ECE 6345

Spring 2015

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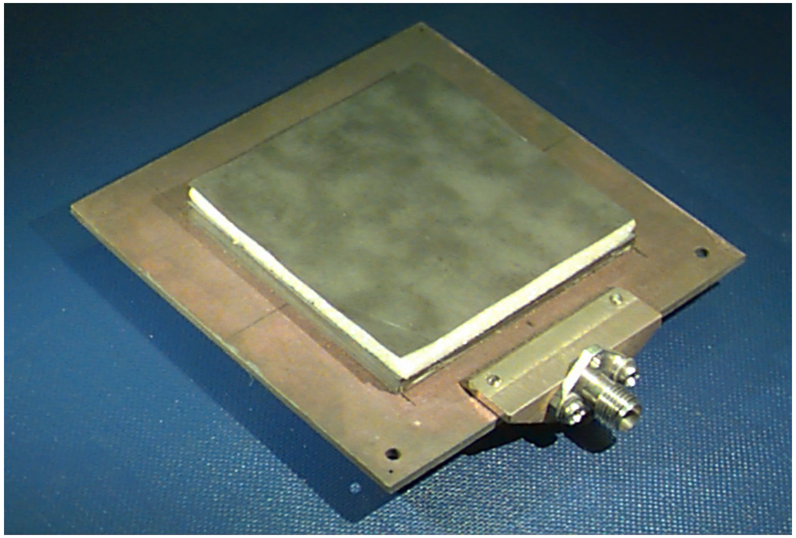
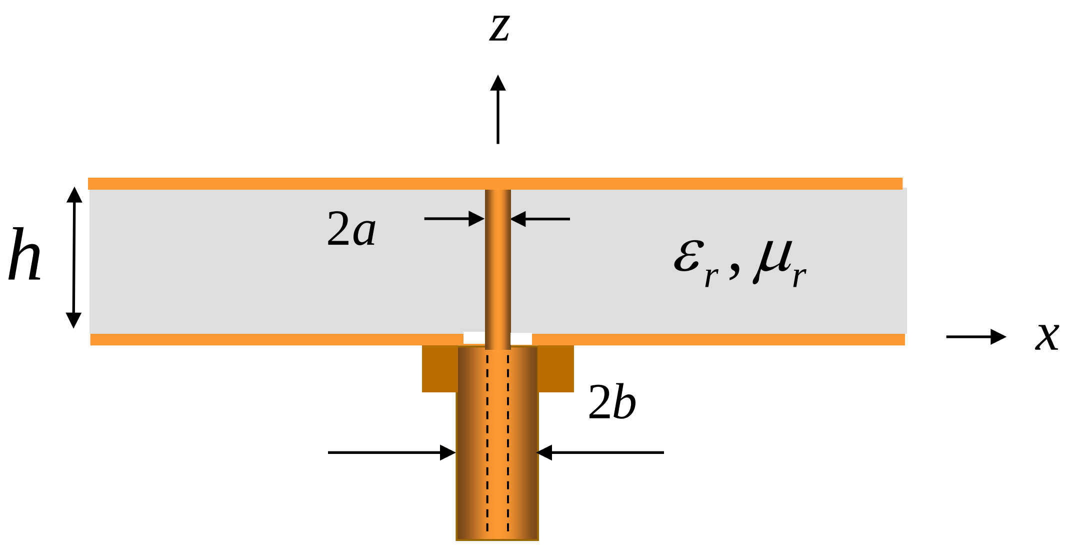
ECE Dept.



Notes 5

Overview

This set of notes discusses improved models of the probe inductance of a coaxially-fed patch (accurate for thicker substrates). A parallel-plate waveguide model is initially assumed (at the end of the notes we will also look at the actual finite patch).



Overview (cont.)

The following models are investigated:

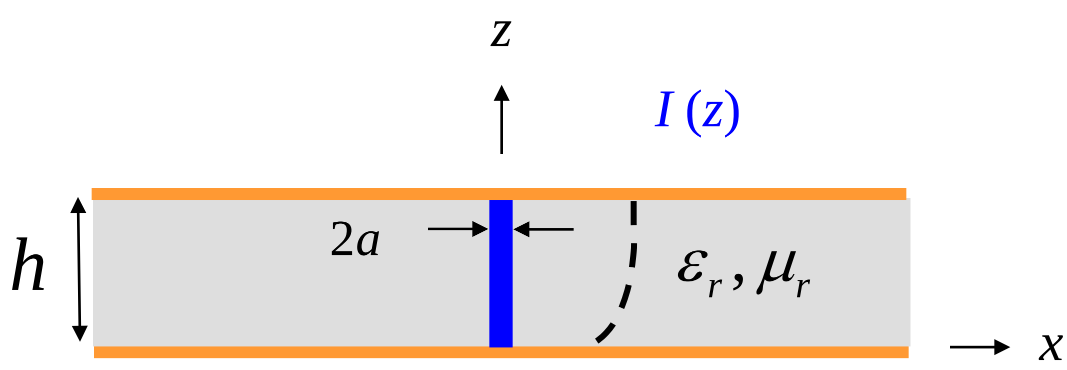
- Cosine-current model
- Gap-source model
- Frill model

Derivations are given in the Appendix.
Even more details may be found in the reference below.

Reference:

H. Xu, D. R. Jackson, and J. T. Williams, "Comparison of Models for the Probe Inductance for a Parallel Plate Waveguide and a Microstrip Patch," *IEEE Trans. Antennas and Propagation*, vol. 53, pp. 3229-3235, Oct. 2005.

Cosine Current Model



We assume a tube of current (as in Notes 4) but with a z variation.

$$I(z) = \cos\left[k(z-h)\right]$$

Note: The derivative of the current is zero at the top conductor (PEC).

$$Z_{in} = \frac{2P_c}{\left|I(0)\right|^2}$$

P_c = complex power radiated by probe current

$$P_c = \frac{1}{2}\int_{S_p} E_z J_{sz}^* \, dS$$

Cosine Current Model (cont.)

Final result:

$$Z_{in} = \frac{1}{8}(k_0 h)\eta_0 \left(\frac{1}{\varepsilon_r}\right) \sec^2\left(k_0 h \sqrt{\varepsilon_r}\right) \sum_{m=0}^{\infty} \left|I_m\right|^2 \bar{k}_{\rho m}^2 (1+\delta_{m0})\, H_0^{(2)}(k_{\rho m} a)\, J_0(k_{\rho m} a)$$

where

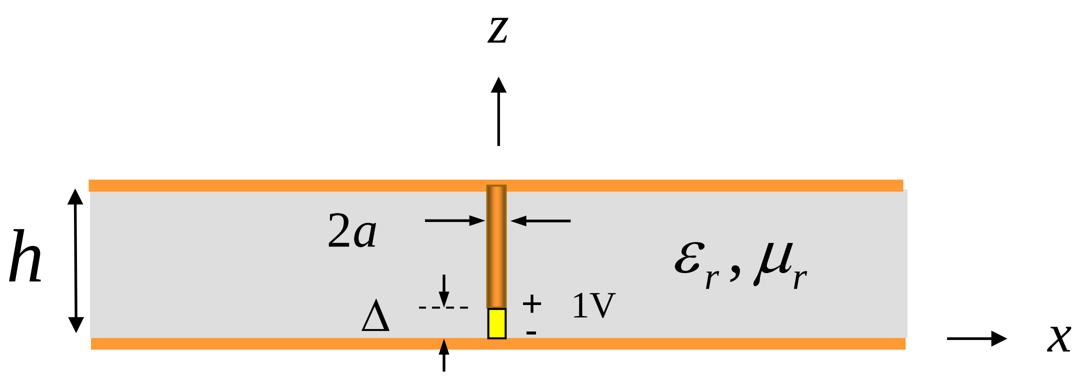
$$I_m = \left(\frac{2}{1+\delta_{mo}}\right)\left[\frac{(kh)}{(kh)^2-(m\pi)^2}\right]\sin(kh)$$

$$k_{\rho m} = \left(k_1^2 - \left(\frac{m\pi}{h}\right)^2\right)^{1/2}$$

Note: The wavenumber $k_{\rho m}$ is chosen to be a positive real number or a negative imaginary number.

$$\bar{k}_{\rho m} = k_{\rho m} / k_0 \qquad \delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

Gap Source Model



An ideal gap voltage source of height Δ is assumed at the bottom of the probe.

$$Z_{in} = \frac{1}{I(\Delta)}$$

Gap Source Model (cont.)

Final result:

$$Y_{in} = j4\pi\left(\frac{1}{\eta_0}\right)\left(\frac{a}{h}\right)\varepsilon_r\sum_{m=0}^{\infty}\left[\frac{H_0^{(2)\prime}(k_{\rho m}a)}{(1+\delta_{m0})\bar{k}_{\rho m}H_0^{(2)}(k_{\rho m}a)}\right]\operatorname{sinc}\left(\frac{2m\pi\Delta}{h}\right)$$

where

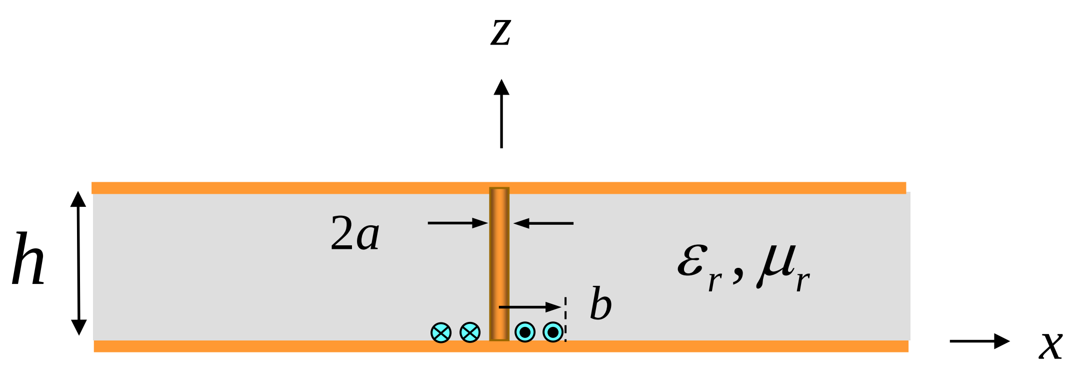
$$k_{\rho m} = \left(k_1^2 - \left(\frac{m\pi}{h}\right)^2\right)^{1/2}$$

Note: The wavenumber $k_{\rho m}$ is chosen to be a positive real number or a negative imaginary number.

$$\bar{k}_{\rho m} = k_{\rho m} / k_0$$

$$\delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

Frill Model



A magnetic frill of radius b is assumed on the mouth of the coax.

$$\underline{M}_s = -\underline{\hat{z}} \times \underline{E} = -\underline{\hat{z}} \times \left(\underline{\hat{\rho}}\, E_\rho \right) \quad \Rightarrow \quad M_{s\phi} = -E_\rho$$

Choose:

$$E_\rho = \frac{1}{\rho} \left[\frac{1}{\ln(b/a)} \right]$$

$$Z_{in} = \frac{1}{I(0)}$$

(TEM mode of coax, assuming 1 V)

Frill Model (cont.)

