ECE 6345

Spring 2015

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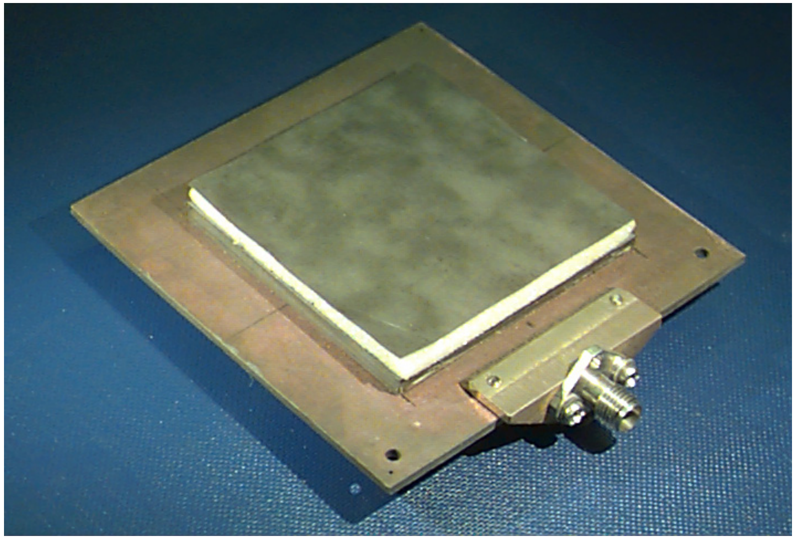
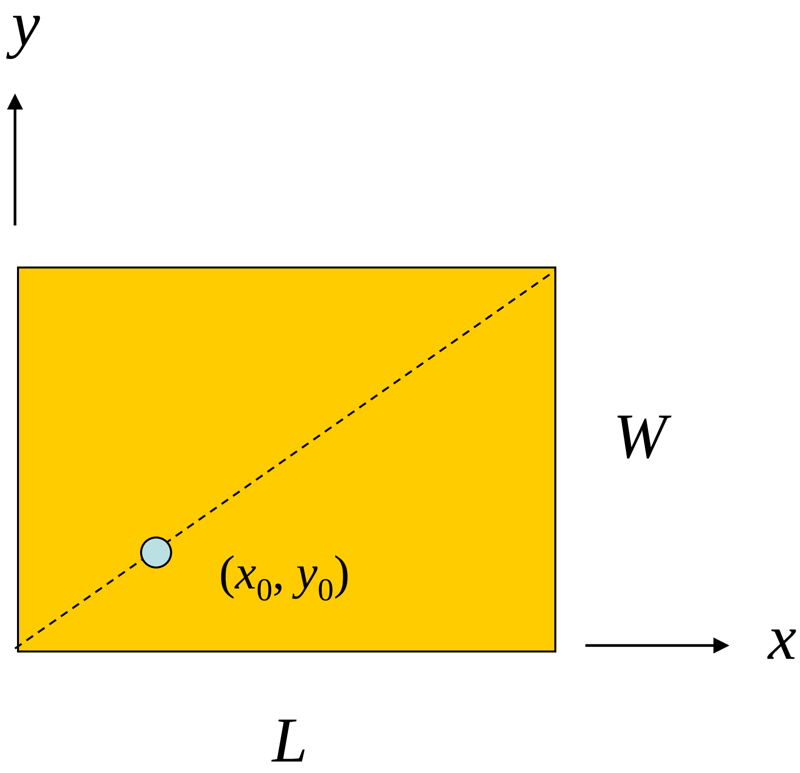
ECE Dept.



Notes 2

Overview

This set of notes treats circular polarization, obtained by using a single feed.



