ECE 6345

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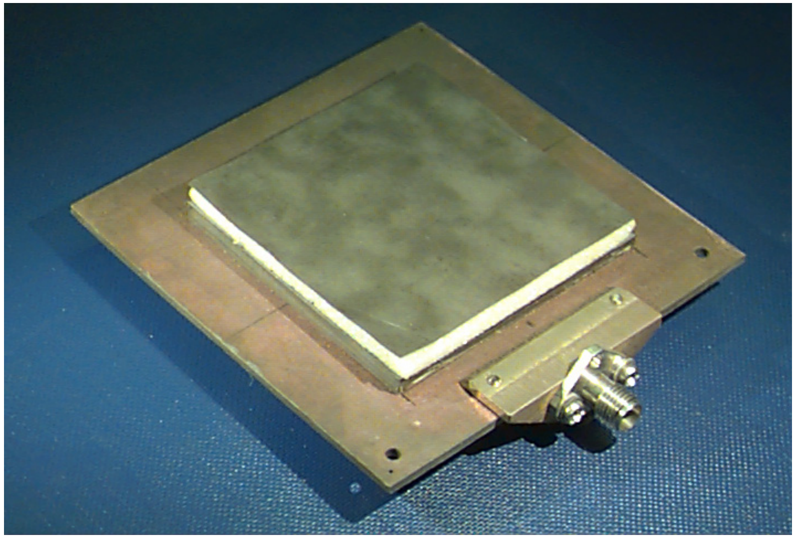
Prof. David R. Jackson
ECE Dept.



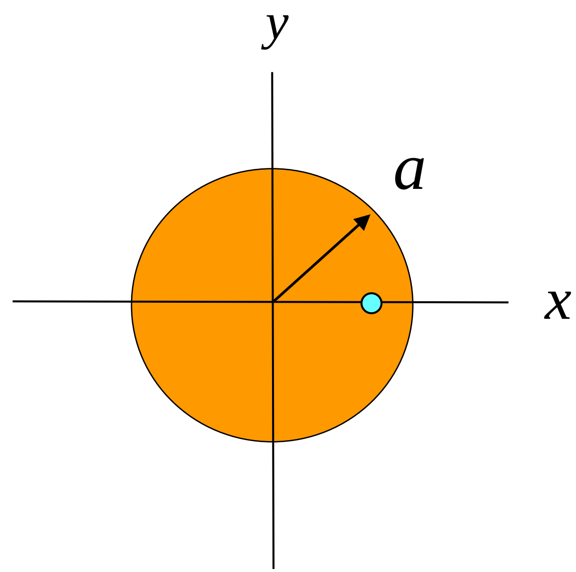
Notes 10

Overview

In this set of notes we derive the far-field pattern of a circular patch operating in the dominant TM_{11} mode.

We use the magnetic current model.

Circular Patch: TM_{11} Mode



$$E_z(\rho,\phi) = A\cos\phi\, J_1(k\rho)$$

This corresponds to a probe on the x axis.

$$k = \frac{x'_{11}}{a}$$

$$x'_{11} = 1.841$$

Circular Patch (cont.)

Magnetic current model:

$$\underline{M}_s = -\underline{\hat{n}} \times \underline{\hat{E}}$$
$$= -\underline{\hat{\rho}} \times \underline{\hat{E}}$$

