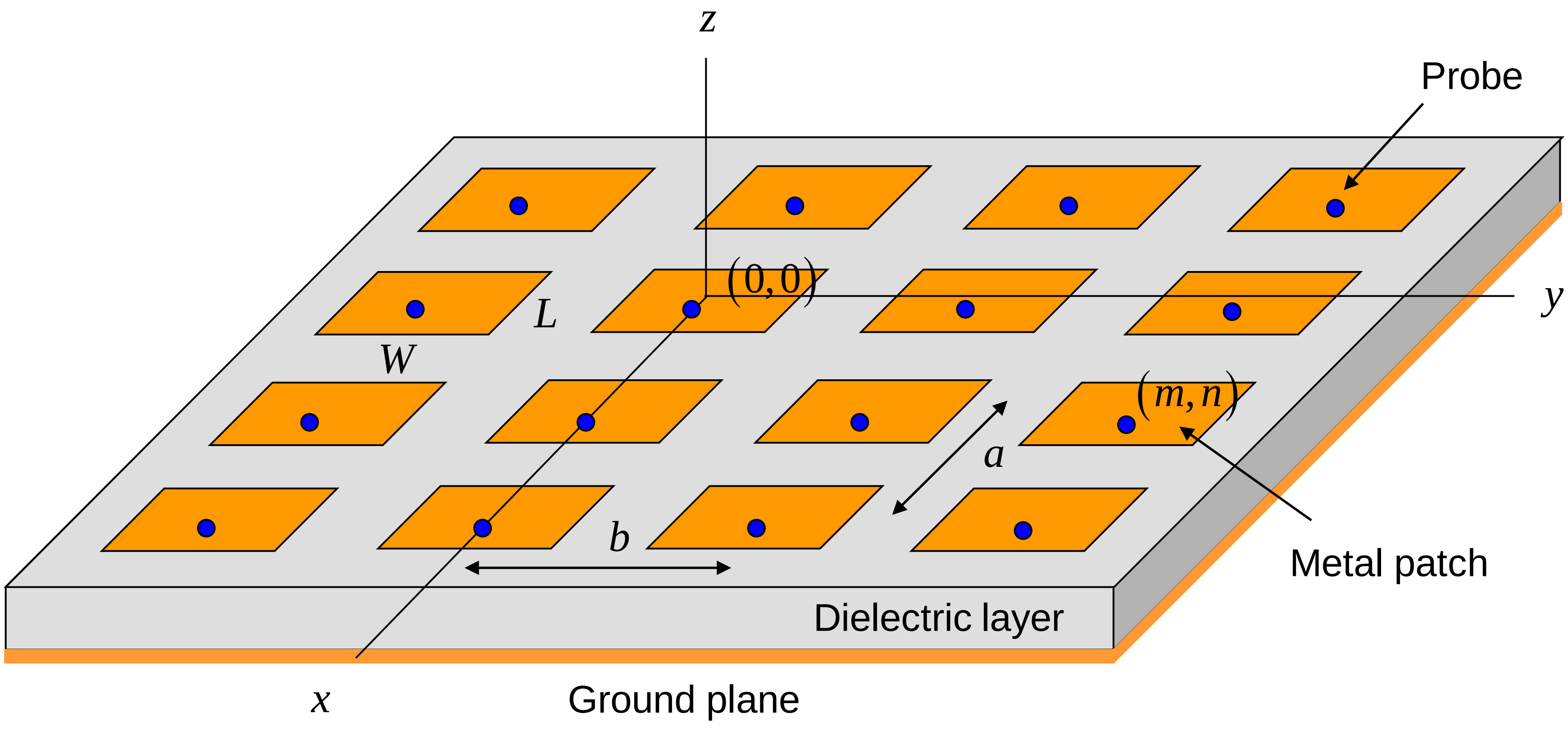


Note:
ψ denotes any field component of interest.

Note: We are following "plane-wave" convention for k_{x0} and k_{y0}, and "transmission-line" convention for k_{z0}.

Microstrip Phased Array Geometry

Probe current mn: $\quad I_{mn} = I_{00}\, e^{-j\left(k_{x0} m a + k_{y0} n b\right)}$



The wavenumbers k_{x0} and k_{y0} are impressed by the feed network.

Microstrip Phased Array Geometry (cont.)

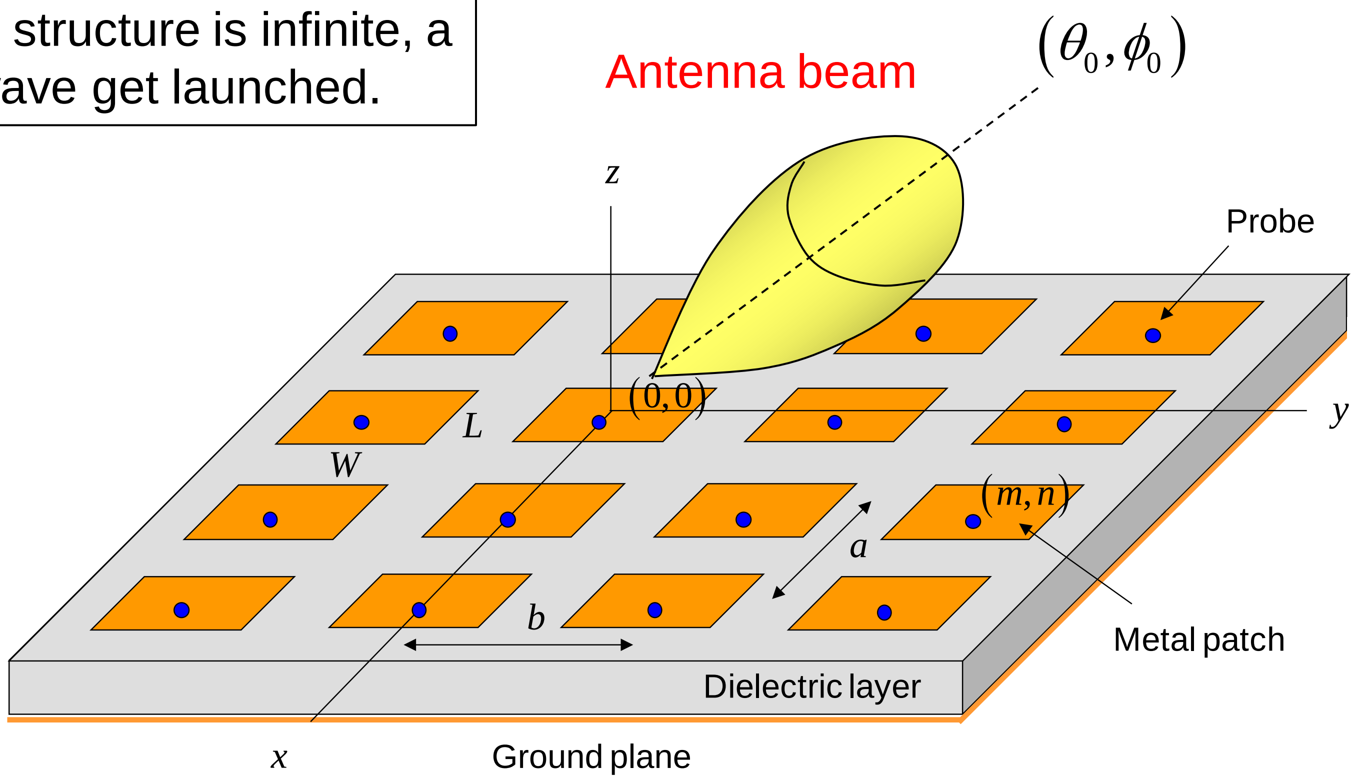
$$I_{mn} = I_{00}\, e^{-j\left(k_{x0} ma + k_{y0} nb\right)}$$

$$k_{x0} = k_0 \sin\theta_0 \cos\phi_0$$

$$k_{y0} = k_0 \sin\theta_0 \sin\phi_0$$

$(\theta_0, \phi_0) =$ radiation angles

Note: If the structure is infinite, a plane wave get launched.

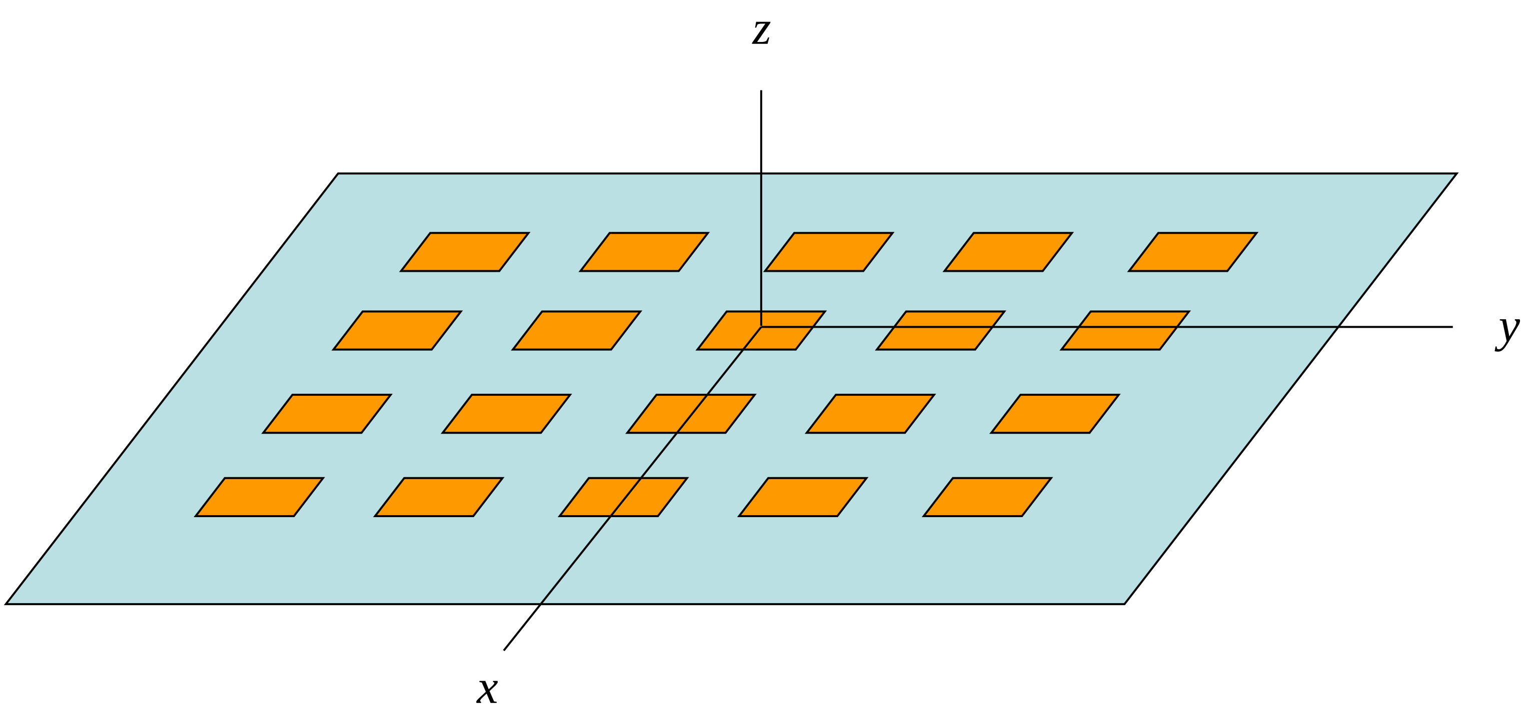


Floquet's Theorem

Fundamental observation:

If the structure is infinite and periodic, and the excitation is periodic except for a phase shift, then all of the currents and radiated fields will also be periodic except for a phase shift.