$$L \approx W$$

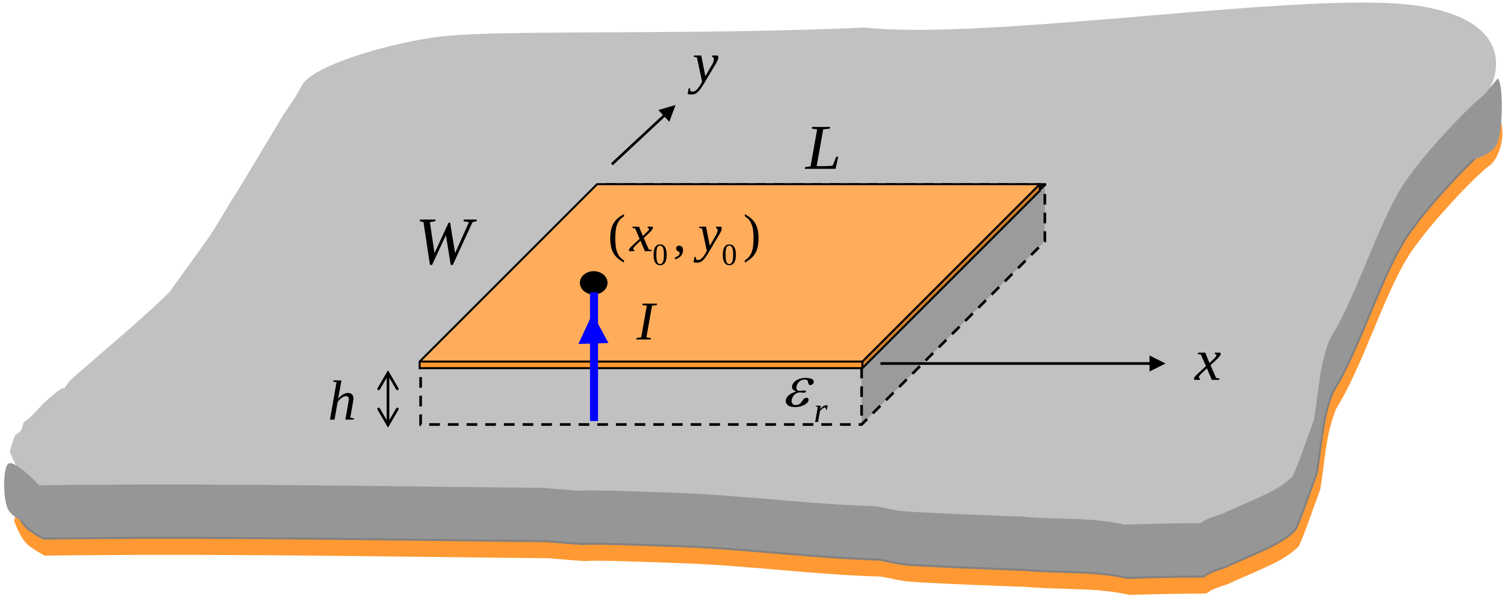
$$y_0 \approx x_0$$

Overview

Goals:

- Find the optimum dimensions of the CP patch
- Find the input impedance of the CP patch
- Find the pattern (axial-ratio) bandwidth
- Find the impedance bandwidth of CP patch

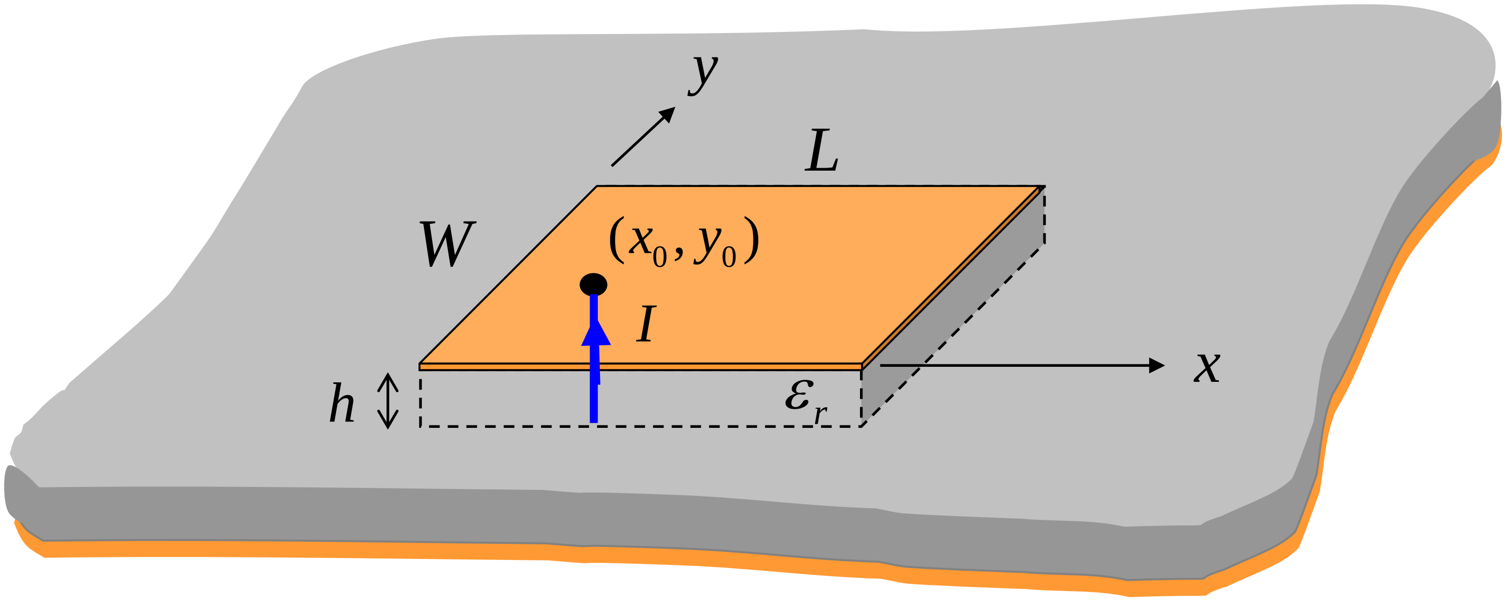
Amplitude of Patch Currents



$$I = 1\,[A]$$

First Step: Find the patch currents (x and y directions), and relate them to the input impedance of the patch.