Final result:

$$Y_{in} = j\left(\frac{1}{\eta_0}\right)\left(\frac{1}{k_0 h}\right)\left(\frac{1}{\ln(b/a)}\right)4\pi\varepsilon_r \sum_{m=0}^{\infty} \frac{H_0^{(2)}(k_{\rho m}b) - H_0^{(2)}(k_{\rho m}a)}{(\overline{k}_{\rho m}^{\,2})(1+\delta_{m0})H_0^{(2)}(k_{\rho m}a)}$$

where

$$k_{\rho m} = \left(k_1^2 - \left(\frac{m\pi}{h}\right)^2\right)^{1/2}$$

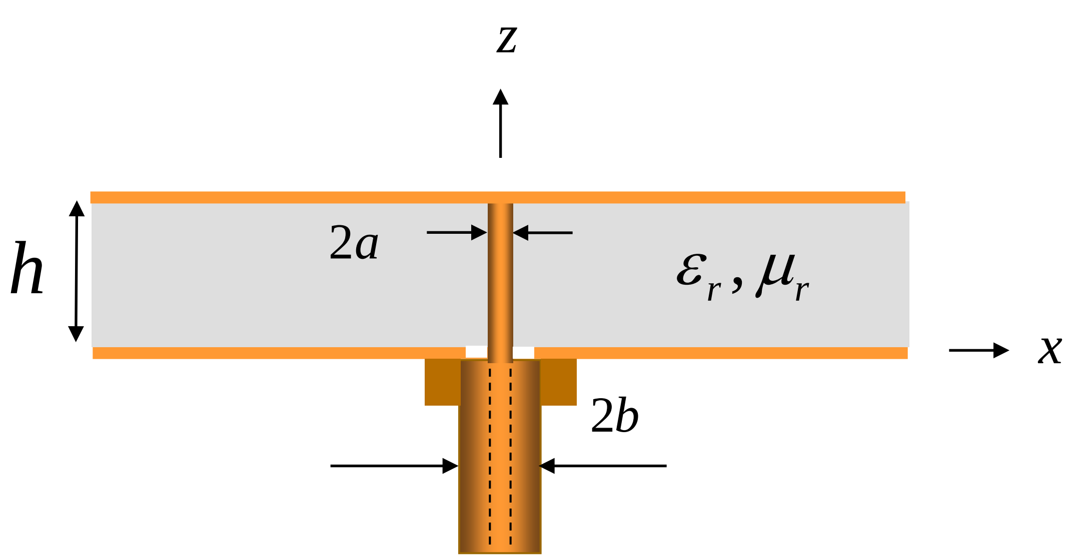
Note: The wavenumber $k_{\rho m}$ is chosen to be a positive real number or a negative imaginary number.

$$\overline{k}_{\rho m} = k_{\rho m} / k_0$$

$$\delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

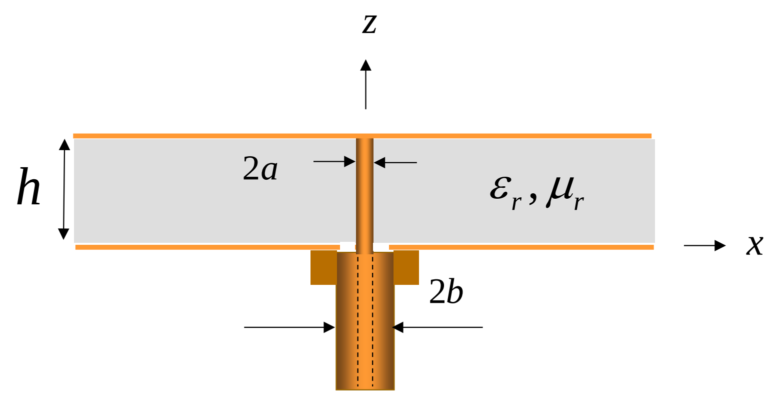
Comparison of Models

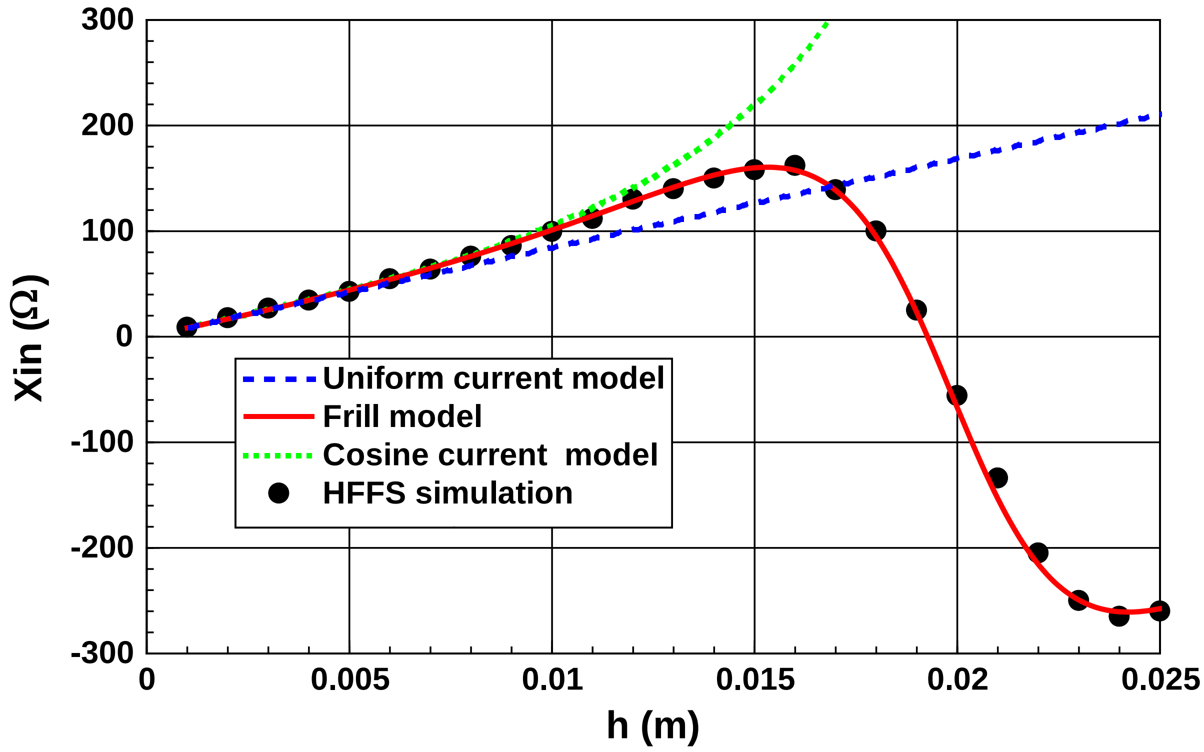
Next, we show results that compare the various models, especially as the substrate thickness increases.



Comparison of Models

Models are compared for changing substrate thickness.



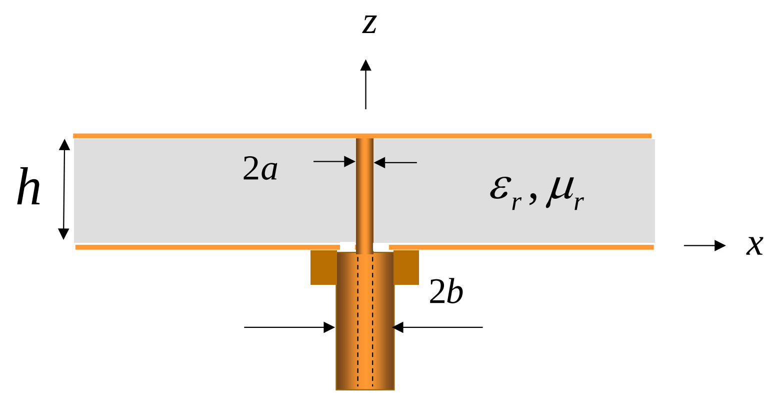


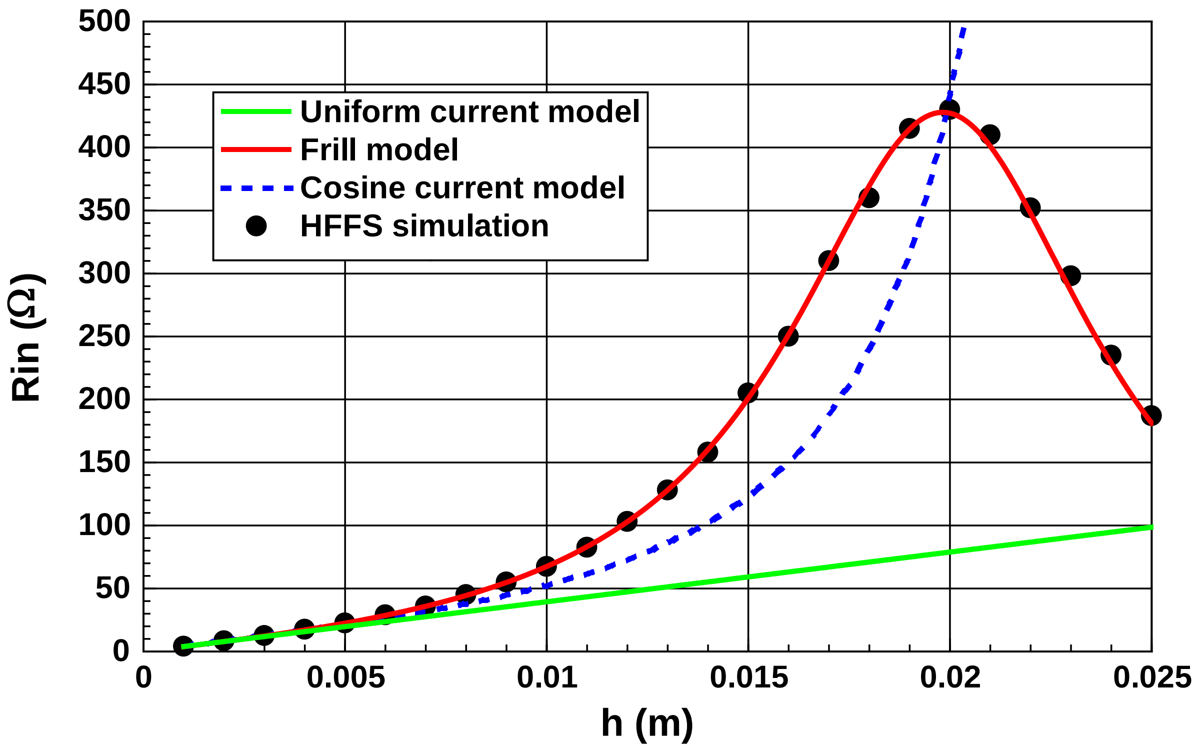
$\varepsilon_r = 2.2$
$a = 0.635$ mm
$f = 2$ GHz
$Z_0 = 50\ \Omega$
($b = 2.19$ mm)

($\lambda_0 = 15$ cm)

Comparison of Models (cont.)