This is sometimes referred to as "*Floquet's theorem*."

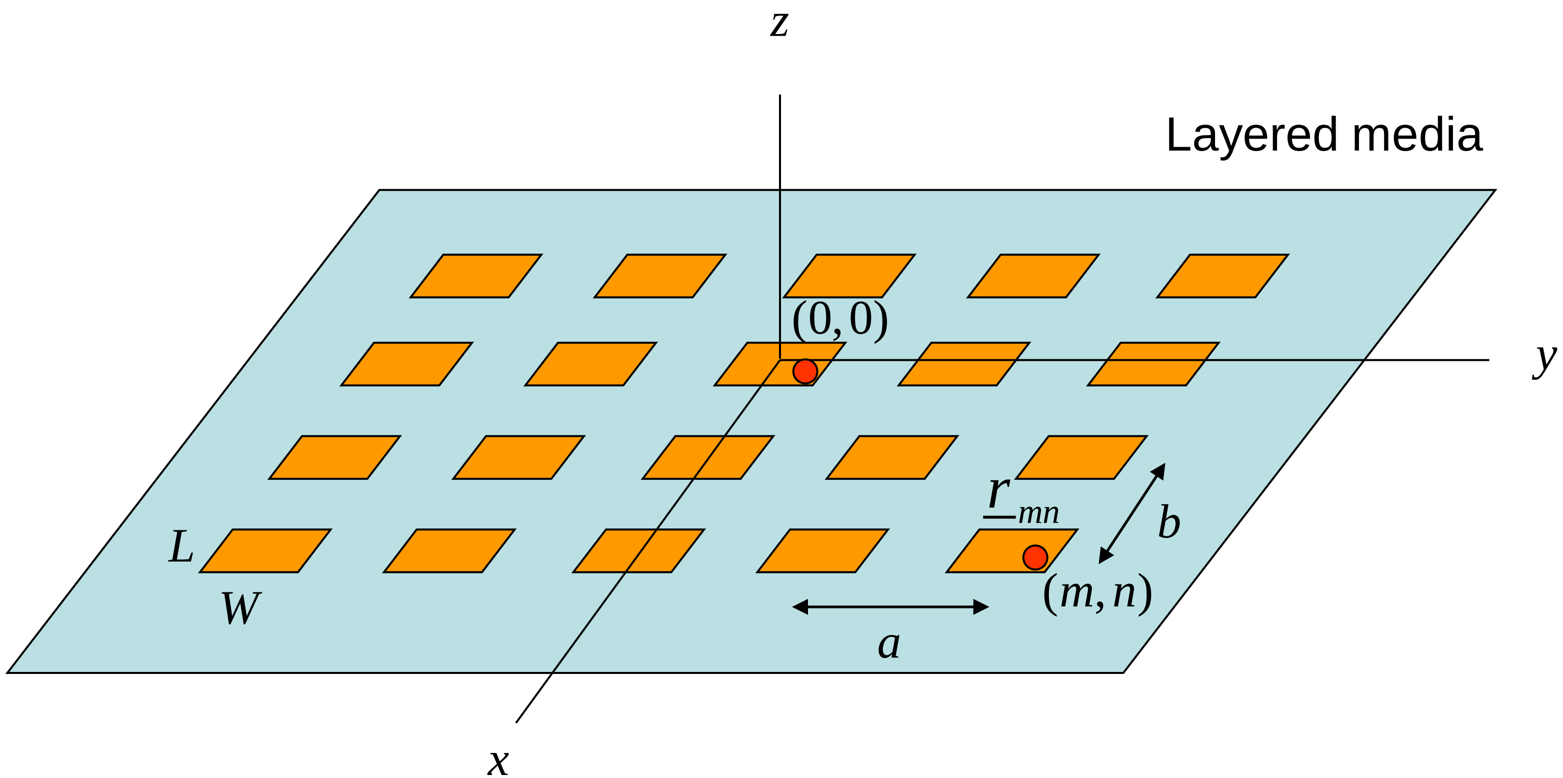


Floquet's Theorem (cont.)

From Floquet's theorem:

$$\underline{J}_s^{mn}\left(\underline{r}\right) = \underline{J}_s^{00}\left(\underline{r} - \underline{r}_{mn}\right) e^{-j\underline{k}_{t00}\cdot\underline{r}_{mn}}$$

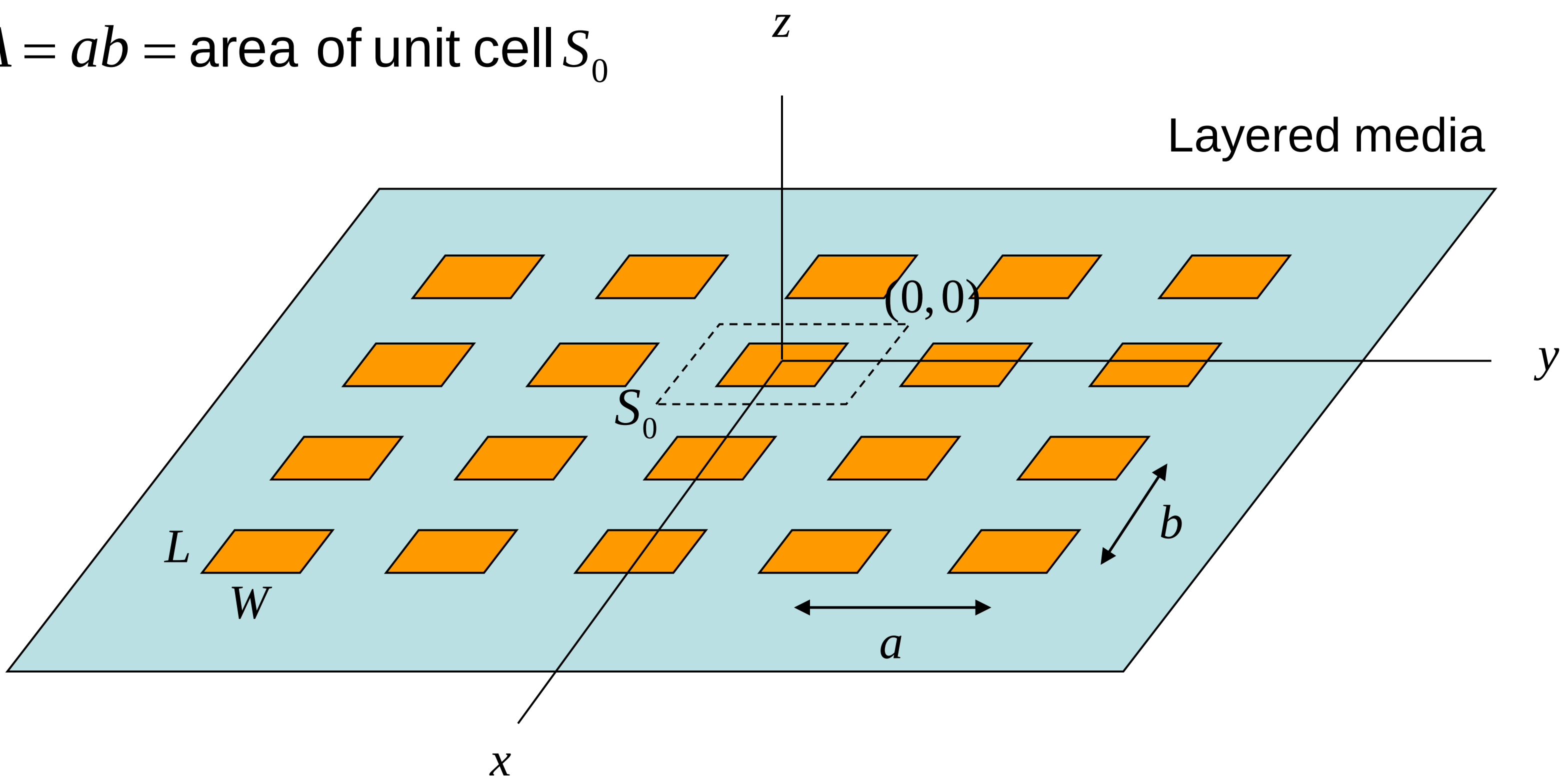
$$\underline{k}_{t00} = \underline{\hat{x}}\, k_{x0} + \underline{\hat{y}}\, k_0$$



Floquet's Theorem (cont.)

If we know the current of field at any point within the (0,0) unit cell, we know the current and field everywhere.

$A = ab = \text{area of unit cell } S_0$



Floquet Waves

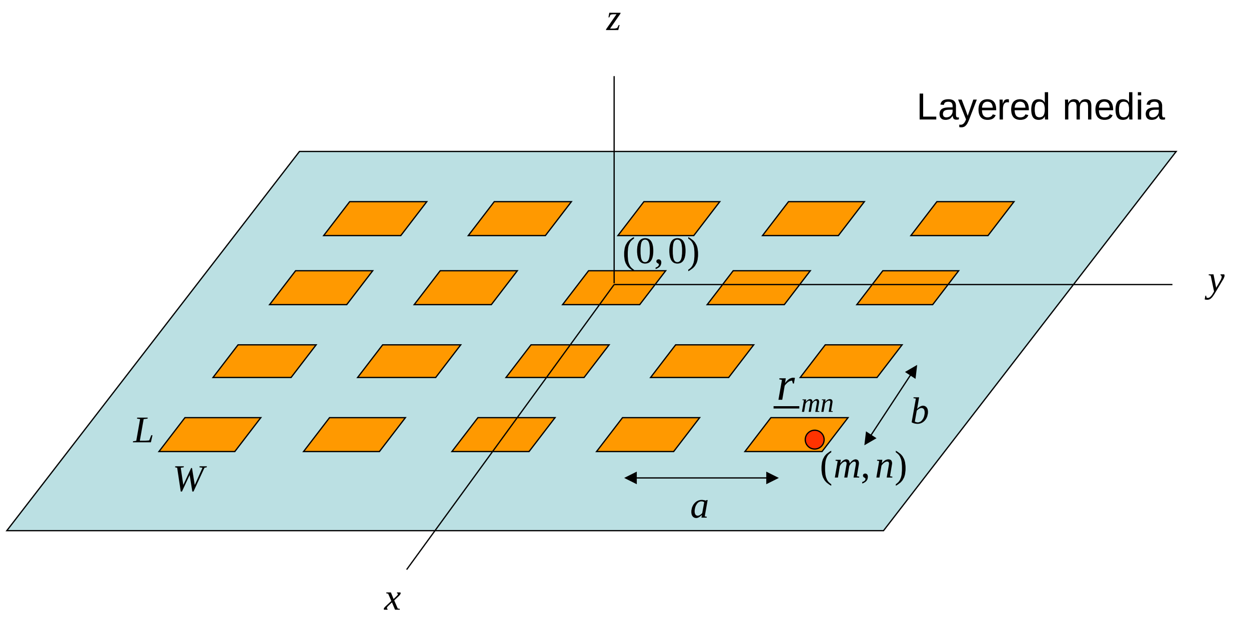
Let ψ denote any component of the surface current or the field (at a fixed value of z).