Amplitude of Patch Currents (cont.)



$$I = 1\,[A]$$

x-directed current mode (1,0): $\underline{J}_s^{x} = \underline{\hat{x}}\, A_x \sin\left(\dfrac{\pi x}{L}\right)$

y-directed current mode (0,1): $\underline{J}_s^{y} = \underline{\hat{y}}\, A_y \sin\left(\dfrac{\pi y}{W}\right)$

Amplitude of Patch Currents (cont.)

The x mode (TM_{10}):

$$\underline{J}_s = \underline{\hat{n}} \times \underline{H} = -\underline{\hat{z}} \times \underline{H}$$

so $J_{sx} = H_y$

$$\Rightarrow \quad \underline{H} = \underline{\hat{y}}\, A_x \sin\left(\frac{\pi x}{L}\right)$$

To find $\underline{E}$, use $\nabla \times \underline{H} = j\omega\varepsilon \underline{E}$

$$E_z = \frac{1}{j\varepsilon_0\varepsilon_r}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] = \frac{1}{j\varepsilon_0\varepsilon_r} A_x \left(\frac{\pi}{L}\right)\cos\left(\frac{\pi x}{L}\right)$$

Amplitude of Patch Currents (cont.)

$$Z_{in} = \frac{V}{I} = V = -hE_z = -h\left(E_z^x + E_z^y\right) = Z_{in}^x + Z_{in}^y$$

For the (1,0) mode we have $$Z_{in}^x = \frac{jh}{\omega\varepsilon_0\varepsilon_r}\left(\frac{\pi}{L}\right)A_x \cos\left(\frac{\pi x_0}{L}\right)$$

or

$$A_x = \frac{Z_{in}^x}{\cos\left(\frac{\pi x_0}{L}\right)}\left[\frac{\omega\varepsilon_0\varepsilon_r L}{j\pi h}\right]$$

A similar derivation holds for the y mode.

Amplitude of Patch Currents (cont.)

The y mode (TM_{10}):

$$A_y = \frac{Z_{in}^{y}}{\cos\left(\dfrac{\pi y_0}{W}\right)}\left[\frac{\omega \varepsilon_0 \varepsilon_r W}{j\pi h}\right]$$

The patch current amplitudes can then be written as:

$$A_x = A_1^{(x)} Z_{in}^{x}$$

$$A_y = A_1^{(y)} Z_{in}^{y}$$

where

$$A_1^{(x)} = \frac{1}{\cos\left(\dfrac{\pi x_0}{L}\right)}\left[\frac{\omega \varepsilon_0 \varepsilon_r L}{j\pi h}\right]$$

$$A_1^{(y)} = \frac{1}{\cos\left(\dfrac{\pi y_0}{W}\right)}\left[\frac{\omega \varepsilon_0 \varepsilon_r W}{j\pi h}\right]$$

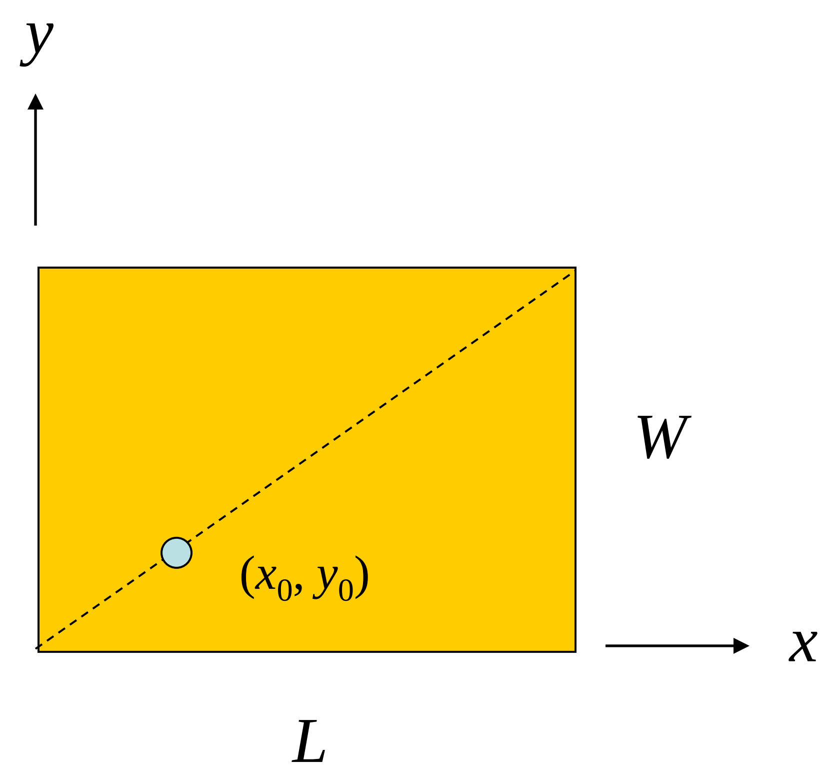
Amplitude of Patch Currents (cont.)

$$A_1^{(x)} = \frac{1}{\cos\left(\dfrac{\pi x_0}{L}\right)}\left[\frac{\omega\varepsilon_0\varepsilon_r L}{j\pi h}\right] \qquad A_1^{(y)} = \frac{1}{\cos\left(\dfrac{\pi y_0}{W}\right)}\left[\frac{\omega\varepsilon_0\varepsilon_r W}{j\pi h}\right]$$

Assume $L \approx W$

$x_0 \approx y_0$

Then $A_1^{(x)} \approx A_1^{(y)} \equiv A_1$



Amplitude of Patch Currents (cont.)