Models are compared for varying substrate thickness.





$\varepsilon_r = 2.2$
$a = 0.635$ mm
$f = 2$ GHz
$Z_0 = 50\ \Omega$
($b = 2.19$ mm)

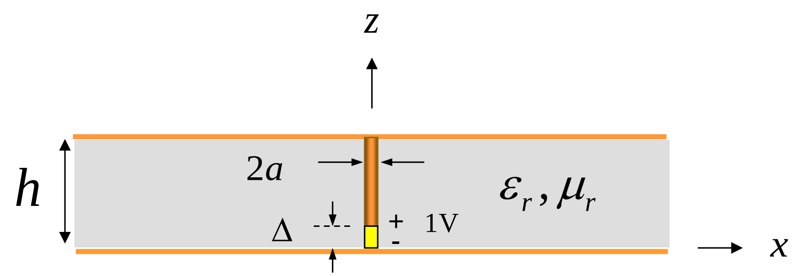
($\lambda_0 = 15$ cm)

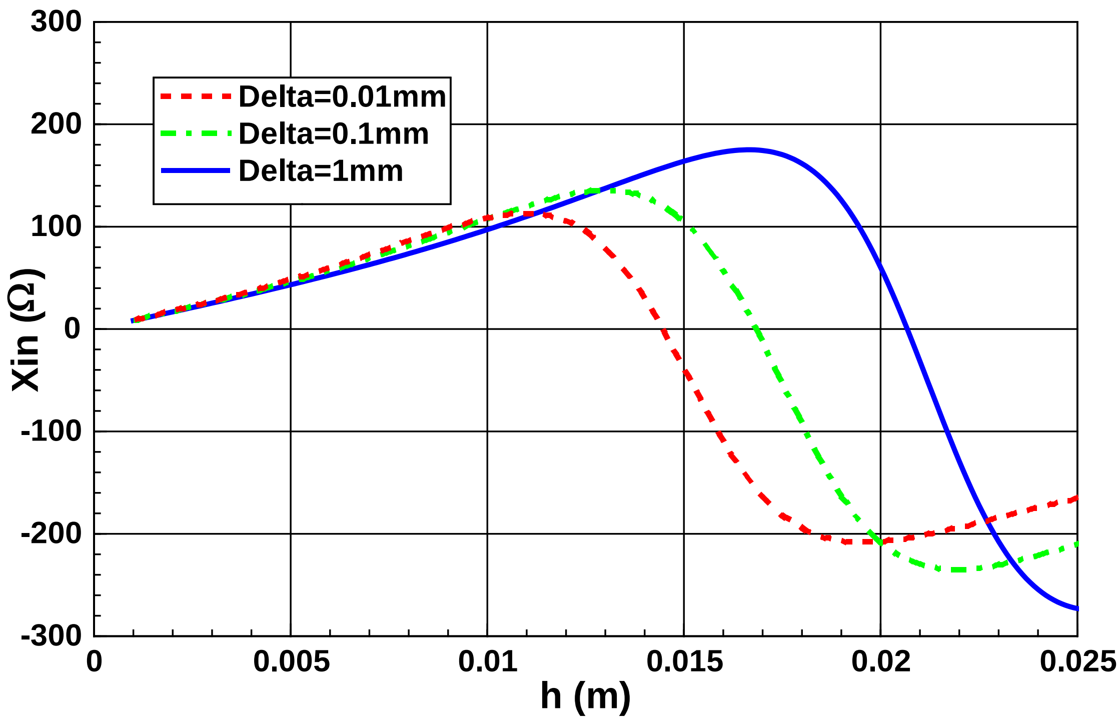
Note:

$$\lambda_d / 4 = 0.025 \text{ [m]}$$

Comparison of Models (cont.)

For the gap-source model, the results depend on Δ.



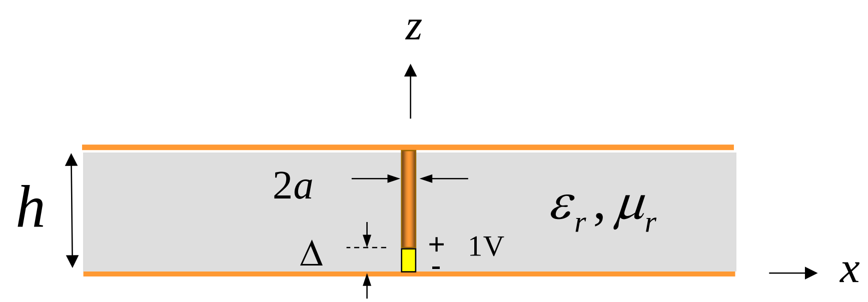


$\varepsilon_r = 2.2$
$a = 0.635$ mm
$f = 2$ GHz
$Z_0 = 50\ \Omega$
($b = 2.19$ mm)

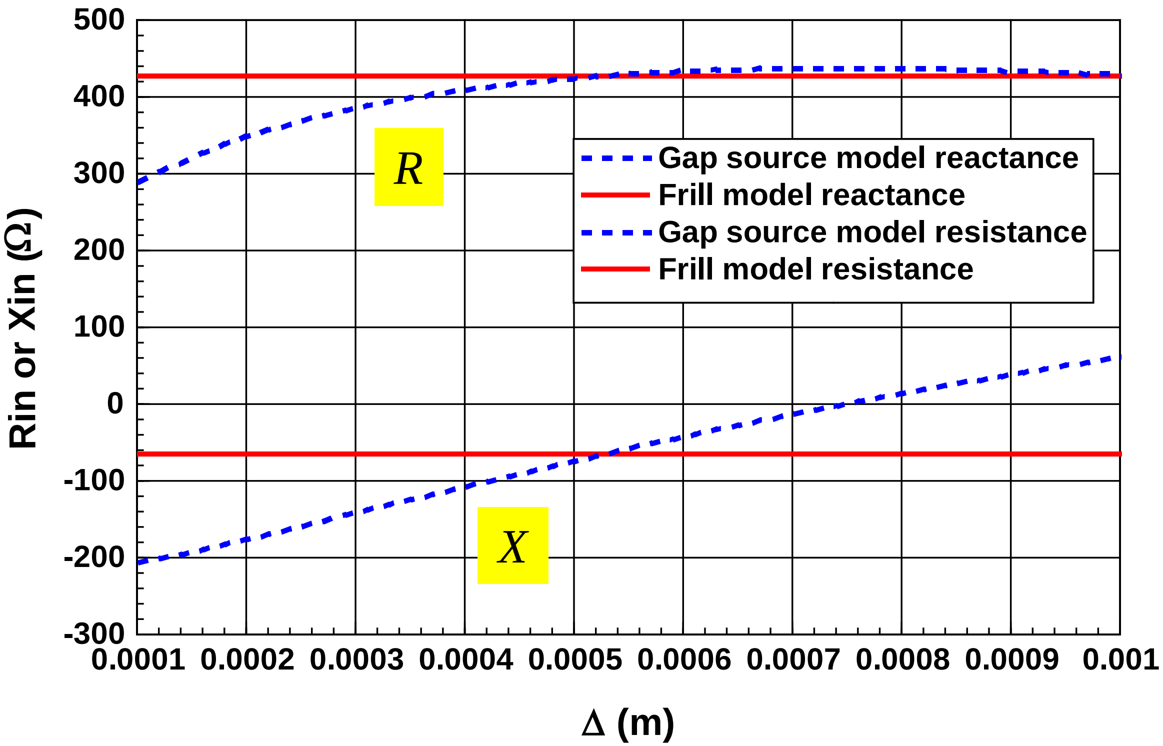
($\lambda_0 = 15$ cm)

Comparison of Models (cont.)

The gap-source model is compared with the frill model, for varying Δ, for a fixed *h*.



h = 20 mm



$\varepsilon_r = 2.2$
$a = 0.635$ mm
$f = 2$ GHz
$Z_0 = 50\ \Omega$
($b = 2.19$ mm)

($\lambda_0 = 15$ cm)

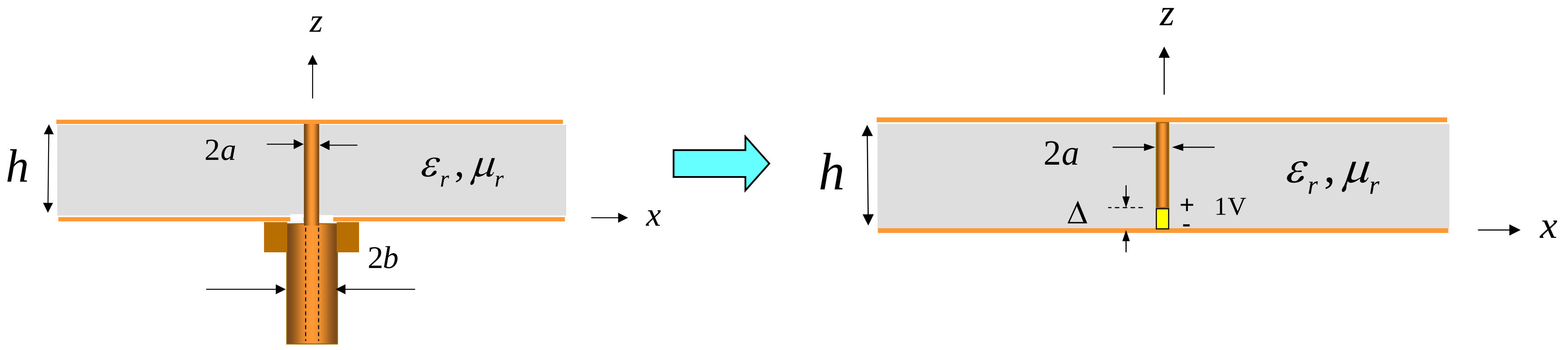
$$\frac{b-a}{3} = 0.52\,\text{mm}$$

Comparison of Models (cont.)