$$\psi(x+a,y)=\psi(x,y)e^{-jk_{x0}a}$$
$$\psi(x,y+b)=\psi(x,y)e^{-jk_{y0}b}$$

$$\Rightarrow \quad \psi(x,y)=e^{-j\left(k_{x0}x+k_{y0}y\right)}P(x,y)$$

where

$$P(x+a,y)=P(x,y)$$
$$P(x,y+b)=P(x,y)$$



Floquet Waves (cont.)

$$\psi(x,y) = e^{-j\left(k_{x0}x + k_{y0}y\right)} P(x,y)$$

From Fourier-series theory, we know that the 2D periodic function P can be represented as

$$P(x,y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} A_{pq}\, e^{-j\left(\frac{2\pi p}{a}x + \frac{2\pi q}{b}y\right)}$$

Hence we have

$$\psi(x,y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} A_{pq}\, e^{-j\left(\left(k_{x0} + \frac{2\pi p}{a}\right)x + \left(k_{y0} + \frac{2\pi q}{b}\right)y\right)}$$

Floquet Waves (cont.)

Hence, the surface current or field can be expanded in a set of Floquet waves:

$$\psi(x,y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} A_{pq}\, e^{-j\left(k_{xp}x + k_{yq}y\right)}$$

$$k_{xp} \equiv k_{x0} + \frac{2\pi p}{a}$$

$$k_{yq} \equiv k_{y0} + \frac{2\pi q}{b}$$

$\left(k_{xp}, k_{yq}\right) =$ wavenumbers of (p,q) Floquet wave

$$\underline{k}_{tpq} = \underline{\hat{x}}\, \underline{k}_{xp} + \underline{\hat{y}}\, \underline{k}_{yq}$$

$$\underline{k}_{tpq} = \left(\underline{\hat{x}} k_{x0} + \underline{\hat{y}} k_{y0}\right) + \left[\left(\frac{2\pi p}{a}\right)\underline{\hat{x}} + \left(\frac{2\pi q}{b}\right)\underline{\hat{y}}\right]$$

Incident part

Periodic part

Floquet Waves (cont.)

Note: Each Floquet wave repeats from one unit cell to the next, except for a phase shift that corresponds to that of the *incident wave*.

$$
\begin{aligned}
\psi_{pq}(x+a,y) &= e^{-j\left(k_{xp}(x+a)+k_{yq}y\right)} \\
&= e^{-j\left(k_{xp}a\right)} e^{-j\left(k_{xp}x+k_{yq}y\right)} \\
&= e^{-j\left(k_{x0}a\right)} e^{-j\left(\frac{2\pi p}{a}a\right)} e^{-j\left(k_{xp}x+k_{yq}y\right)} \\
&= e^{-j\left(k_{x0}a\right)} e^{-j\left(k_{xp}x+k_{yq}y\right)} \\
&= e^{-j\left(k_{x0}a\right)} \psi_{pq}(x,y)
\end{aligned}
$$

Hence

$$
\psi_{pq}(x+a,y) = e^{-j\left(k_{x0}a\right)} \psi_{pq}(x,y)
$$

Similarly,

$$
\psi_{pq}(x,y+b) = e^{-j\left(k_{y0}b\right)} \psi_{pq}(x,y)
$$

Periodic SDI

The surface current on the periodic structure is represented in terms of Floquet waves:

$$\underline{J}_s(x,y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \underline{a}_{pq}\, e^{-j\underline{k}_{tpq}\cdot\underline{r}} \qquad \underline{r} = \underline{\hat{x}}\, x + \underline{\hat{y}}\, y$$

To solve for the unknown coefficients, multiply both sides by $e^{j\underline{k}_{tp'q'}\cdot\underline{r}}$ and integrate over the $(0,0)$ unit cell S_0:

$$\int_{S_0} \underline{J}_s(x,y)\, e^{j\underline{k}_{tp'q'}\cdot\underline{r}}\, dS = \int_{S_0} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \underline{a}_{pq}\, e^{-j\underline{k}_{tpq}\cdot\underline{r}}\, e^{j\underline{k}_{tp'q'}\cdot\underline{r}}\, dS$$

$$= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \int_{S_0} \underline{a}_{pq}\, e^{-j\left(\frac{2\pi}{a}(p-p')x + \frac{2\pi}{b}(q-q')y\right)} dS$$

Use orthogonality: $\displaystyle\int_{S_0} \underline{J}_s(x,y)\, e^{j\underline{k}_{tp'q'}\cdot\underline{r}}\, dS = \underline{a}_{p'q'}\, A$

Periodic SDI (cont.)

Hence, we have:

$$\underline{a}_{pq} = \frac{1}{A} \int_{S_0} \underline{J}_s(x,y) e^{j\underline{k}_{tpq}\cdot \underline{r}} \, dS$$

Therefore we have:

$$\underline{a}_{pq} = \frac{1}{A} \int_{S_0} \underline{J}_s(x,y) \, e^{j\underline{k}_{tpq}\cdot \underline{r}} \, dS$$

$$= \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{J}_s^{00}(x,y) \, e^{-j\left(k_{xp}x + k_{yq}y\right)} \, dS$$

$$= \frac{1}{A} \underline{\tilde{J}}_s^{00}\left(k_{xp}, k_{yq}\right)$$

$$= \frac{1}{A} \underline{\tilde{J}}_s^{00}\left(\underline{k}_{tpq}\right)$$

The current $\underline{J}_s^{00}$ is the current on the (0,0) patch.

so

$$\underline{a}_{pq} = \frac{1}{A} \underline{\tilde{J}}_s^{00}\left(k_{xp}, k_{yq}\right)$$

Periodic SDI (cont.)

Hence the current on the 2D periodic structure can be represented as

$$\underline{J}_s(x,y)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\tilde{J}}_s^{00}\left(k_{xp},k_{yq}\right)e^{-j\underline{k}_{tpq}\cdot\underline{r}}$$

We now calculate the Fourier transform of the 2D periodic current $\underline{J}_s(x, y)$ (this is what we need in the SDI method):

$$F\left[e^{-j\underline{k}_{tpq}\cdot\underline{r}}\right]=F\left[e^{-jk_{xp}x}e^{-jk_{yq}y}\right]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[e^{-jk_{xp}x}\,e^{-jk_{yq}y}\right]e^{+j\left(k_x x+k_y y\right)}\,dxdy$$

$$=\int_{-\infty}^{\infty}e^{-jk_{xp}x}\,e^{+jk_x x}\,dx\int_{-\infty}^{\infty}e^{-jk_{yq}y}\,e^{+jk_y y}\,dy$$

$$=2\pi\delta\left(k_x-k_{xp}\right)2\pi\delta\left(k_y-k_{yq}\right)$$

Periodic SDI (cont.)

Hence

$$\underline{\tilde{J}}_s\left(k_x,k_y\right)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\tilde{J}}_s^{00}\left(k_{xp},k_{yq}\right)2\pi\delta\left(k_x-k_{xp}\right)2\pi\delta\left(k_y-k_{yq}\right)$$

Next, we calculate the field produced by the periodic patch currents:

$$\underline{\tilde{E}}\left(k_x,k_y,z\right)=\underline{\underline{\tilde{G}}}\left(k_x,k_y;z,z'\right)\cdot\underline{\tilde{J}}_s\left(k_x,k_y\right)$$

$$\underline{E}\left(x,y,z\right)=\frac{1}{\left(2\pi\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\underline{\tilde{G}}}\left(k_x,k_y;z,z'\right)\cdot\underline{\tilde{J}}_s\left(k_x,k_y\right)e^{-j\left(k_xx+k_yy\right)}dk_xdk_y$$

Periodic SDI (cont.)

Hence, we have

$$\underline{E}(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{\underline{\tilde{G}}}(k_x, k_y; z, z') \cdot$$

$$\left[\frac{1}{A} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \underline{\tilde{J}}_s^{00}(k_{xp}, k_{yq}) 2\pi\delta(k_x - k_{xp})\, 2\pi\delta(k_y - k_{yq}) \right]$$

$$e^{-j(k_x x + k_y y)}\, dk_x dk_y$$

Periodic SDI (cont.)

Therefore, integrating over the delta functions, we have:

$$\underline{E}(x,y,z) = \frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \underline{\underline{\tilde{G}}}\left(k_{xp},k_{yq};z,z'\right)\cdot \underline{\tilde{J}}_s^{00}\left(k_{xp},k_{yq}\right)e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

The field is thus in the form of a double summation of Floquet waves.

Periodic SDI (cont.)

Compare:

Single element (non-periodic):

$$\underline{E}(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\underline{\tilde{G}}}(k_x,k_y;z,z')\cdot\underline{\tilde{J}}_s(k_x,k_y)\,e^{-j(k_x x+k_y y)}dk_x dk_y$$

Periodic array of phased elements:

$$\underline{E}(x,y,z)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\underline{\tilde{G}}}(k_{xp},k_{yq};z,z')\cdot\underline{\tilde{J}}_s^{00}(k_{xp},k_{yq})e^{-j(k_{xp}x+k_{yq}y)}$$

Periodic SDI (cont.)