Because of the nearly equal dimensions and the feed along the diagonal, we also have

$$R^x \approx R^y \equiv R$$ (The resistance in the circuit model for either mode is the same.)

R^i = resonant input resistance of the mode i, when excited by itself (R^x only depends on x_0, R^y only depends on y_0).

We then have:

$$A_x = A_2 \, \bar{Z}_{in}^{x}$$

$$A_y = A_2 \, \bar{Z}_{in}^{y}$$

where

$$A_2 \equiv A_1 R$$

$$\Rightarrow A_2 = \frac{R}{\cos\left(\frac{\pi x_0}{L}\right)} \left[\frac{\omega \varepsilon_0 \varepsilon_r L}{j \pi h} \right]$$

Reminder: The bar denotes impedances that are normalized by R (either R_x or R_y).

Amplitude of Patch Currents (cont.)

The A_2 coefficient can be written as

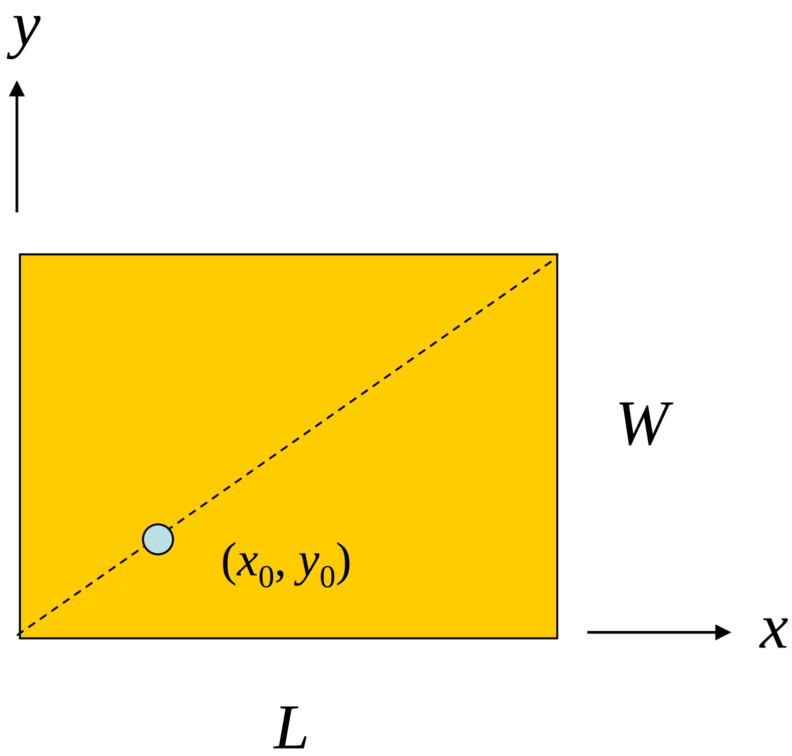
$$A_2 = \frac{R}{\cos\left(\dfrac{\pi x_0}{L}\right)}\left[\frac{\omega \varepsilon_0 \varepsilon_r L}{j\pi h}\right]$$

$$= \frac{R_{edge}\cos^2\left(\dfrac{\pi x_0}{L}\right)}{\cos\left(\dfrac{\pi x_0}{L}\right)}\left[\frac{\omega \varepsilon_0 \varepsilon_r L}{j\pi h}\right]$$

so

$$A_2 = R_{edge}\cos\left(\frac{\pi x_0}{L}\right)\left[\frac{\omega \varepsilon_0 \varepsilon_r L}{j\pi h}\right]$$

Circular Polarization Condition



Let

$$L = W\left(1+\delta\right)$$

$$y_0 = x_0$$

The CP condition is

$$\frac{A_y}{A_x} = \pm j$$

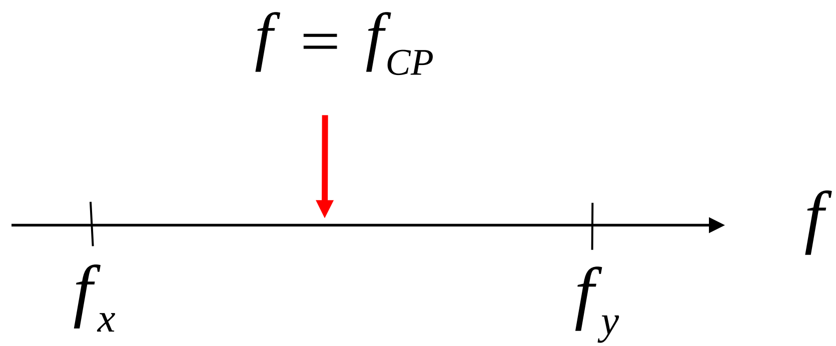
- for RHCP
+ for LHCP

A_x = amplitude of x mode

A_y = amplitude of y mode

Circular Polarization Condition (cont.)