These results suggest the “1/3” rule: The best Δ is chosen as

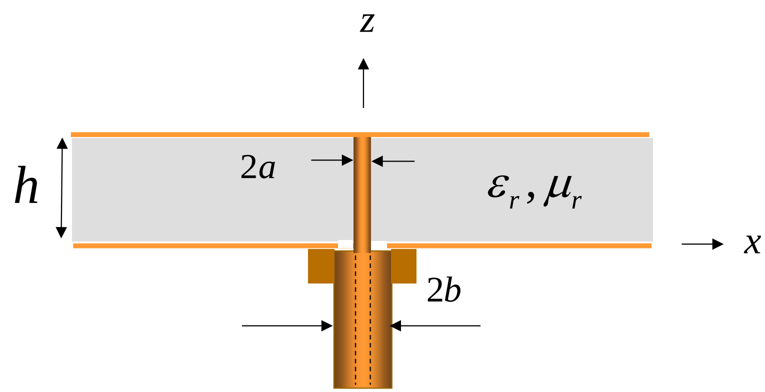
$$\Delta = \frac{b-a}{3}$$

This rule applies for a coax feed that has a 50 Ω impedance.

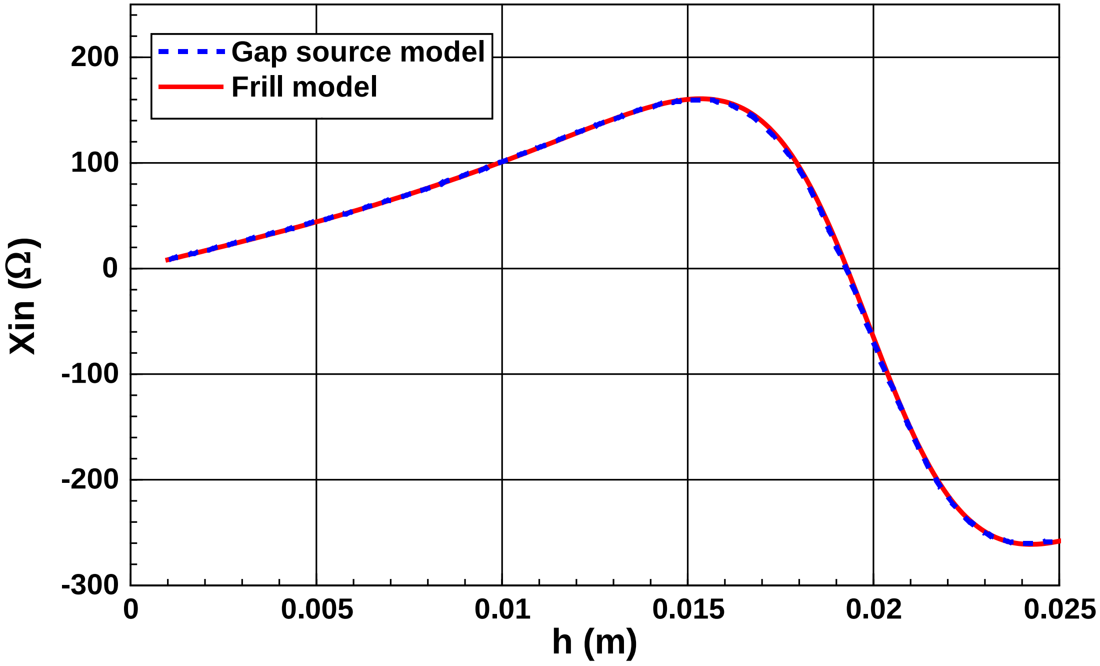


Comparison of Models (cont.)

The gap-source model is compared with the frill model, using the optimum gap height (1/3 rule).



Reactance

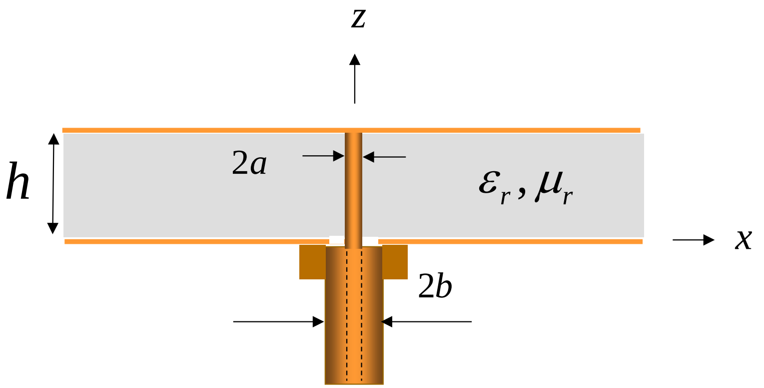


$\varepsilon_r = 2.2$
$a = 0.635$ mm
$f = 2$ GHz
$Z_0 = 50\ \Omega$
($b = 2.19$ mm)

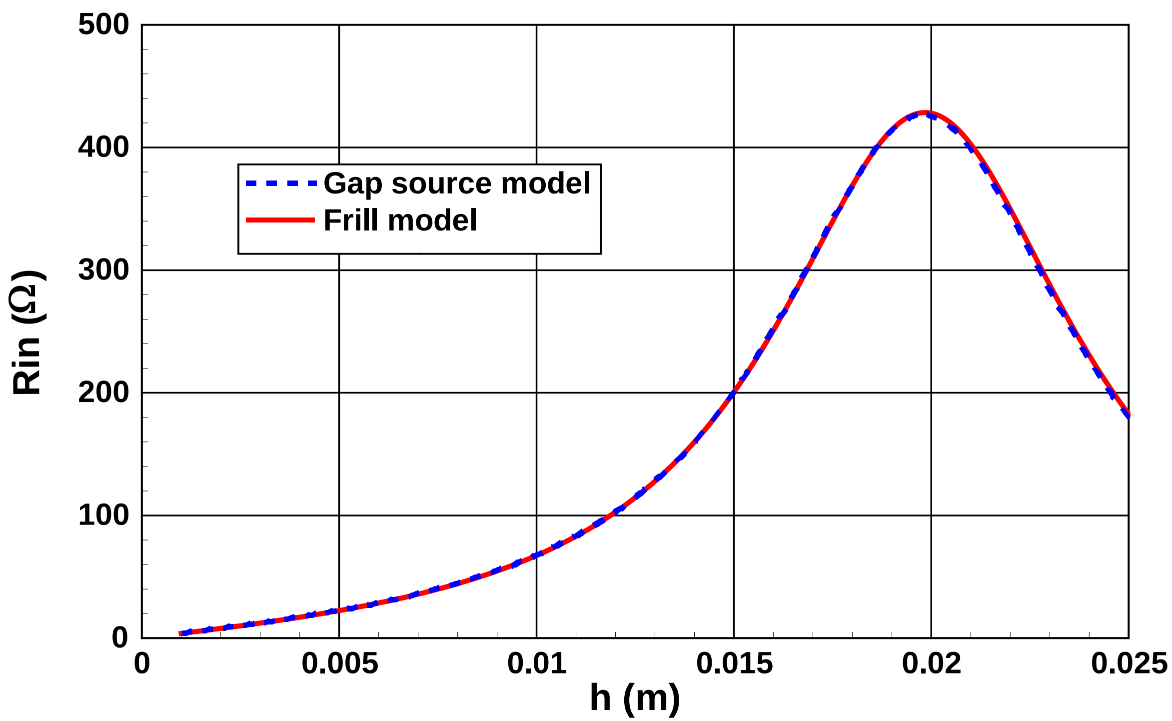
($\lambda_0 = 15$ cm)

Comparison of Models (cont.)

The gap-source model is compared with the frill model, using the optimum gap height (1/3 rule).



Resistance

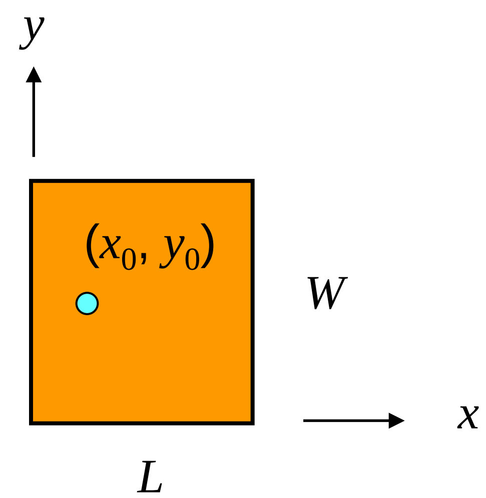


$\varepsilon_r = 2.2$
$a = 0.635$ mm
$f = 2$ GHz
$Z_0 = 50\ \Omega$
($b = 2.19$ mm)

($\lambda_0 = 15$ cm)

Probe in Patch

A probe in a patch does not see an infinite parallel-plate waveguide.



Exact calculation of probe reactance:

$$Z_{in} = jX_p + Z_{in}^{cavity}$$

$$\Rightarrow \quad X_p = \operatorname{Im}\left(Z_{in}\right)_{f_0}$$

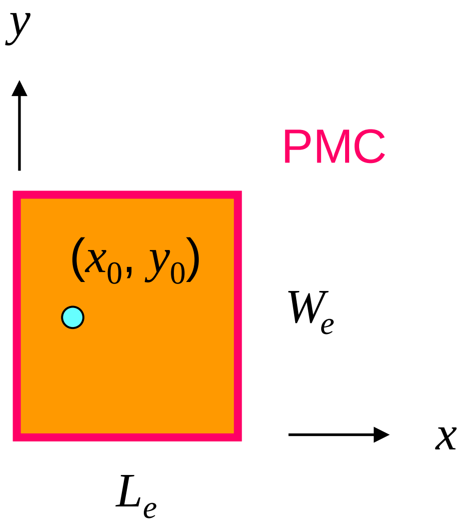
Z_{in} may be calculated by HFSS or any other software, or it may be measured.

f_0 = frequency at which R_{in} is maximum $\left(X_{in}^{cavity} = 0\right)$

Probe in Patch (cont.)

Cavity Model

Using the cavity model, we can derive an expression for the probe reactance (derivation given later)



This formula assumes that there is no z variation of the probe current or cavity fields (thin-substrate approximation), but it does accurately account for the actual patch dimensions.

Probe in Patch (cont.)