Conclusion:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F\left(k_x, k_y\right) dk_x dk_y \rightarrow \frac{(2\pi)^2}{A} \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} F\left(k_{xp}, k_{yq}\right)$$

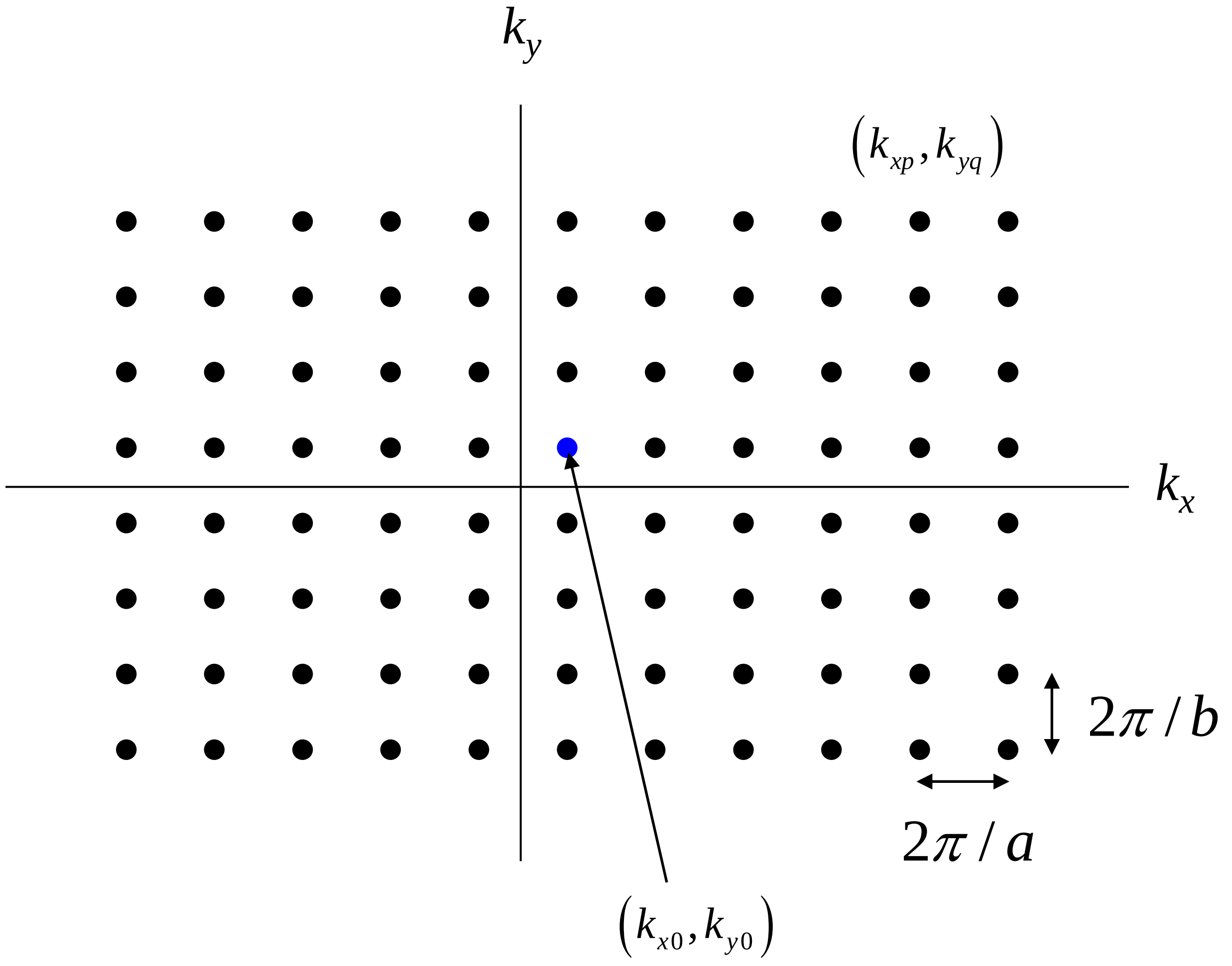
where

$$\begin{aligned} \underline{k}_{tpq} &= \left(\underline{\hat{x}} k_{xp} + \underline{\hat{y}} k_{yq}\right) \\ &= \left(\underline{\hat{x}} k_{x0} + \underline{\hat{y}} k_{y0}\right) + \left[\left(\frac{2\pi p}{a}\right)\underline{\hat{x}} + \left(\frac{2\pi q}{b}\right)\underline{\hat{y}}\right] \end{aligned}$$

The double integral is replaced by a double sum, and a factor is introduced.

Periodic SDI (cont.)

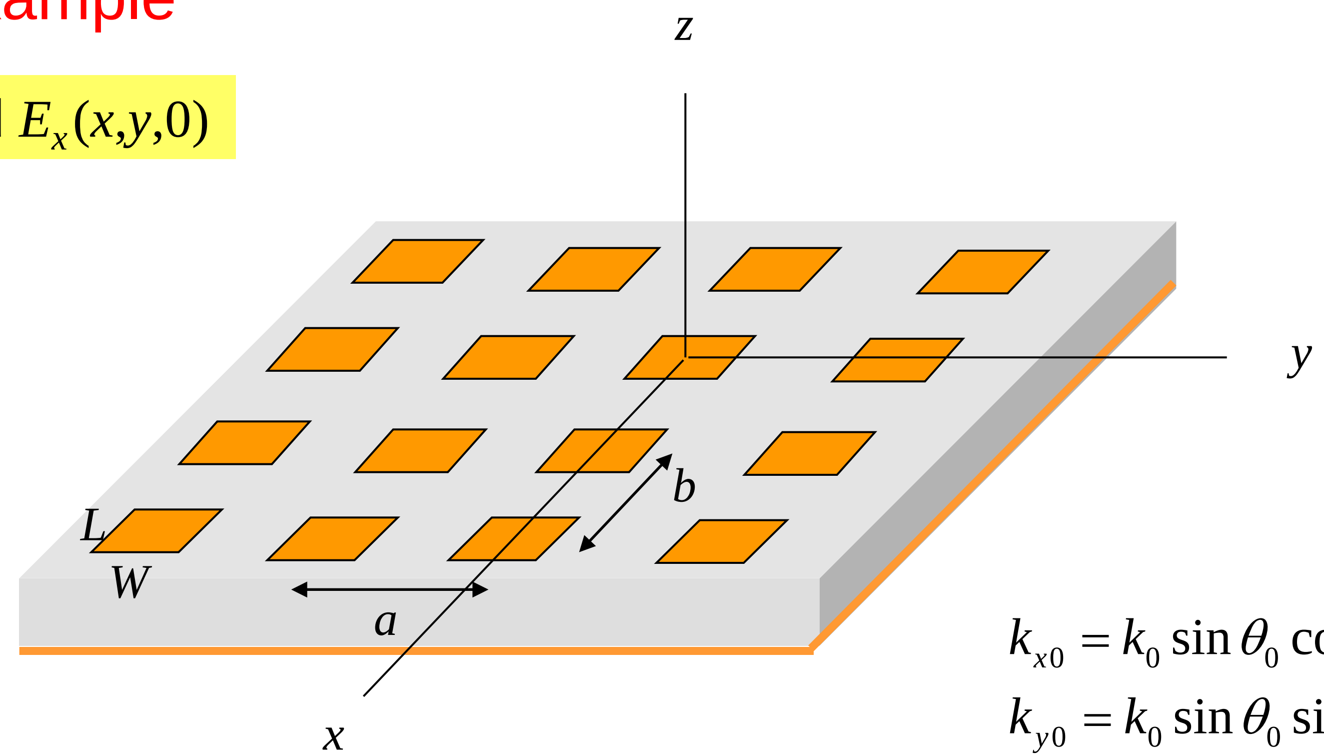
Sample points in the (k_x, k_y) plane



Microstrip Patch Phased Array

Example

Find $E_x(x,y,0)$



$$k_{x0} = k_0 \sin\theta_0 \cos\phi_0$$

$$k_{y0} = k_0 \sin\theta_0 \sin\phi_0$$

Microstrip Patch Phased Array

Phased Array (cont.)

Single patch:

$$E_x(x,y,0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} -\frac{1}{k_t^2} \tilde{J}_{sx}(k_x, k_y) \cdot \left[\frac{k_x^2}{D_m(k_t)} + \frac{k_y^2}{D_e(k_t)} \right] e^{-j(k_x x + k_y y)} \, dk_x dk_y$$

$$J_{sx}(x,y) = \cos\left(\frac{\pi x}{L}\right)$$

$$\tilde{J}_{sx}(k_x, k_y) = \left(\frac{\pi}{2} LW\right) \operatorname{sinc}\left(k_y \frac{W}{2}\right) \left[\frac{\cos\left(k_x \frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(k_x \frac{L}{2}\right)^2} \right]$$

Phased Array (cont.)

2D phased array of patches:

$$E_x(x,y,0)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}-\frac{1}{k_{tpq}^2}\tilde{J}_{sx}^{00}\left(k_{xp},k_{yq}\right)\left[\frac{k_{xp}^2}{D_m\left(k_{tpq}\right)}+\frac{k_{yq}^2}{D_e\left(k_{tpq}\right)}\right]e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

where

$$J_{sx}^{00}(x,y)=\cos\left(\frac{\pi x}{L}\right)$$

$$\tilde{J}_{sx}^{00}\left(k_{xp},k_{yq}\right)=\left(\frac{\pi}{2}LW\right)\mathrm{sinc}\left(k_{yq}\frac{W}{2}\right)\left[\frac{\cos\left(k_{xp}\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_{xp}\frac{L}{2}\right)^2}\right]$$

$$k_{tpq}=\sqrt{k_{xp}^2+k_{yq}^2}$$

Phased Array (cont.)

The field is of the form

$$E_x(x,y,0) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} A_{pq}\, e^{-j\left(k_{xp}x + k_{yq}y\right)}$$

$$= \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} A_{pq}\, e^{-j\underline{k}_{tpq}\cdot\underline{r}}$$

$$= \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} A_{pq}\, \psi_{pq}(x,y)$$