The frequency f_{CP} is defined as the frequency for which we get CP at broadside.



We have:

$$A_x = \frac{A_2}{1 + j2Q\left(f_{rx} - 1\right)}$$

$$A_y = \frac{A_2}{1 + j2Q\left(f_{ry} - 1\right)}$$

where

$$f_{rx} \equiv \frac{f}{f_x} \qquad f_{ry} \equiv \frac{f}{f_y}$$

f_x = resonance frequency of (1,0) mode

f_y = resonance frequency of (0,1) mode

Circular Polarization Condition (cont.)

LHCP:

$$\frac{A_y}{A_x} = +j$$

at $f = f_{CP}$

Choose:

$$f_{rx} - 1 = \frac{1}{2Q}$$

$$f_{ry} - 1 = -\frac{1}{2Q}$$

Then we have

$$A_x = \frac{A}{1+j}$$

$$A_y = \frac{A}{1-j}$$

$$\frac{A_y}{A_x} = \frac{1+j}{1-j} = \frac{\sqrt{2}e^{j\frac{\pi}{4}}}{\sqrt{2}e^{-j\frac{\pi}{4}}} = e^{j\frac{\pi}{2}} = j$$

(LHCP)

Circular Polarization Condition (cont.)

The frequency conditions for $f = f_{CP}$ can be written as:

$$f_{rx} - 1 = \frac{1}{2Q} \;\Rightarrow\; \frac{f_{CP}}{f_x} = 1 + \frac{1}{2Q} \;\Rightarrow\; \frac{f_x}{f_{CP}} \approx 1 - \frac{1}{2Q}$$

$$f_{ry} - 1 = -\frac{1}{2Q} \;\Rightarrow\; \frac{f_{CP}}{f_y} = 1 - \frac{1}{2Q} \;\Rightarrow\; \frac{f_y}{f_{CP}} \approx 1 + \frac{1}{2Q}$$

so $$\frac{f_x + f_y}{f_{CP}} = 2$$

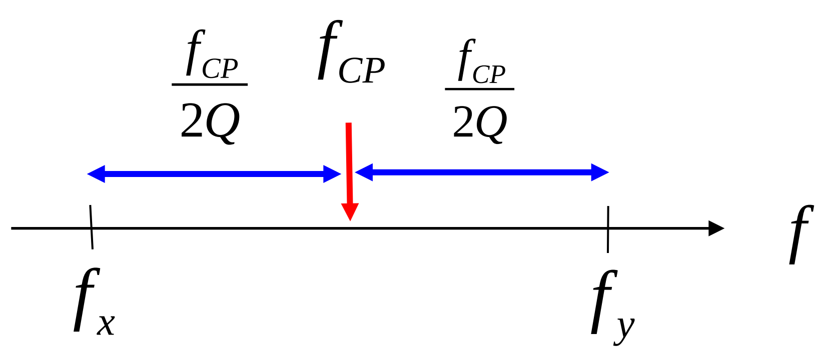
or $$f_{CP} = \frac{1}{2}\left(f_x + f_y\right)$$

Circular Polarization Condition (cont.)

Also, $$\frac{f_y}{f_{CP}} - \frac{f_x}{f_{CP}} = \frac{1}{2Q} - \left(-\frac{1}{2Q}\right) = \frac{1}{Q}$$

Let $$\Delta f \equiv f_y - f_x$$

Then we have $$\frac{\Delta f}{f_{CP}} = \frac{1}{Q}$$



Circular Polarization Condition (cont.)