Final result:

$$X_p = -\omega \mu_0 \mu_r h \sum_{(m,n)\neq(1,0)} \frac{\cos^2\left(\dfrac{m\pi x_0}{L_e}\right)\cos^2\left(\dfrac{n\pi y_0}{W_e}\right)}{\left(\dfrac{W_e L_e}{4}\right)\left(1+\delta_{m0}\right)\left(1+\delta_{n0}\right)} \frac{\operatorname{sinc}^2\left(\dfrac{n\pi w_p}{2}\right)}{k^2 - \left(\dfrac{m\pi}{L_e}\right)^2 - \left(\dfrac{n\pi}{W_e}\right)^2}$$

$$w_p = e^{3/2} a$$

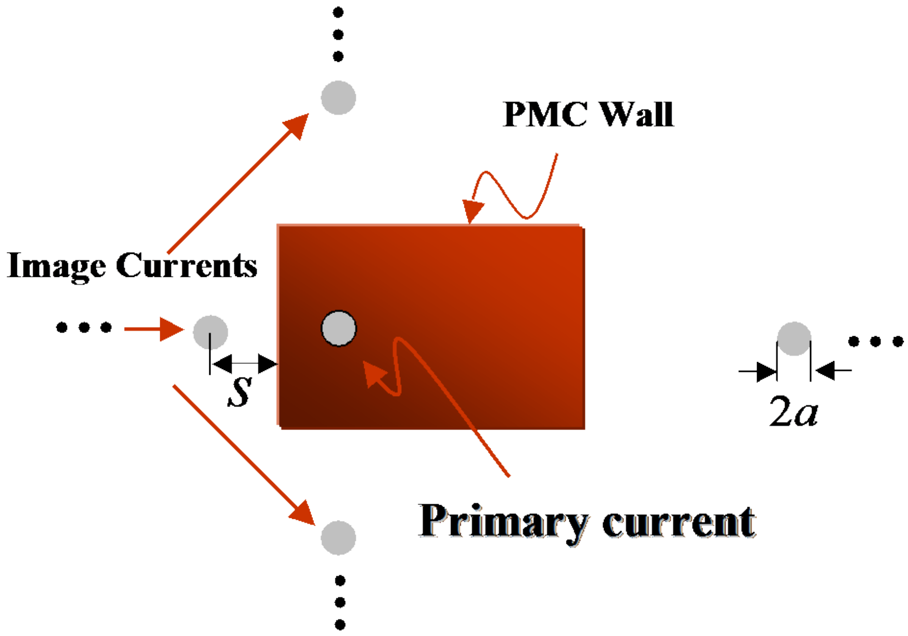
a = probe radius

(x_0, y_0) = probe location

Probe in Patch (cont.)

Image Theory

Image theory can be used to improve the simple parallel-plate waveguide model when the probe gets close to the patch edge.



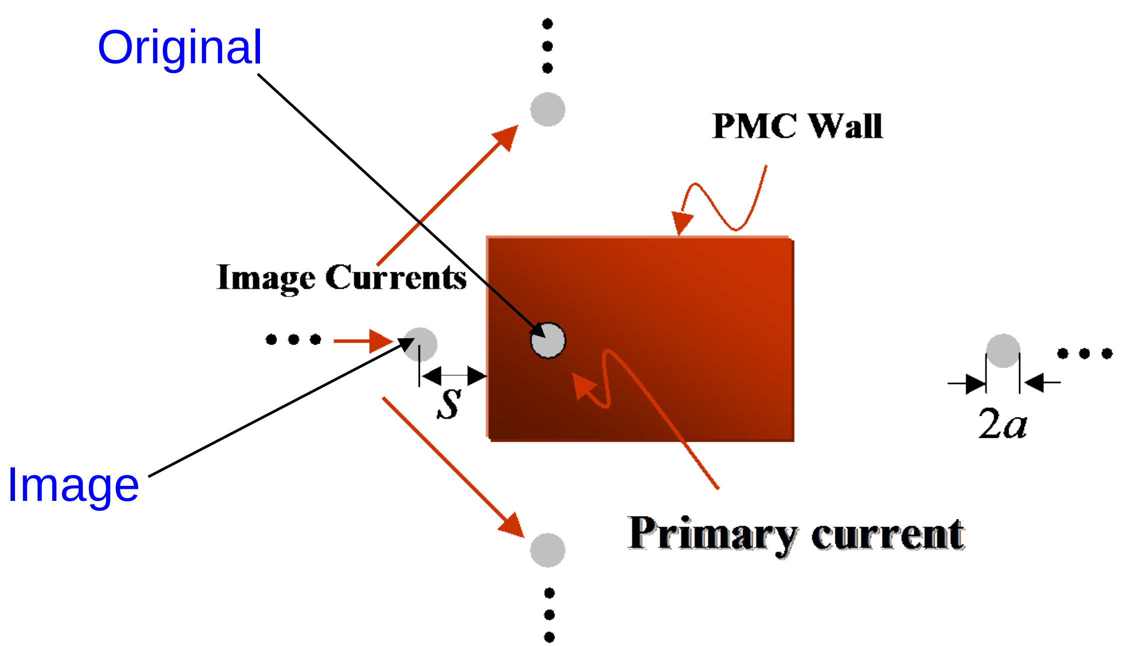
Using image theory, we have an infinite set of “image probes.”

Probe in Patch (cont.)

A simple approximate formula is obtained by using two terms: the original probe current in a parallel-plate waveguide and one image. This should be an improvement when the probe is close to an edge.

$$X_{in}^{\text{two}} = X_{in}^{\text{probe}} + X_{in}^{\text{image}}$$

$$X_{in}^{\text{two}} = -\tfrac{1}{4}\eta\, kh\, Y_0\left(ka\right) J_0\left(ka\right) - \tfrac{1}{4}\eta\, kh\, Y_0\left(2ks\right) J_0\left(ka\right)$$



$$k = k_0\sqrt{\varepsilon_r}$$

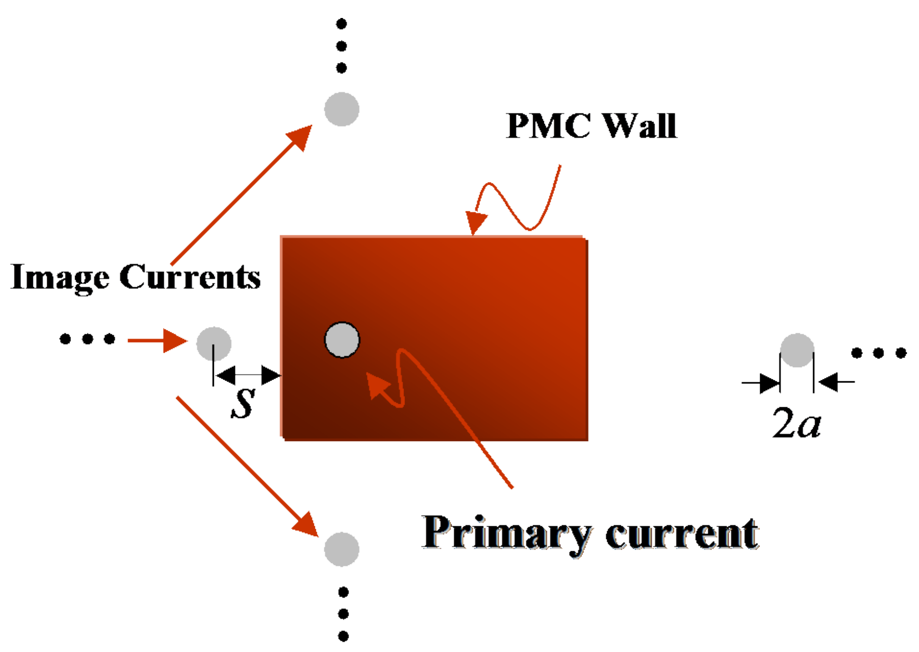
$$\eta = \eta_0 / \sqrt{\varepsilon_r}$$

Probe in Patch (cont.)

As shown on the next plot, the best overall approximation in obtained by using the following formula:

$$X_{in} = \max\left(X_{in}^{\text{probe}}, X_{in}^{\text{two}}\right)$$

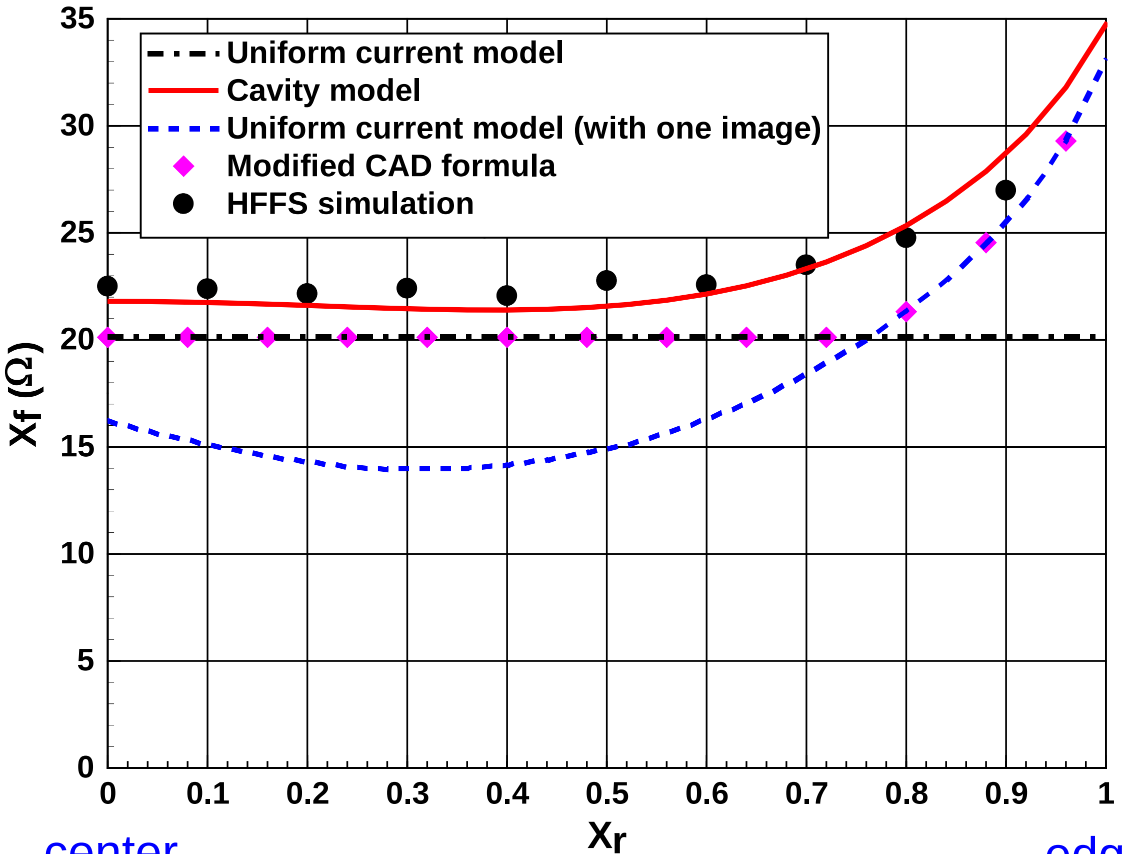
"modified CAD formula"



Probe in Patch (cont.)