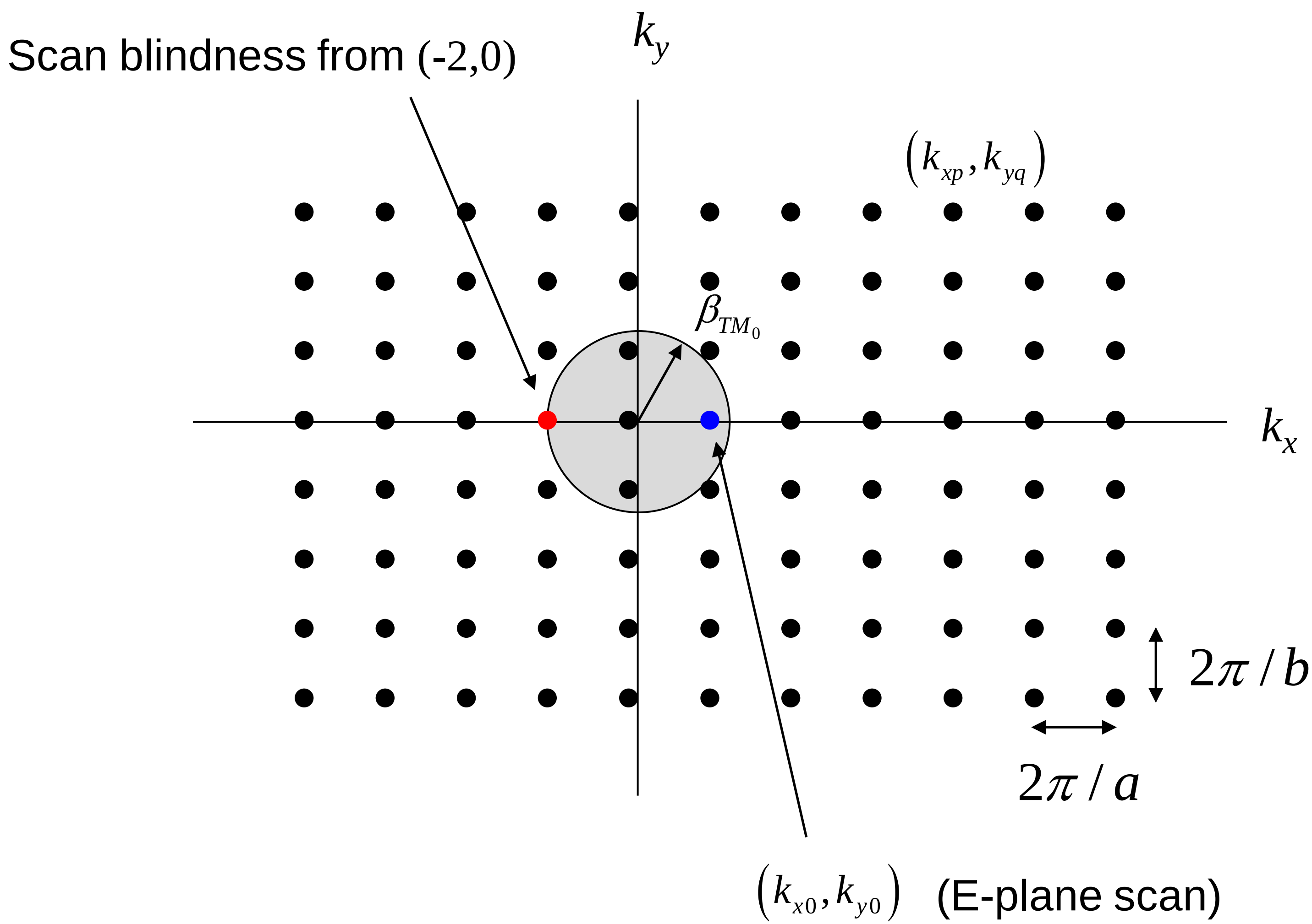
where

$$\underline{k}_{tpq} = \underline{\hat{x}}k_{xp} + \underline{\hat{y}}k_{yq} = \left(\underline{\hat{x}}k_{x0} + \underline{\hat{y}}k_{y0}\right) + \left[\left(\frac{2\pi p}{a}\right)\underline{\hat{x}} + \left(\frac{2\pi q}{b}\right)\underline{\hat{y}}\right]$$

The field is thus represented as a “sum of Floquet waves.”

Scan Blindness in Phased Array

This occurs when one of the sample points (p,q) lies on the surface-wave circle (shown for (-2, 0)).



Scan Blindness (cont.)

The scan blindness condition is:

$$k_{tpq} = \left|\underline{k}_{tpq}\right| = \beta_{TM_0} \;\; \text{(for some } (p,q))$$

$$E_x(x,y,0) = \frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} -\frac{1}{k_{tpq}^2}\tilde{J}_{sx}^{00}\left(k_{xp},k_{nq}\right)\left[\frac{k_{xp}^2}{D_m\left(k_{tpq}\right)} + \frac{k_{yq}^2}{D_e\left(k_{tpq}\right)}\right]e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

$$D_m\left(k_{tpq}\right) = D_m\left(\beta_{TM0}\right) = 0$$

The field produced by an *impressed* set of infinite periodic phased surface-current sources will be infinite.

Scan Blindness (cont.)

Physical interpretation: All of the surface-wave fields excited from the patches add up in phase in the direction of the transverse phasing vector:

$$\underline{k}_{tpq} = \underline{\hat{x}} k_{xp} + \underline{\hat{y}} k_{yq} \quad \Rightarrow \quad \cos\phi = \left(\frac{k_{xp}}{k_{tpq}} \right)$$

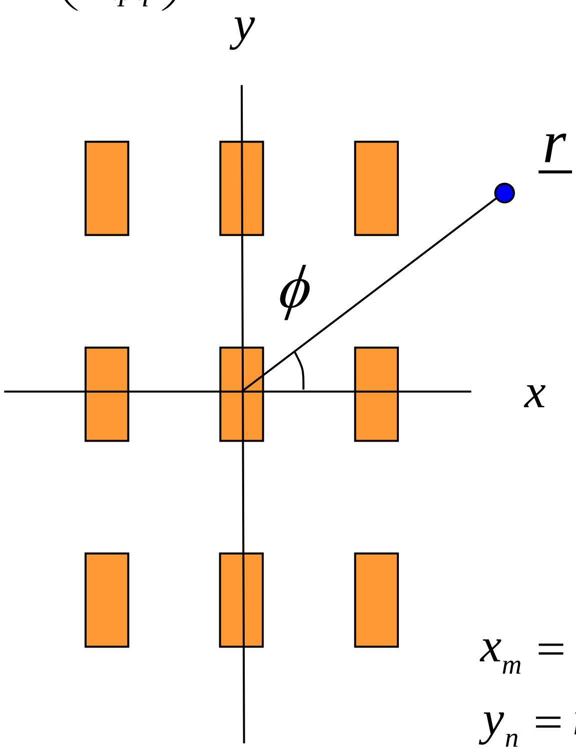
Proof:

Start with the *surface-wave array factor*:

$$AF_{sw} = \sum_{m,n} A_{mn} \, e^{+j\left(\beta_{TM_0} x_m \cos\phi + \beta_{TM_0} y_n \sin\phi\right)}$$

$$= \sum_{m,n} A_{mn} \, e^{+j\left(\beta_{TM_0} x_m \left(\frac{k_{xp}}{k_{tpq}}\right) + \beta_{TM_0} y_n \left(\frac{k_{yq}}{k_{tpq}}\right)\right)}$$

$$\sum_{m,n} A_{mn} \, e^{+j\left(\beta_{TM_0} x_m \left(\frac{k_{xp}}{\beta_{TM_0}}\right) + \beta_{TM_0} y_n \left(\frac{k_{yq}}{\beta_{TM_0}}\right)\right)}$$



$x_m = ma$

$y_n = nb$

N elements

Scan Blindness (cont.)

Hence we have, in this direction, that

$$AF_{sw} = \sum_{m,n} A_{mn}\, e^{+j\left(k_{xp}x_m + k_{yq}y_n\right)}$$

$$= \sum_{m,n} A_{mn}\, e^{+j\left(\left(k_{x0}+\frac{2\pi p}{a}\right)x_m + \left(k_{y0}+\frac{2\pi q}{b}\right)y_n\right)}$$

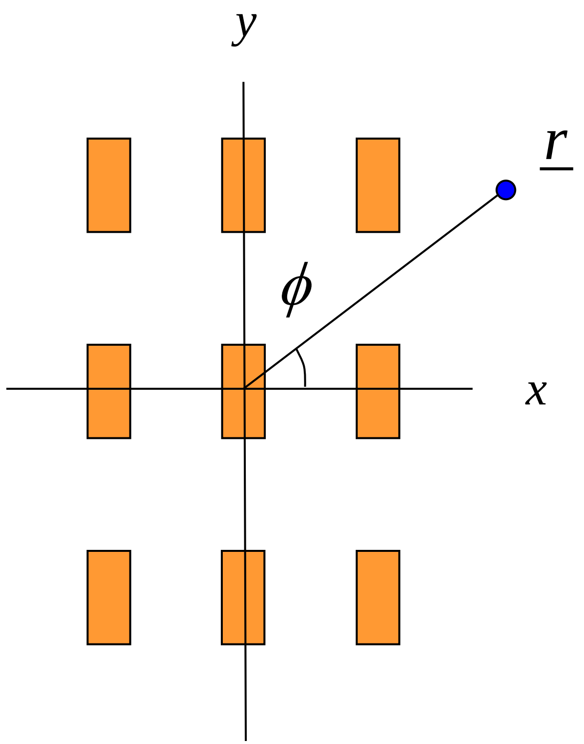
$$= \sum_{m,n} A_{mn}\, e^{+j\left(\left(k_{x0}+\frac{2\pi p}{a}\right)(x_0+ma) + \left(k_{y0}+\frac{2\pi q}{b}\right)(y_0+nb)\right)}$$

$$= e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} \sum_{m,n} A_{mn}\, e^{+j\left(k_{x0}ma+k_{y0}nb\right)} e^{+j\left(\frac{2\pi p}{a}\right)ma} e^{+j\left(\frac{2\pi q}{b}\right)nb}$$

$$= e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} \sum_{m,n} A_{mn}\, e^{+j\left(k_{x0}ma+k_{y0}nb\right)}$$

$$e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} \sum_{m,n} A_{00}\, e^{-j\left(k_{x0}ma+k_{y0}nb\right)} e^{+j\left(k_{x0}ma+k_{y0}nb\right)}$$

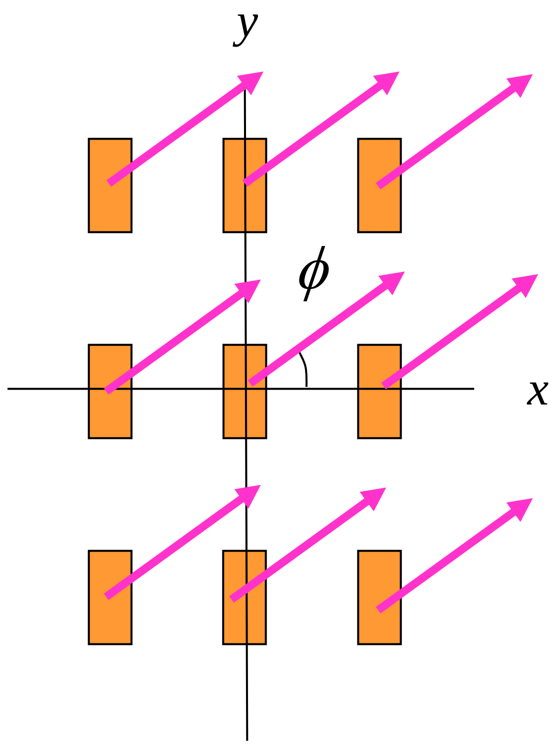
$$e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} A_{00} N$$



N elements

$$\cos\phi = \left(\frac{k_{xp}}{k_{tpq}}\right)$$

Scan Blindness (cont.)



TM_0 surface wave

In the direction ϕ, the surface fields from each patch add up in phase.

$$\cos\phi = \left(\frac{k_{xp}}{\beta_{TM_0}}\right)$$

N elements

$$\Rightarrow \quad AF_{sw} = N\left(A_{00}e^{+jk_{xp}x_0}e^{+jk_{yq}y_0}\right)$$

Note: There is also an element pattern as well, with the field decaying as $1/\rho^{1/2}$, but this is ignored here.