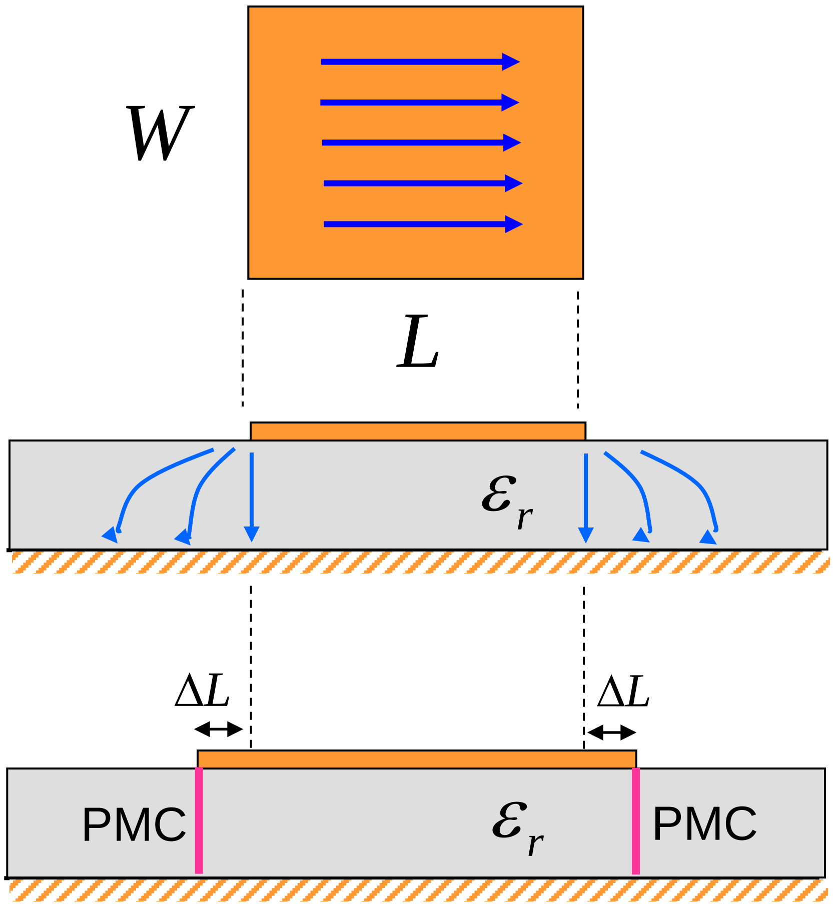
Summary of frequencies

$$f_x = f_{CP}\left[1-\frac{1}{2Q}\right] \qquad f_y = f_{CP}\left[1+\frac{1}{2Q}\right] \qquad \text{(LHCP)}$$

$$f_x = f_{CP}\left[1+\frac{1}{2Q}\right] \qquad f_y = f_{CP}\left[1-\frac{1}{2Q}\right] \qquad \text{(RHCP)}$$

f_{CP} = frequency for which we get CP at broadside.

Patch Dimensions for CP



$$\Delta L = f\left(h, \varepsilon_r, W\right)$$ (Hammerstad formula)

Physical Dimensions for CP (cont.)

$$\Delta L = f\left(h, \varepsilon_r, W\right)$$

$$L^e = L + 2\Delta L$$

$$k_0 \sqrt{\varepsilon_r} L^e = \pi$$ (resonance condition)

Let

$$k_{0x} \equiv 2\pi\, f_x \sqrt{\mu_0 \varepsilon_0}$$
$$k_{0y} \equiv 2\pi\, f_y \sqrt{\mu_0 \varepsilon_0}$$

(known wavenumbers)

Physical Dimensions for CP (cont.)

$$k_{0x} L^e \sqrt{\varepsilon_r} = \pi \quad \Rightarrow \quad k_{0x}\left(L + 2\Delta L\right)\sqrt{\varepsilon_r} = \pi$$

Similarly, we have

$$k_{0y}\left(W + 2\Delta W\right)\sqrt{\varepsilon_r} = \pi$$

Hence

$$L = \frac{\pi}{k_{0x}\sqrt{\varepsilon_r}} - 2\Delta L\left(h, \varepsilon_r, W\right)$$

$$W = \frac{\pi}{k_{0y}\sqrt{\varepsilon_r}} - 2\Delta W\left(h, \varepsilon_r, L\right)$$

Note: For ΔW, we use the same formula as ΔL , but replace $W \to L$.

Physical Dimensions for CP (cont.)

Since the patch is nearly square, the two fringing extensions are nearly equal ($\Delta L \approx \Delta W$). Hence we have

$$L = \frac{\pi}{k_{0x}\sqrt{\varepsilon_r}} - 2\Delta L$$

$$W = \frac{\pi}{k_{0y}\sqrt{\varepsilon_r}} - 2\Delta L$$

where

$$k_{0x} \equiv 2\pi\, f_x \sqrt{\mu_0 \varepsilon_0}$$

$$k_{0y} \equiv 2\pi\, f_y \sqrt{\mu_0 \varepsilon_0}$$

(known wavenumbers)

Note: The fringing length ΔL depends on W, so these equations must be solved numerically (using, e.g., iteration).

Hammerstad's Formula

$$\Delta L = 0.412h \left[\frac{\frac{W}{h} + 0.262}{\frac{W}{h} + 0.813} \right] \left[\frac{\varepsilon_{re} + 0.300}{\varepsilon_{re} - 0.258} \right]$$

$$\varepsilon_{re} = \left(\frac{\varepsilon_r + 1}{2} \right) + \left(\frac{\varepsilon_r - 1}{2} \right) \frac{1}{\sqrt{1 + 12\frac{h}{W}}}$$

Input Impedance of CP Patch

$$Z_{in}(f) = Z_{in}^{x}(f) + Z_{in}^{y}(f) = \frac{R}{1 + j2Q\left(f_{rx} - 1\right)} + \frac{R}{1 + j2Q\left(f_{ry} - 1\right)}$$

At f_{CP}: $f_{rx} - 1 = -1/(2Q)$ and $f_{ry} - 1 = 1/(2Q)$ (LHCP)

so $$Z_{in}(f_0) = \frac{R}{1-j} + \frac{R}{1+j}$$

$$= \frac{R(1+j) + R(1-j)}{(1-j)(1+j)} = \frac{R(1+j) + R(1-j)}{2}$$

or $$Z_{in} = R$$

The CP frequency f_{CP} is also the resonance frequency where the input impedance is real (if we neglect the probe inductance).