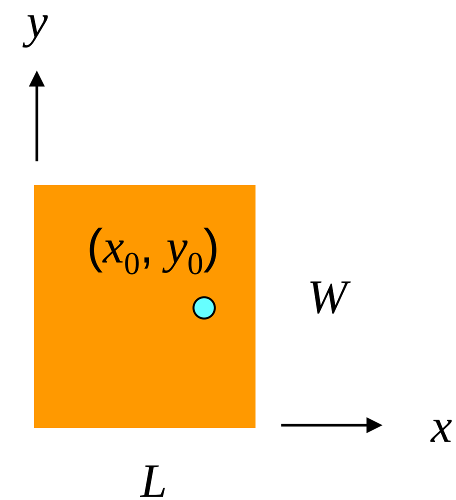
Results show that the simple formula ("modified CAD formula") works fairly well.

$$X_{in} = \max\left(X_{in}^{\text{probe}}, X_{in}^{\text{two}} \right)$$



center

edge



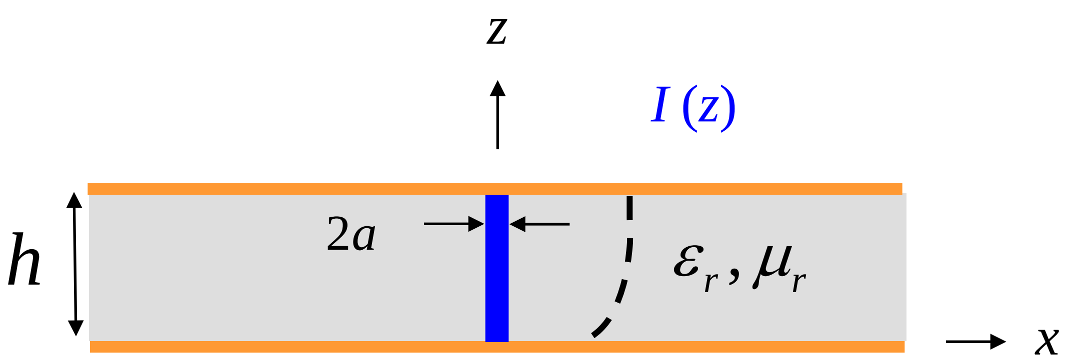
$$x_r = \frac{x_0 - L/2}{L/2}$$

Appendix

Next, we investigate each of the improved probe models in more detail:

- Cosine-current model
- Gap-source model
- Frill model

Cosine Current Model

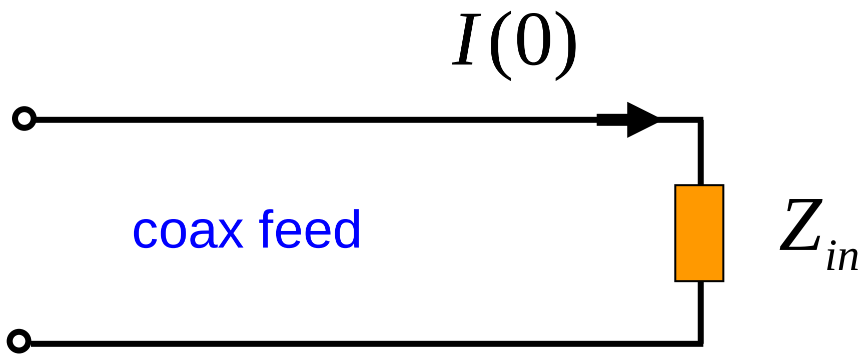


Assume that $$I(z) = \cos k(z - h)$$

Note: $$I(0) = \cos(kh)$$

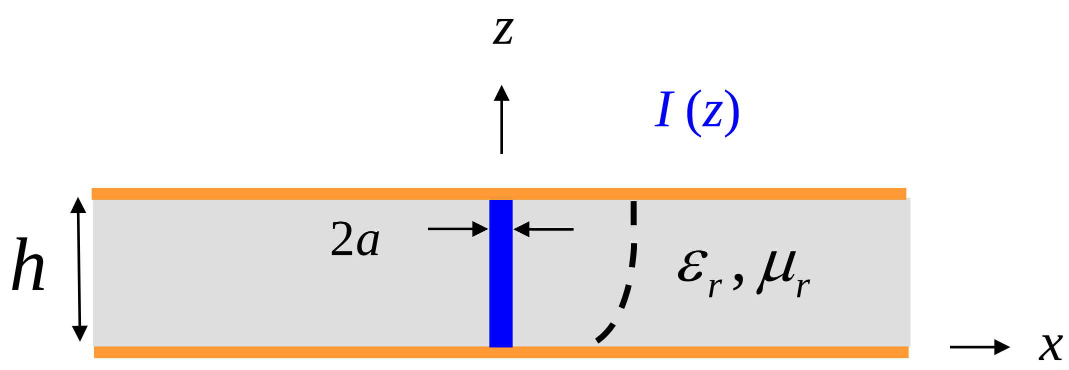
Cosine Current Model (cont.)

Circuit Model:



$$P_c = \frac{1}{2} Z_{in} \left| I(0) \right|^2 \quad \Rightarrow \quad Z_{in} = \frac{2P_c}{\left| I(0) \right|^2}$$

Cosine Current Model (cont.)



$$Z_{in} = \frac{2P_c}{\left|I(0)\right|^2}$$

Represent the probe current as:

$$I(z) = \sum_{m=0}^{\infty} I_m \cos\left(\frac{m\pi z}{h}\right)$$

This will allow us to find the fields and hence the power radiated by the probe current.

Cosine Current Model (cont.)

Using Fourier-series theory:

$$\int_0^h I(z)\cos\left(\frac{m'\pi z}{h}\right)dz = \sum_{m=0}^{\infty} I_m \int_0^h \cos\left(\frac{m\pi z}{h}\right)\cos\left(\frac{m'\pi z}{h}\right)dz$$

The integral is zero unless $m = m'$.

Hence

$$\int_0^h I(z)\cos\left(\frac{m\pi z}{h}\right)dz = I_m\left[\frac{h}{2}\left(1+\delta_{m0}\right)\right]$$

$$\Rightarrow \quad I_m = \frac{2}{h\left(1+\delta_{m0}\right)}\int_0^h I(z)\cos\left(\frac{m\pi z}{h}\right)dz$$

Cosine Current Model (cont.)

or

$$I_m = \frac{2}{h\left(1+\delta_{mo}\right)} \int_0^h \cos k(z-h) \cos\left(\frac{m\pi z}{h}\right) dz$$

Result:

$$I_m = \left(\frac{2}{1+\delta_{mo}}\right)\left[\frac{(kh)}{(kh)^2-(m\pi)^2}\right]\sin(kh)$$

(derivation omitted)

Cosine Current Model (cont.)

Note: We have both E_z and E_ρ

To see this:

$$\nabla \cdot \underline{E} = 0 \quad \text{(Time-Harmonic Fields)}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} = 0$$

so $E_\rho \neq 0$

Cosine Current Model (cont.)

For E_z, we represent the field as follows:

$\rho < a$

$$E_z^- = \sum_{m=0}^{\infty} A_m^- \cos\left(\frac{m\pi z}{h}\right) J_0\left(k_{\rho m}\rho\right)$$

$\rho > a$

$$E_z^+ = \sum_{m=0}^{\infty} A_m^+ \cos\left(\frac{m\pi z}{h}\right) H_0^{(2)}\left(k_{\rho m}\rho\right)$$

where

$$k_{\rho m} = \left(k^2 - \left(\frac{m\pi}{h}\right)^2\right)^{1/2}$$

$$= \left(k^2 - k_{zm}^2\right)^{1/2}$$

Cosine Current Model (cont.)

At $\rho = a$

$$E_z^+ = E_z^- \quad \text{(BC 1)}$$

so

$$A_m^+ H_0^{(2)}\left(k_{\rho m} a\right) = A_m^- J_0\left(k_{\rho m} a\right)$$

Cosine Current Model (cont.)

Also we have

$$H_{\phi 2} - H_{\phi 1} = J_{sz} \qquad \text{(BC 2)}$$

where

$$H_{\phi} = \frac{-1}{j\omega\mu}\left(\frac{\partial E_{\rho}}{\partial z} - \frac{\partial E_z}{\partial \rho}\right)$$

To solve for E_{ρ}, use

$$\nabla \times \underline{H} = j\omega\varepsilon \underline{E}$$

Cosine Current Model (cont.)

so $$j\omega\varepsilon E_\rho = \cancel{\frac{1}{\rho}\frac{\partial H_z}{\partial\phi}} - \frac{\partial H_\phi}{\partial z}$$

$$E_\rho = -\frac{1}{j\omega\varepsilon}\frac{\partial H_\phi}{\partial z}$$

Hence we have $$H_\phi = -\frac{1}{j\omega\mu}\left[-\frac{1}{j\omega\varepsilon}\frac{\partial^2 H_\phi}{\partial z^2} - \frac{\partial E_z}{\partial\rho}\right]$$

For the m^{th} Fourier term:

$$H_\phi^{(m)} = -\frac{1}{j\omega\mu}\left[-\frac{1}{j\omega\varepsilon}\left(-k_{zm}^2\right)H_\phi^{(m)} - \frac{\partial E_z^{(m)}}{\partial\rho}\right]$$

Cosine Current Model (cont.)

so that

$$k^2 H_\phi^{(m)} - k_{zm}^2 H_\phi^{(m)} = -j\omega\varepsilon \frac{\partial E_z^{(m)}}{\partial \rho}$$

where $$k^2 - k_{zm}^2 = k_{\rho m}^2$$

Hence

$$H_\phi^{(m)} = -\frac{j\omega\varepsilon}{k_{\rho m}^2} \frac{\partial E_z^{(m)}}{\partial \rho}$$

Cosine Current Model (cont.)

$$H_{\phi 2} - H_{\phi 1} = J_{sz} = \frac{I}{2\pi a}$$

For the m^{th} Fourier term:

$$H_{\phi 2}^{(m)} - H_{\phi 1}^{(m)} = J_{sz}^{(m)} = \frac{I_m}{2\pi a}$$

where

$$H_{\phi}^{(m)} = -\frac{j\omega\varepsilon}{k_{\rho m}^2} \frac{\partial E_z^{(m)}}{\partial \rho}$$

Cosine Current Model (cont.)