Scan Blindness (cont.)

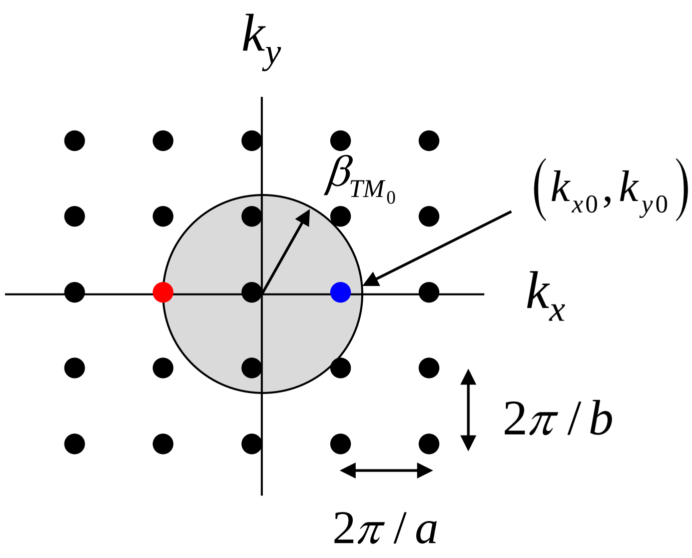
Example

$$p = -2, \quad q = 0$$

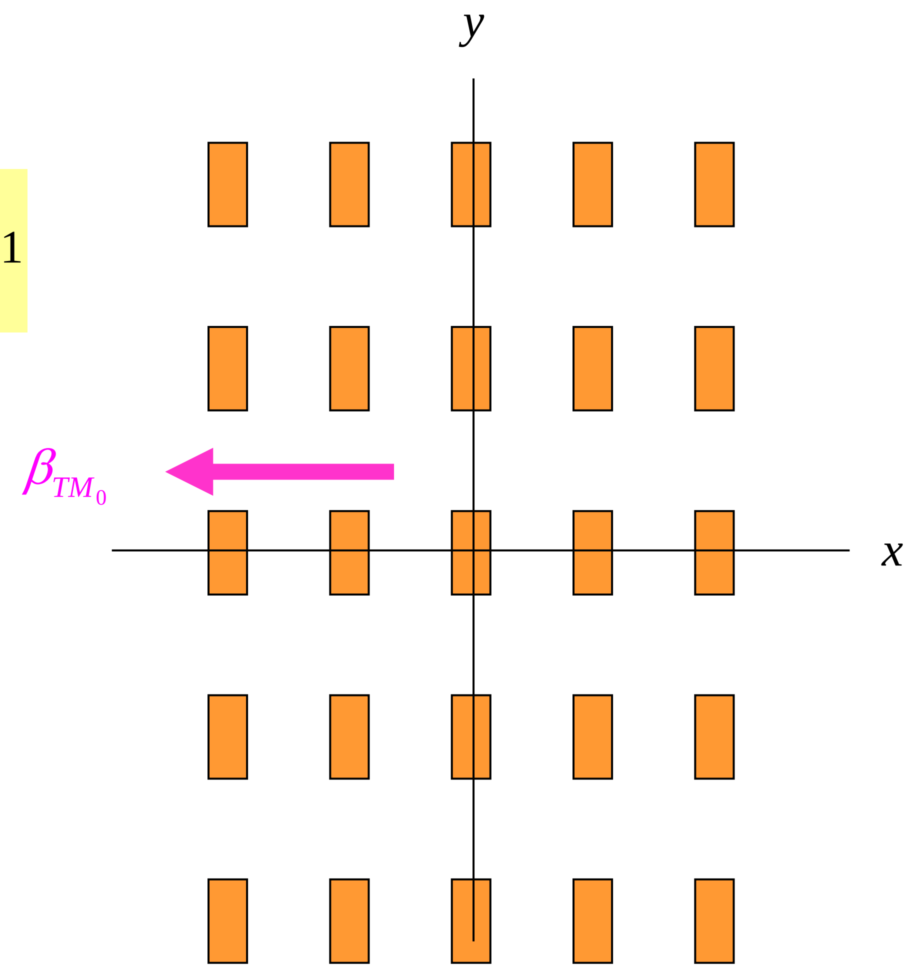
$$k_{xp} = -\beta_{TM_0}$$

$$k_{yq} = 0$$

$$\cos\phi = \left(\frac{-\beta_{TM_0}}{\beta_{TM_0}}\right) = -1$$



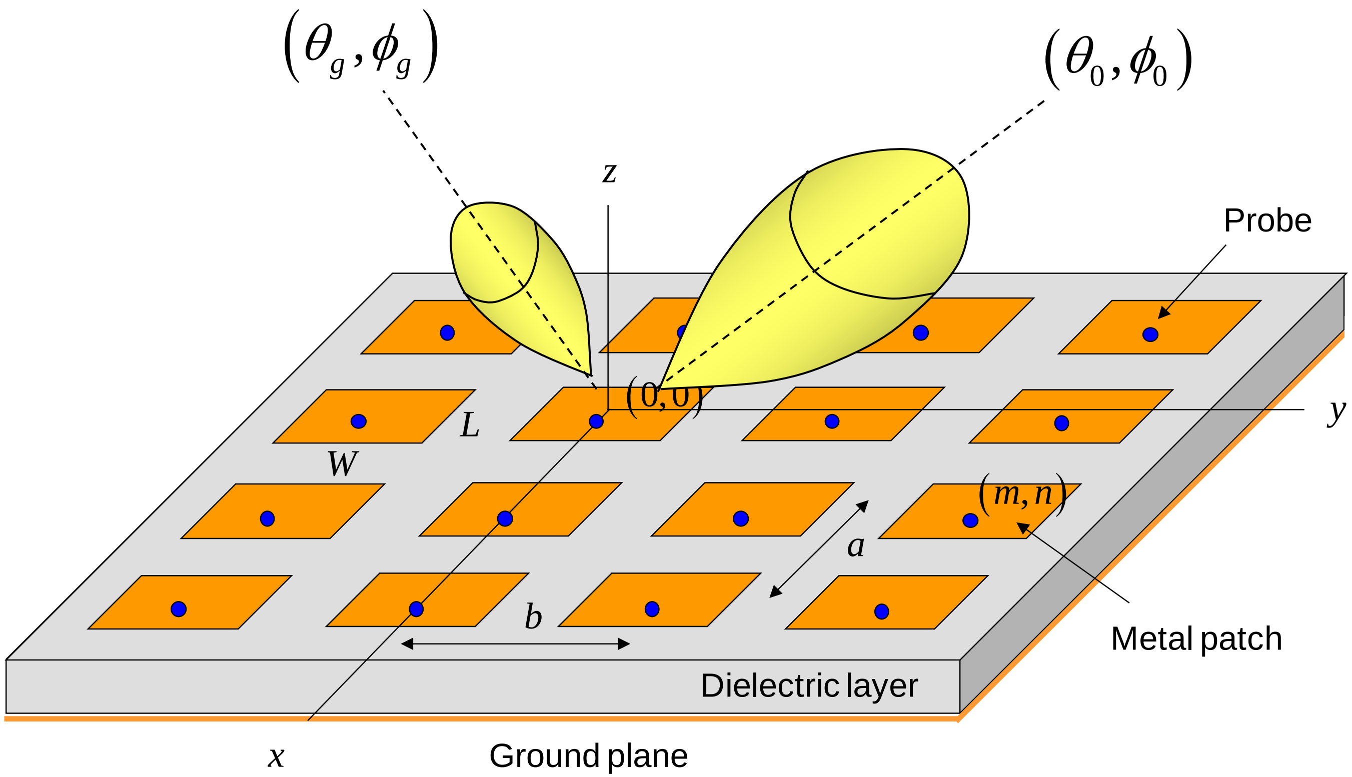
E-plane scan



Grating Lobes

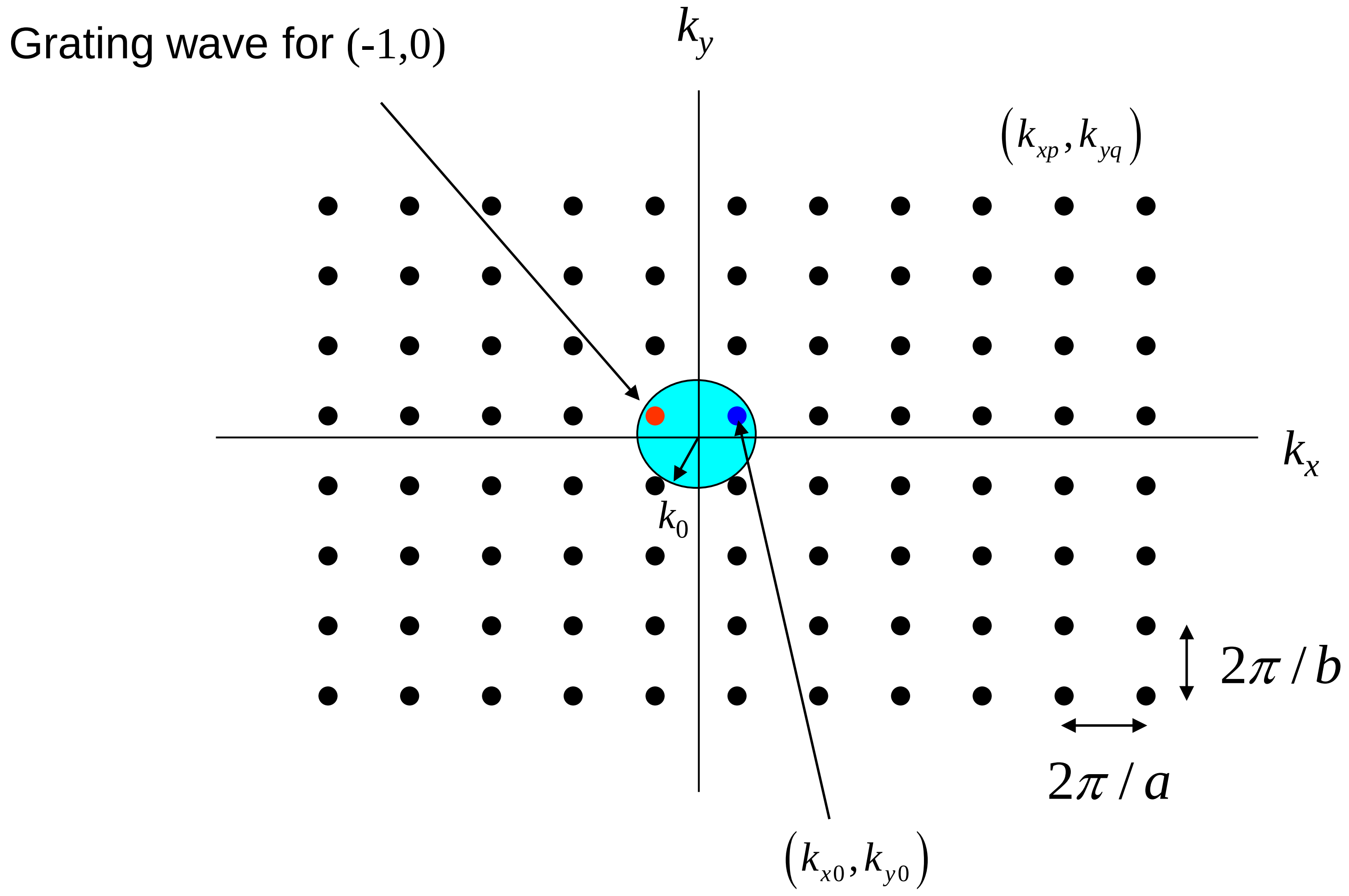
❖ Grating lobes occur when one or more of the higher-order Floquet waves propagates in space.