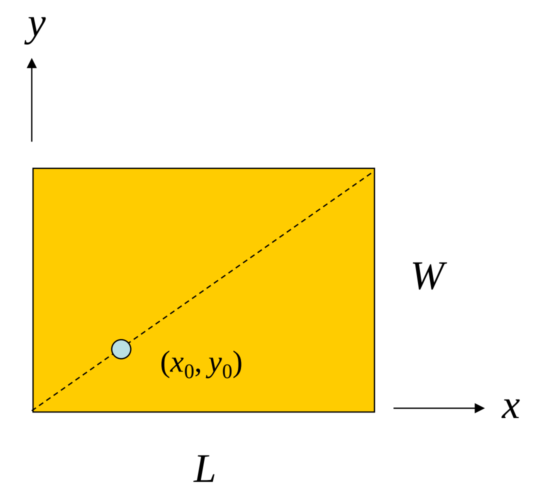
Recall: $R = R^{x} = R$ for TM_{10} mode

Input Impedance of CP Patch

Hence, at the resonance (CP) frequency f_{CP} we have

$$Z_{in} = R_{edge} \cos^2\left(\frac{\pi x_0}{L}\right)$$

$R_{edge} = R_{edge}$ for TM_{10} mode



Note: We have a CAD formula for R_{edge}.

The fed position x_0 can be chosen to give the desired input resistance at the resonance frequency f_{CP}.

CP (Axial Ratio) Bandwidth

We now examine the frequency dependence of the term A_y / A_x.

$$A_x = \frac{A_2}{1 + j2Q\left(f_{rx} - 1\right)}$$

$$A_y = \frac{A_2}{1 + j2Q\left(f_{ry} - 1\right)}$$

$$\frac{A_y}{A_x} = \frac{1 + j2Q(f_{rx} - 1)}{1 + j2Q(f_{ry} - 1)}$$

where

$$f_{rx} \equiv \frac{f}{f_x} = \frac{f}{f_{CP}\left(1 - \frac{1}{2Q}\right)} \approx \frac{f}{f_{CP}}\left(1 + \frac{1}{2Q}\right) \quad \text{(LHCP)}$$

CP Bandwidth (cont.)

Define

$$f_r \equiv \frac{f}{f_{CP}}$$

This is the ratio of the operating frequency to the CP frequency.

Then

$$f_{rx} = f_r\left(1 + \frac{1}{2Q}\right)$$

Similarly,

$$f_{ry} = f_r\left(1 - \frac{1}{2Q}\right)$$

CP Bandwidth (cont.)

Hence

$$\frac{A_y}{A_x} = \frac{1 + j2Q\left(f_{rx} - 1\right)}{1 + j2Q\left(f_{ry} - 1\right)} = \frac{1 + j2Q\left(f_r\left(1 + \frac{1}{2Q}\right) - 1\right)}{1 + j2Q\left(f_r\left(1 - \frac{1}{2Q}\right) - 1\right)}$$

Note: $f_r = 1 \quad \Rightarrow \quad \frac{A_y}{A_x} = \frac{1 + j}{1 - j} = j$

Let $x \equiv 2Q\left(f_r - 1\right)$

Then

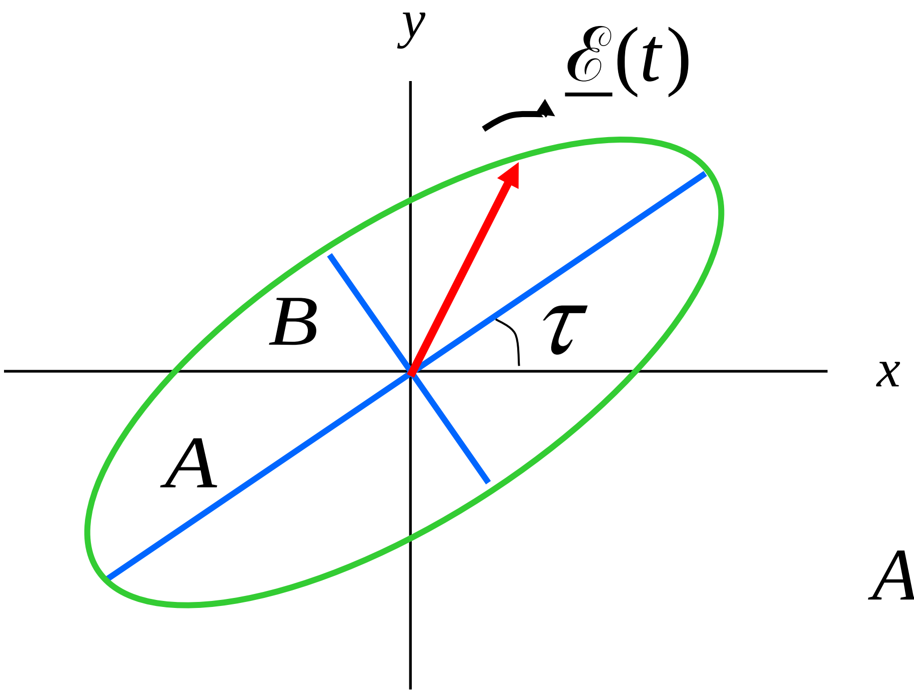
$$\frac{A_y}{A_x} = \frac{1 + j(f_r + x)}{1 - j(f_r - x)} \approx \frac{1 + j(1 + x)}{1 - j(1 - x)}$$