Hence

$$\left(\frac{-j\omega\varepsilon}{k_{\rho m}^{2}}\right)(k_{\rho m})\left[A_m^{+}H_0^{(2)\prime}\left(k_{\rho m}a\right)-A_m^{-}J_0{}'\left(k_{\rho m}a\right)\right]=\frac{I_m}{2\pi a}$$

Using $A_m^{+}H_0^{(2)}\left(k_{\rho m}a\right)=A_m^{-}J_0\left(k_{\rho m}a\right)$ (BC 1)

we have

$$A_m^{+}H_0^{(2)\prime}(k_{\rho m}a)-A_m^{+}\left(\frac{H_0^{(2)}(k_{\rho m}a)}{J_0(k_{\rho m}a)}\right)J_0{}'(k_{pm}a)=\left(\frac{I_m}{2\pi a}\right)\left(\frac{k_{\rho m}}{-j\omega\varepsilon}\right)$$

Cosine Current Model (cont.)

or

$$A_m^+\left[J_0(k_{\rho m}a)H_0^{(2)\prime}(k_{\rho m}a)-H_0^{(2)}(k_{\rho m}a)J_0{}'(k_{\rho m}a)\right]=\left(\frac{I_m}{2\pi a}\right)\left(\frac{k_{\rho m}}{-j\omega\varepsilon}\right)J_0(k_{\rho m}a)$$

or

$$A_m^+\left[-j\left(\frac{2}{\pi k_{\rho m}a}\right)\right]=\left(\frac{I_m}{2\pi a}\right)\left(\frac{k_{\rho m}}{-j\omega\varepsilon}\right)J_0(k_{\rho m}a)$$

(using the Wronskian identity)

Hence

$$A_m^+=I_m\left[-\frac{1}{4}\left(\frac{k_{\rho m}^2}{\omega\varepsilon}\right)J_0\left(k_{\rho m}a\right)\right]$$

Cosine Current Model (cont.)

We now find the complex power radiated by the probe:

$$P_c = \frac{-1}{2} \oint_S \underline{E} \cdot \underline{J}_s^* \, dS$$

$$= \frac{-1}{2} \int_0^{2\pi} \int_0^h E_z(a)\, J_{sz}^* \, a \, dz \, d\phi$$

$$= -\pi a \int_0^h E_z(a)\, J_{sz}^* \, dz$$

$$= -\frac{\pi a}{2\pi a} \int_0^h E_z(a)\, I^*(z)\, dz$$

$$= -\frac{1}{2} \int_0^h \left(\sum_{m=0}^{\infty} A_m^+ H_0^{(2)}(k_{\rho m} a) \cos\left(\frac{m\pi z}{h} \right) \right) \cdot \left(\sum_{m'=0}^{\infty} I_{m'}^* \cos\left(\frac{m'\pi z}{h} \right) \right) dz$$

Cosine Current Model (cont.)

Integrating in z and using orthogonality, we have:

A_m^+ coefficient

$$P_c = -\frac{1}{2}\sum_{m=0}^{\infty} A_m^+ I_m^* H_0^{(2)}(k_{\rho m}a)\left(\frac{h}{2}\right)\left(1+\delta_{m0}\right)$$

$$= -\left(\frac{h}{4}\right)\sum_{m=0}^{\infty}\left(1+\delta_{m0}\right)H_0^{(2)}(k_{\rho m}a)\left[I_m\left(-\frac{1}{4}\right)\left(\frac{k_{\rho m}^2}{\omega\varepsilon}\right)J_0(k_{\rho m}a)\right]I_m^*$$

Hence, we have:

$$P_c = +\frac{h}{16}\left(\frac{1}{\omega\varepsilon}\right)\sum_{m=0}^{\infty}\left|I_m\right|^2 k_{pm}^2 (1+\delta_{m0})\, H_0^{(2)}(k_{\rho m}a)\, J_0(k_{\rho m}a)$$

Cosine Current Model (cont.)

$$Z_{in} = \frac{2P_c}{\cos^2(kh)}$$

Therefore,

$$Z_{in} = \frac{h}{8}\left(\frac{1}{\omega\varepsilon}\right)\sec^2(kh)\sum_{m=0}^{\infty}\left|I_m\right|^2 k_{\rho m}^2\left(1+\delta_{m0}\right)H_0^{(2)}(k_{\rho m}a)\,J_0(k_{\rho m}a)$$

Define: $$\bar{k}_{\rho m} = \frac{k_{\rho m}}{k_0}$$

$$= \sqrt{\varepsilon_r\mu_r - \left(\frac{m\pi}{k_0 h}\right)^2}$$

Cosine Current Model (cont.)

Also, use $$\frac{k_0}{\omega\varepsilon} = \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\varepsilon} = \frac{\eta_0}{\varepsilon_r}$$

We then have

$$Z_{in} = \frac{1}{8}(k_0 h)\eta_0\left(\frac{1}{\varepsilon_r}\right)\sec^2\left(k_0 h\sqrt{\varepsilon_r}\right)\sum_{m=0}^{\infty}\left|I_m\right|^2 \bar{k}_{\rho m}^{-2}(1+\delta_{m0})\,H_0^{(2)}(k_{\rho m}a)\,J_0(k_{\rho m}a)$$

The probe reactance is: $$X_p = \mathrm{Im}(Z_{in})$$

Cosine Current Model (cont.)

Thin substrate approximation

$$Z_{in} = \frac{1}{8}(k_0 h)\eta_0 \left(\frac{1}{\varepsilon_r}\right)\sec^2\left(k_0 h\sqrt{\varepsilon_r}\right)\sum_{m=0}^{\infty}\left|I_m\right|^2 \bar{k}_{\rho m}^{-2}(1+\delta_{m0})\, H_0^{(2)}(k_{\rho m}a)\, J_0(k_{\rho m}a)$$

$k_0 h << 1$: Keep only the m = 0 term

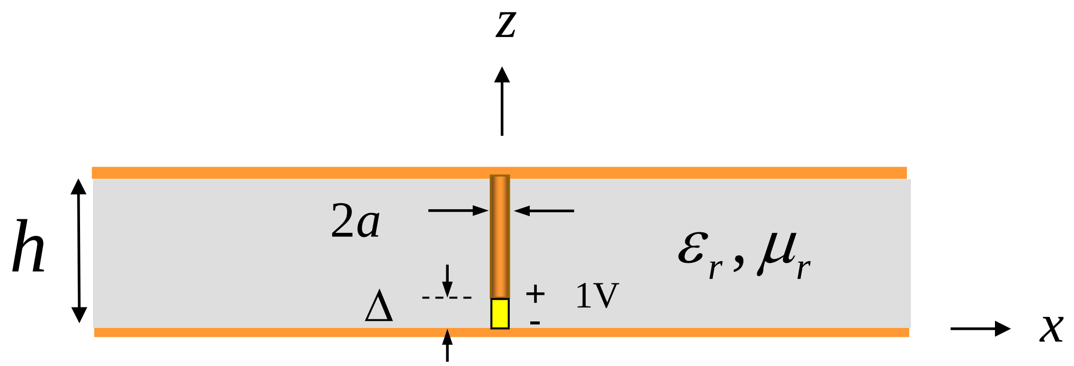
$$I_m = \left(\frac{2}{1+\delta_{mo}}\right)\left[\frac{(kh)}{(kh)^2-(m\pi)^2}\right]\sin(kh)$$

The result is

$$Z_{in} \approx \frac{1}{4}\eta_0 (k_0 h)\mu_r J_0(ka) H_0^{(2)}(ka)$$

(same as previous result using uniform model)

Gap Model

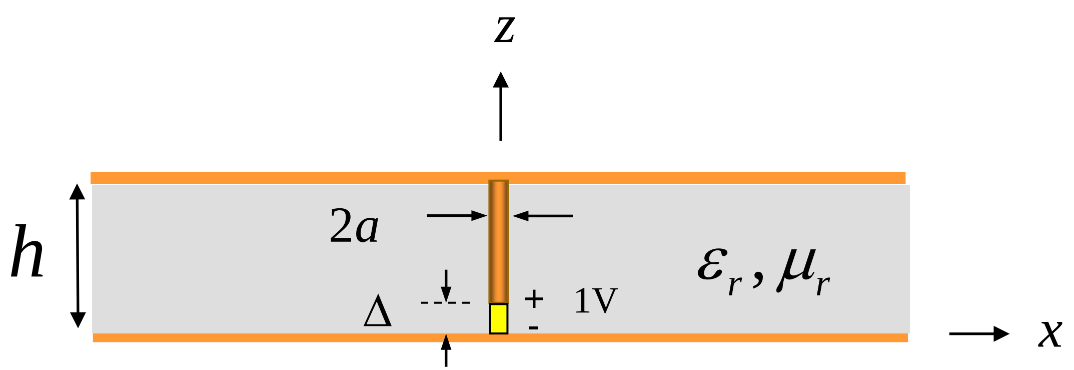


$$E_z(z,\rho) = \sum_{m=0}^{\infty} B_m H_0^{(2)}(k_{\rho m}\rho)\cos\left(\frac{m\pi z}{h}\right)$$

$$E_z(z,a) = \begin{cases} -1/\Delta, & 0 < z < \Delta \\ 0, & \text{otherwise}. \end{cases}$$

Note: It is not clear how best to choose Δ, but this will be re-visited later.

Gap Model (cont.)