❖ For a finite-size array, this corresponds to a secondary beam that gets radiated.



Grating Lobes (cont.)

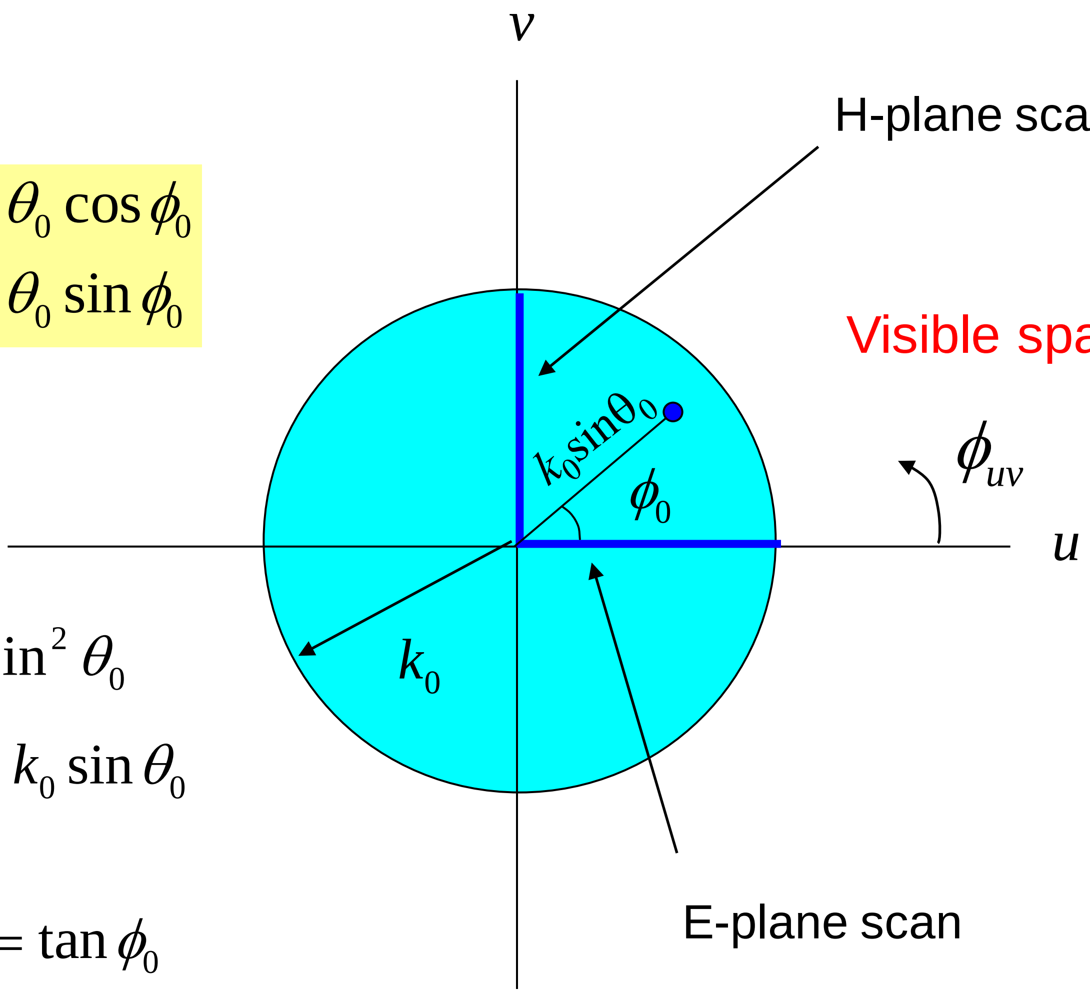
$$k_{tpq} < k_0 \quad \left(\text{for some } (p,q) \neq (0,0)\right)$$



Pozar Circle Diagram

Define

$$u \equiv k_{x0} = k_0 \sin\theta_0 \cos\phi_0$$
$$v \equiv k_{y0} = k_0 \sin\theta_0 \sin\phi_0$$



$$u^2 + v^2 = k_0^2 \sin^2\theta_0$$

$$\Rightarrow \quad \sqrt{u^2 + v^2} = k_0 \sin\theta_0$$

$$\tan\phi_{uv} = v / u = \tan\phi_0$$

$$\Rightarrow \quad \phi_{uv} = \phi_0$$

Pozar Circle Diagram (cont.)

Grating Lobes

$k_{tpq} < k_0$ for $(u, v) \in$ visible space circle

So, we require that

$$k_{xp}^2 + k_{yq}^2 < k_0^2 \qquad \text{for } u^2 + v^2 < k_0^2$$

The first inequality gives us

$$\left(k_{x0} + \frac{2\pi p}{a} \right)^2 + \left(k_{y0} + \frac{2\pi q}{b} \right)^2 < k_0^2$$

or

$$\left(k_0 \sin\theta_0 \cos\phi_0 + \frac{2\pi p}{a} \right)^2 + \left(k_0 \sin\theta_0 \sin\phi_0 + \frac{2\pi q}{b} \right)^2 < k_0^2$$

or

$$\left(u + \frac{2\pi p}{a} \right)^2 + \left(v + \frac{2\pi q}{b} \right)^2 < k_0^2$$

Pozar Circle Diagram (cont.)

or

$$\left(u-\left(-\frac{2\pi p}{a}\right)\right)^2+\left(v-\left(-\frac{2\pi q}{b}\right)\right)^2<k_0^2$$

or

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2<k_0^2$$ (we are inside a shifted (p,q) visible space circle)

$$u_p\equiv-\frac{2\pi p}{a},\quad v_q\equiv-\frac{2\pi q}{b}$$

Summary of grating lobe condition:

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2<k_0^2$$
$$u^2+v^2<k_0^2$$

Part of the *interior* of the (p,q) circle is also inside the visible space circle.

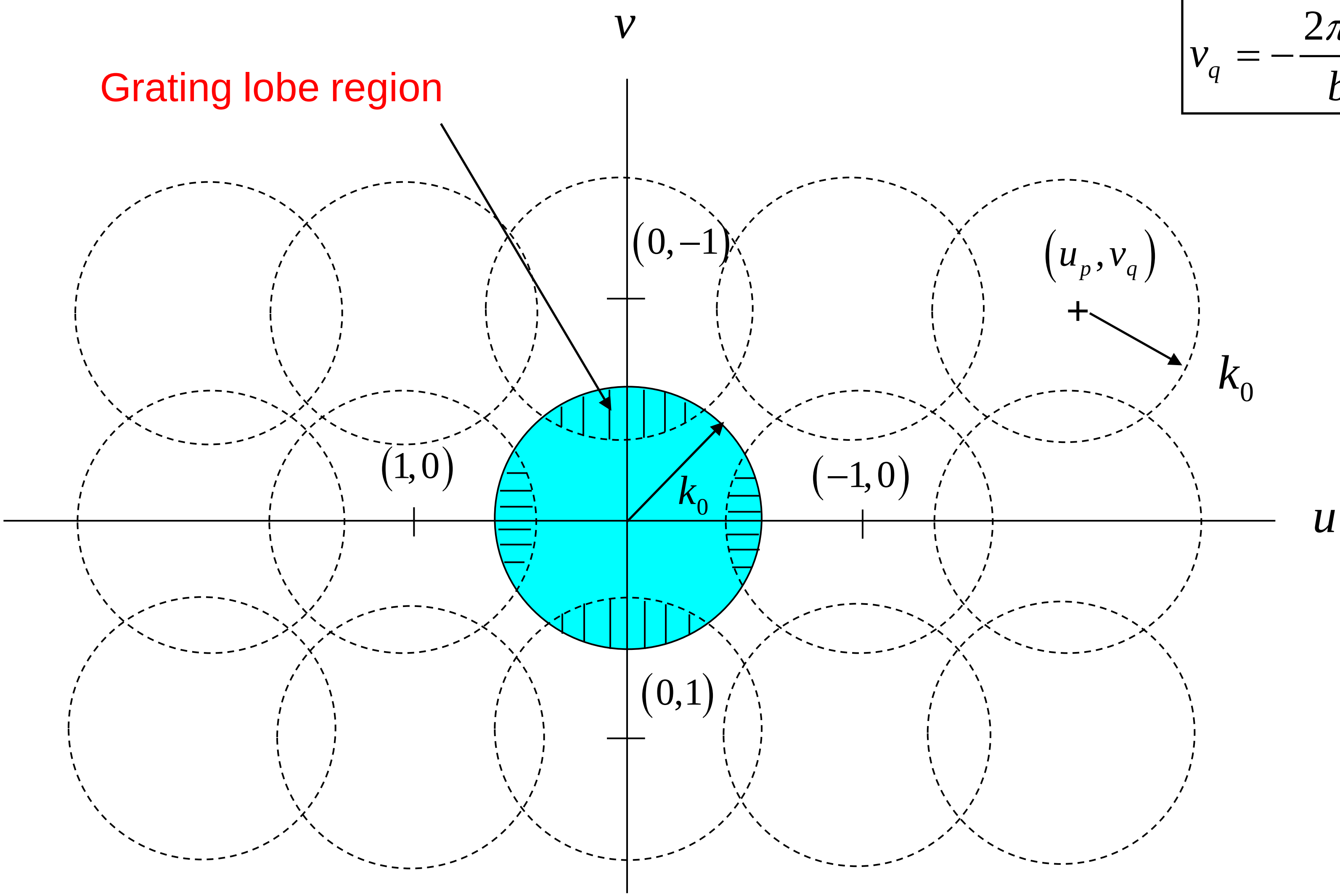
Pozar Circle Diagram (cont.)