CP Bandwidth (cont.)

$$\frac{A_y}{A_x} = \frac{1 + j(1+x)}{1 - j(1-x)}$$

$$x \equiv 2Q\left(f_r - 1\right)$$

$$f_r \equiv \frac{f}{f_{CP}}$$



$$AR \equiv \frac{A}{B}$$

CP Bandwidth (cont.)

From ECE 6340: $$AR = \cot \xi$$

where

$$\sin 2\xi = \sin(2\gamma)\sin\phi$$

$$\gamma = \tan^{-1}\left|\frac{A_y}{A_x}\right|$$

$$\phi = \arg\left(\frac{A_y}{A_x}\right)$$

In our case,

$$\gamma = \tan^{-1}\sqrt{\frac{1+(1+x)^2}{1+(1-x)^2}}$$

$$\phi = \tan^{-1}(1+x) + \tan^{-1}(1-x)$$

CP Bandwidth (cont.)

Set $$AR = \sqrt{2} \quad \left(AR = 3 \text{ dB} \right)$$

From a numerical solution: $$x = \pm 0.348$$

CP Bandwidth (cont.)

Hence $$2Q\left(f_r - 1\right) = \pm 0.348$$

$$f_r = 1 \pm \frac{0.348}{2Q}$$

so

$$f_r^+ = 1 + \frac{0.348}{2Q}$$

$$f_r^- = 1 - \frac{0.348}{2Q}$$

Therefore, $$\Delta f_r = f_r^+ - f_r^- = \frac{0.348}{Q}$$

$$BW^{AR\,CP} \equiv \frac{f^+ - f^-}{f_{CP}} = f_r^+ - f_r^-$$

Hence, we have

$$BW^{AR\,CP} = \frac{0.348}{Q}$$

Impedance Bandwidth

$$Z_{in}(f) = \frac{R}{1 + j2Q\left(f_{rx} - 1\right)} + \frac{R}{1 + j2Q\left(f_{ry} - 1\right)}$$

$$= \frac{R}{1 + j2Q\left(f_r\left(1 + \frac{1}{2Q}\right) - 1\right)} + \frac{R}{1 + j2Q\left(f_r\left(1 - \frac{1}{2Q}\right) - 1\right)}$$

$$= \frac{R}{1 + j(f_r + x)} + \frac{R}{1 - j(f_r - x)}$$

$$\approx \frac{R}{1 + j(1 + x)} + \frac{R}{1 - j(1 - x)} \qquad \left(\text{for } f_r \approx 1\right)$$

where $x \equiv 2Q\left(f_r - 1\right)$

Note: At $x = 0$ we have $Z_{in} = \frac{R}{1 + j} + \frac{R}{1 - j} = R$

Impedance Bandwidth (cont.)

$$\overline{Z}_{in} \equiv \frac{Z_{in}}{R} = \frac{1}{1+j(1+x)} + \frac{1}{1-j(1-x)}$$

$$\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_{in} - R}{Z_{in} + R} = \frac{\overline{Z}_{in} - 1}{\overline{Z}_{in} + 1}$$

$$S = SWR = \frac{1+|\Gamma|}{1-|\Gamma|}$$

Set $S = S_0 = 2$ (bandwidth limits)

⇨ $x_0 = \pm\sqrt{2}$ (derivation omitted)

Impedance Bandwidth (cont.)

Hence $$2Q\left(f_r - 1\right) = \pm\sqrt{2}$$

so $$f_r = 1 \pm \frac{\sqrt{2}}{2Q}$$

The band edges (in normalized frequency) are then

$$f_r^{+} = 1 + \frac{1}{\sqrt{2}Q}$$

$$f_r^{-} = 1 - \frac{1}{\sqrt{2}Q}$$

Impedance Bandwidth (cont.)

$$BW^{imp\ CP} \equiv \frac{f^{+} - f^{-}}{f_{CP}} = f_r^{+} - f_r^{-}$$

Hence $$BW^{imp\ CP} = 2\left(\frac{1}{\sqrt{2}Q}\right)$$

Hence $$BW^{imp\ CP} = \frac{\sqrt{2}}{Q}$$

Summary

CP antenna

$$BW^{AR\ CP} = \frac{0.348}{Q}$$

$$BW^{imp\ CP} = \frac{\sqrt{2}}{Q} = \frac{1.414}{Q}$$

Linear antenna

$$BW^{imp\ Lin} = \frac{1}{\sqrt{2}Q} = \frac{0.707}{Q}$$