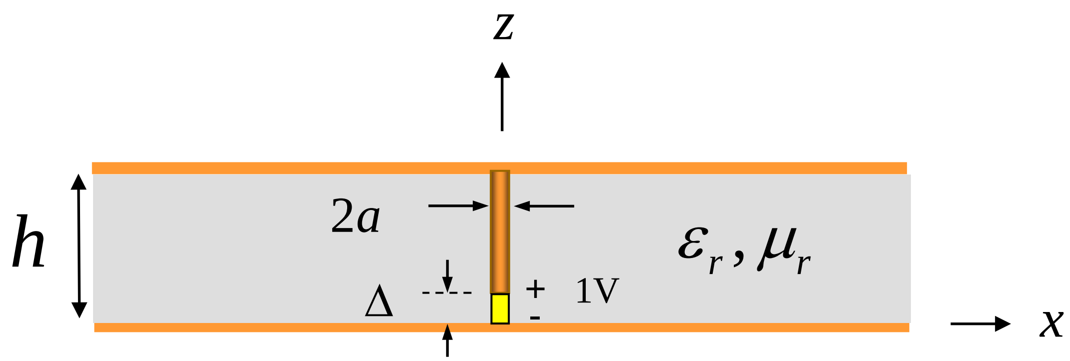
At $\rho = a$:

$$E_z(z,a) = \sum_{m=0}^{\infty} B_m H_0^{(2)}(k_{\rho m} a)\cos\left(\frac{m\pi z}{h}\right) = \begin{cases} -1/\Delta, & 0 < z < \Delta \\ 0, & \text{otherwise}. \end{cases}$$

From Fourier series analysis (details omitted):

$$B_m = \frac{-2}{h(1+\delta_{m0}) H_0^{(2)}(k_{\rho m} a)} \operatorname{sinc}\left(\frac{m\pi\Delta}{h}\right)$$

Gap Model (cont.)



$$Y_{in} = 2\pi a\, J_{sz}(\Delta) \quad \text{where} \quad J_{sz}(z) = H_\phi(z)$$

The magnetic field is found from E_z, with the help of the magnetic vector potential A_z (the field is TM_z):

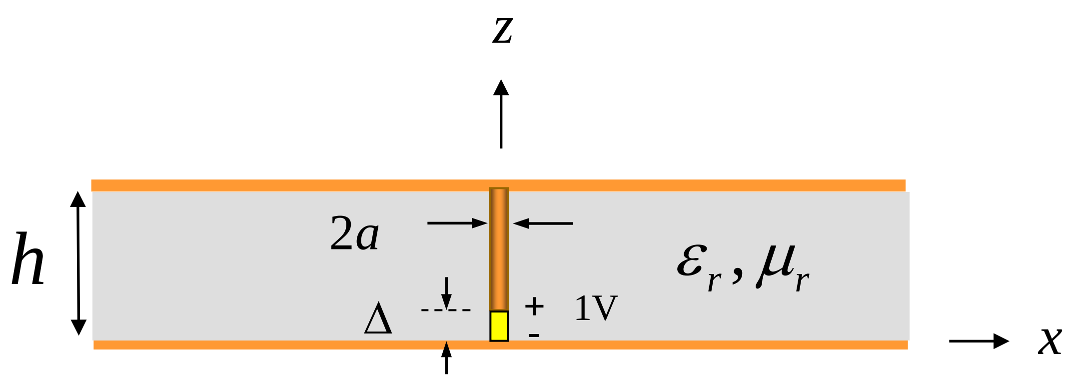
$$H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho}$$

Use: $$A_z(z,\rho) = \sum_{m=0}^{\infty} A_m\, H_0^{(2)}(k_{\rho m}\rho)\cos\left(\frac{m\pi z}{h}\right)$$

where $$E_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)A_z$$

Setting $\rho = a$ allows us to solve for the coefficients A_m.

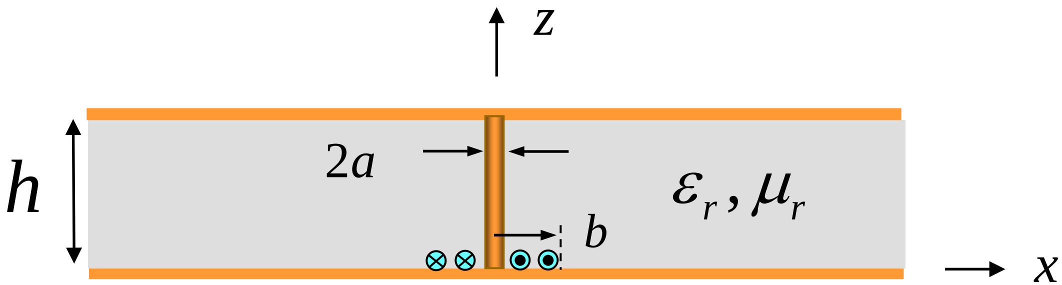
Gap Model (cont.)



Final result:

$$Y_{in} = j4\pi\left(\frac{1}{\eta}\right)\left(\frac{a}{h}\right)k\sum_{m=0}^{\infty}\left[\frac{H_0^{(2)\prime}(k_{\rho m}a)}{(1+\delta_{m0})k_{\rho m}H_0^{(2)}(k_{\rho m}a)}\right]\operatorname{sinc}\left(\frac{2m\pi\Delta}{h}\right)$$

Frill Model



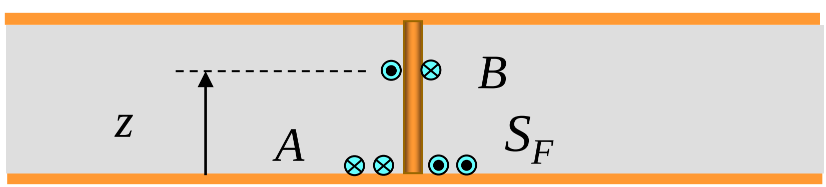
1V frill

$$M_{s\phi} = -\frac{1}{\rho}\left[\frac{1}{\ln(b/a)}\right]$$

$$Z_{in} = \frac{1}{I(0)}$$

To find the current $I(z)$, use reciprocity.

Introduce a ring of magnetic current $K = 1$ in the ϕ direction at z (the testing current "B").

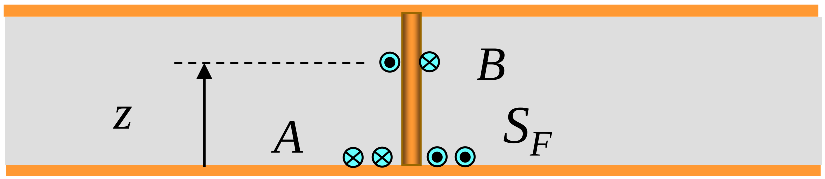


$$I(z) = \int_V \underline{H}^a \cdot \underline{M}^b \, dV = -\langle A,B \rangle = -\langle B,A \rangle$$

$$= \int_V \underline{H}^b \cdot \underline{M}^a \, dV$$

$$= \int_{S_F} \underline{H}^b \cdot \underline{M}_s \, dS$$

Frill Model (cont.)



$$I(z) = \int_{S_F} \underline{H}^{b} \cdot \underline{M}_{s} \, dS$$

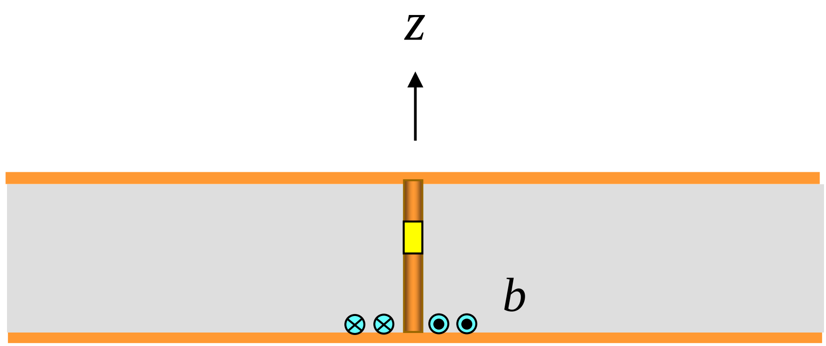
$$= \int_{0}^{2\pi} \int_{a}^{b} H_{\phi}^{b}(\rho, 0) M_{s\phi} \, \rho \, d\rho \, d\phi$$

$$= 2\pi \int_{a}^{b} H_{\phi}^{b}(\rho, 0) M_{s\phi} \, \rho \, d\rho$$

$$= 2\pi \int_{a}^{b} H_{\phi}^{b}(\rho, 0) \left[-\frac{1}{\rho} \frac{1}{\ln(b/a)} \right] \rho \, d\rho$$

$$= -\frac{2\pi}{\ln(b/a)} \int_{a}^{b} H_{\phi}^{b}(\rho, 0) \, d\rho$$

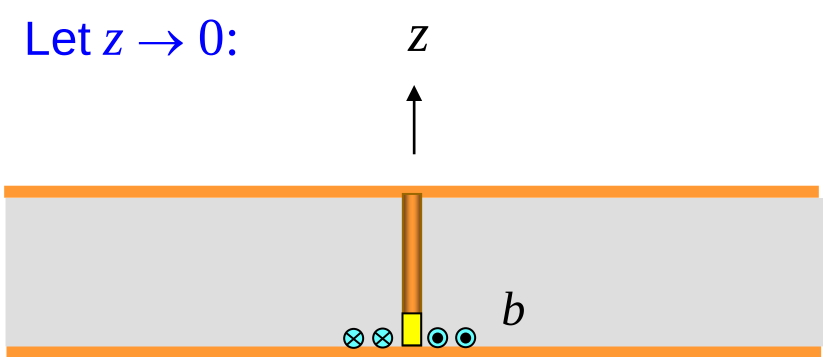
Frill Model (cont.)



The magnetic current ring B may be replaced by a 1V gap source of zero height (by the equivalence principle).

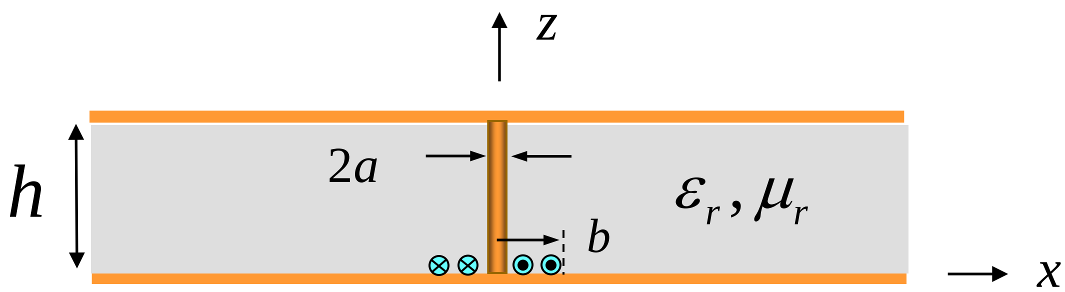
$$I(z) = -\frac{2\pi}{\ln(b/a)} \int_a^b H_\phi^{gap}(\rho, 0)\, d\rho$$

Let $z \to 0$:



The field of the gap source is then calculated as was done in the gap-source model, using $\Delta = 0$.

Frill Model (cont.)



Final result:

$$Y_{in} = j\left(\frac{1}{h\eta}\right)\left(\frac{1}{\ln\left(b/a\right)}\right)4\pi k\sum_{m=0}^{\infty}\frac{H_0^{(2)}(k_{\rho m}b)-H_0^{(2)}(k_{\rho m}a)}{(k_{\rho m}^2)(1+\delta_{m0})H_0^{(2)}(k_{\rho m}a)}$$