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2<k_0^2$$

$$u^2+v^2<k_0^2$$

$$u_p=-\frac{2\pi p}{a}$$

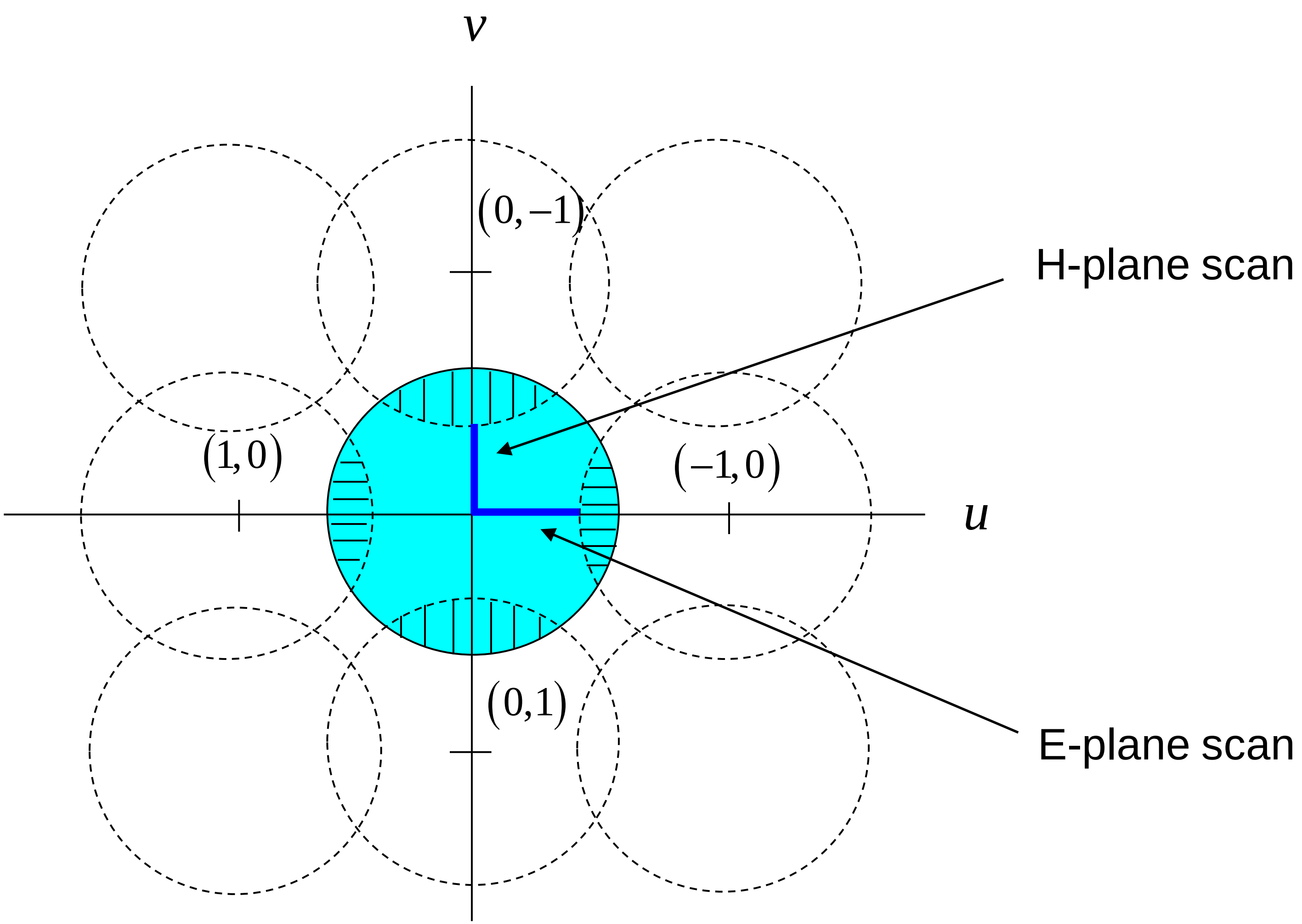
$$v_q=-\frac{2\pi q}{b}$$



Pozar Circle Diagram (cont.)

This diagram shows when grating lobes occur in the principal scan planes.

$$u = k_0 \sin\theta_0 \cos\phi_0 \qquad v = k_0 \sin\theta_0 \sin\phi_0$$



Pozar Circle Diagram (cont.)

To avoid grating lobes for all scan angles, we require:

$$u_{-1} > 2k_0$$
$$v_{-1} > 2k_0$$

(The circles do not overlap.)

or

$$\frac{2\pi}{a} > 2k_0$$
$$\frac{2\pi}{b} > 2k_0$$

or

$$k_0 a < \pi$$
$$k_0 b < \pi$$

Hence

$$a < \lambda_0 / 2$$
$$b < \lambda_0 / 2$$

Pozar Circle Diagram (cont.)

Scan Blindness

$$k_{tpq} = \beta_{TM_0} \text{ for } (u,v) \in \text{visible space circle}$$

So, we require that

$$k_{xp}^2 + k_{yq}^2 = \beta_{TM_0}^2 \qquad \text{for } u^2 + v^2 < k_0^2$$

The equation gives us

$$\left(k_{x0} + \frac{2\pi p}{a}\right)^2 + \left(k_{y0} + \frac{2\pi q}{b}\right)^2 = \beta_{TM_0}^2$$

or

$$\left(k_0 \sin\theta_0 \cos\phi_0 + \frac{2\pi p}{a}\right)^2 + \left(k_0 \sin\theta_0 \sin\phi_0 + \frac{2\pi q}{b}\right)^2 = \beta_{TM_0}^2$$

or

$$\left(u + \frac{2\pi p}{a}\right)^2 + \left(v + \frac{2\pi q}{b}\right)^2 = \beta_{TM_0}^2$$

Pozar Circle Diagram (cont.)

or

$$\left(u-\left(-\frac{2\pi p}{a}\right)\right)^2+\left(v-\left(-\frac{2\pi q}{b}\right)\right)^2=\beta_{TM_0}^2$$

or

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2=\beta_{TM_0}^2$$

where

$$u_p=-\frac{2\pi p}{a}$$

$$v_q=-\frac{2\pi q}{b}$$

Summary of scan blindness condition:

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2=\beta_{TM0}^2$$

$$u^2+v^2<k_0^2$$

Part of the *boundary* of the (p,q) circle is inside the visible space circle.

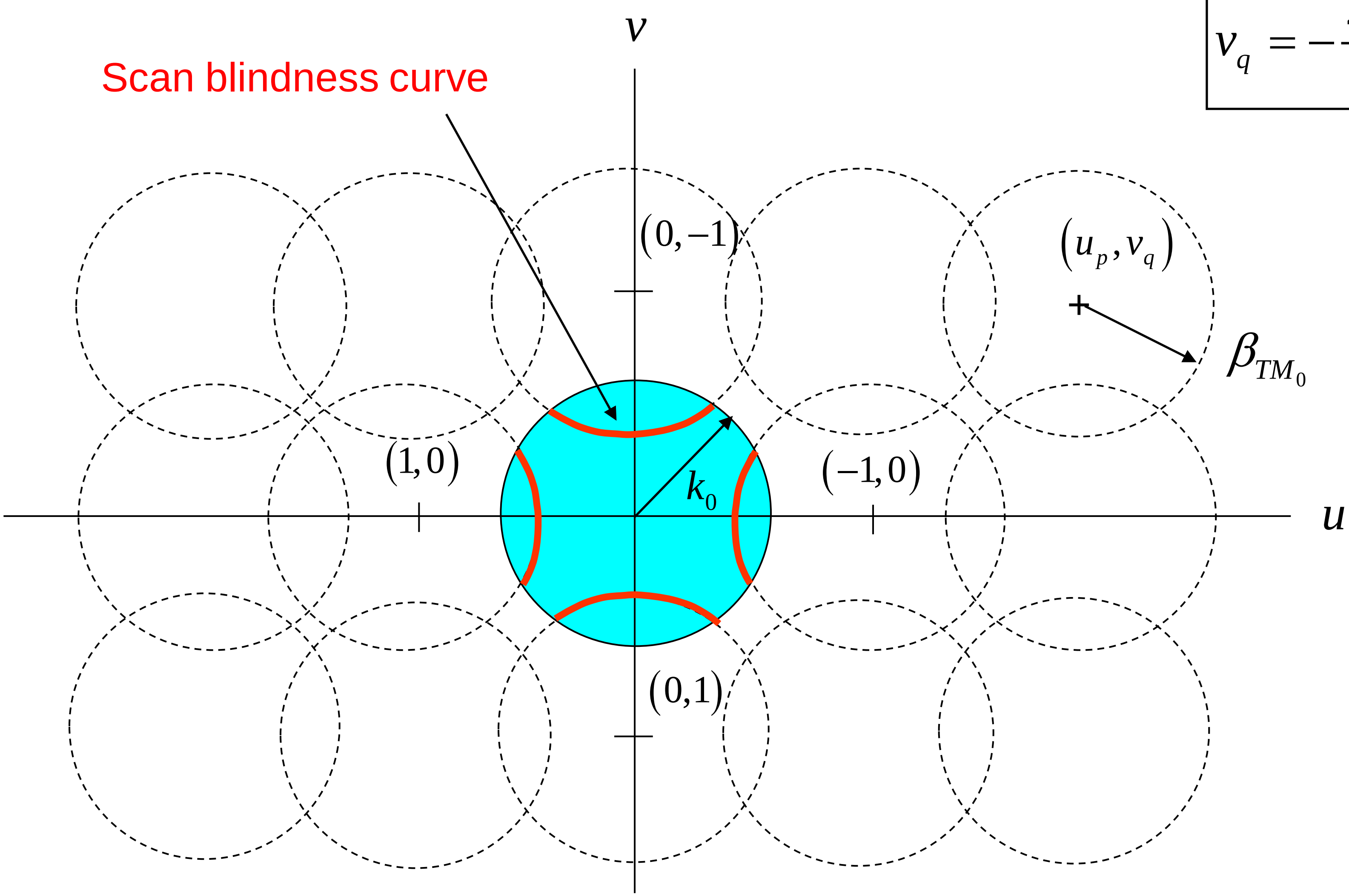
Pozar Circle Diagram (cont.)

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2=\beta_{TM0}^2$$

$$u^2+v^2<k_0^2$$

$$u_p=-\frac{2\pi p}{a}$$

$$v_q=-\frac{2\pi q}{b}$$



Pozar Circle Diagram (cont.)

To avoid scan blindness for all scan angles, we require

$$u_{-1} > \beta_{TM_0} + k_0$$
$$v_{-1} > \beta_{TM_0} + k_0$$

(The circles do not overlap.)

or

$$\frac{2\pi}{a} > \beta_{TM_0} + k_0$$
$$\frac{2\pi}{b} > \beta_{TM_0} + k_0$$

or

$$\frac{2\pi}{k_0 a} > \beta_{TM_0} / k_0 + 1$$
$$\frac{2\pi}{k_0 b} > \beta_{TM_0} / k_0 + 1$$

Hence

$$a / \lambda_0 < \frac{1}{\beta_{TM_0} / k_0 + 1}$$
$$b / \lambda < \frac{1}{\beta_{TM_0} / k_0 + 1}$$