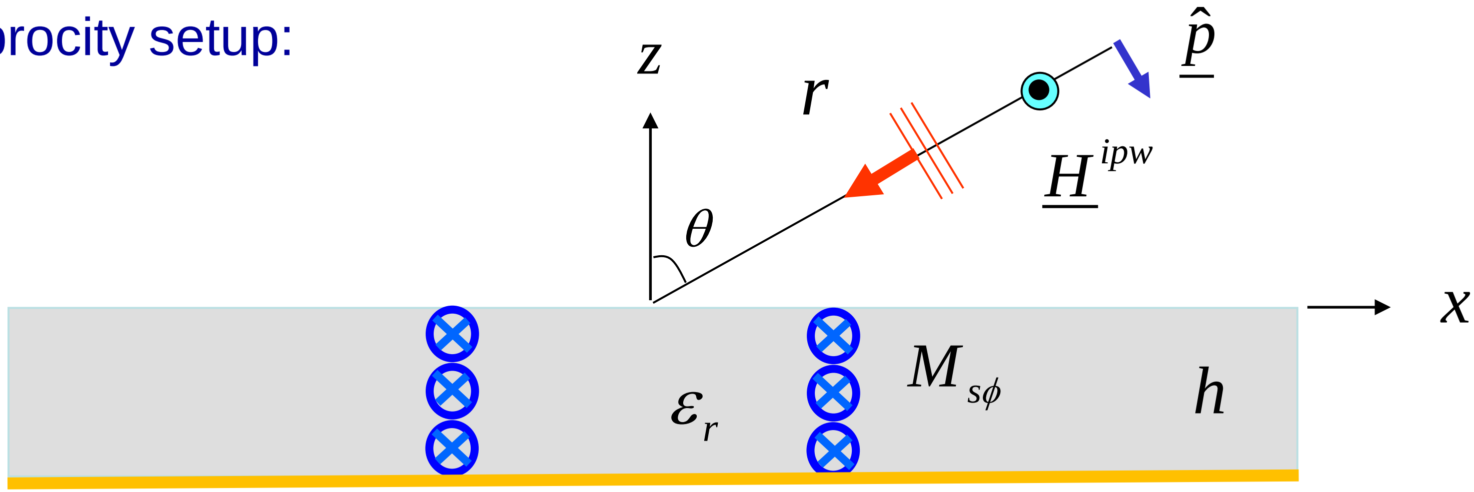
$$M_{s\phi} = E_z(a,\phi) = A\cos\phi\, J_1(ka)$$

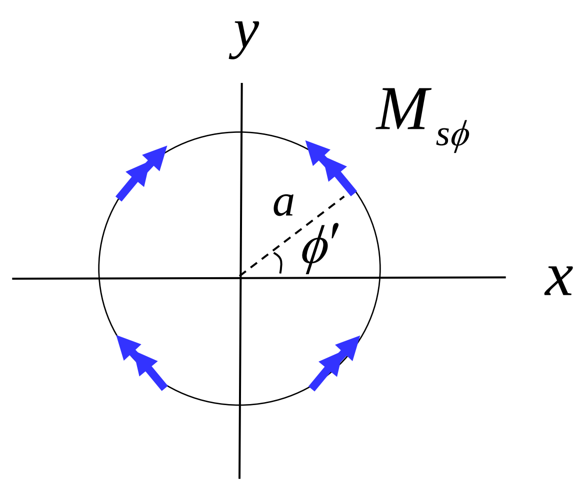
Choose $A = \dfrac{1}{J_1(ka)}$ $\qquad M_{s\phi} = \cos\phi$

$\left(V_0 = -h \text{ [V] at patch edge on } x \text{ axis}\right)$

Far Field of Circular Patch

Reciprocity setup:





Far Field of Circular Patch (cont.)

Far-field:

$$
\begin{aligned}
E_p^{FF}(r,\theta,\phi) &= \langle a,b \rangle \\
&= \langle b,a \rangle \\
&= -\int_S \underline{H}^{pw} \cdot \underline{M}_s^a \, dS \\
&= -\int_S H_{\phi'}^{pw}(\rho',\phi',z') \cos\phi' \, dS' \\
&= -\int_0^{2\pi}\int_{-h}^{0} H_{\phi'}^{pw}(a,\phi',z') \cos\phi' \, a \, dz' d\phi'
\end{aligned}
$$

The primes here denotes source coordinates.

$$
E_p^{FF}(r,\theta,\phi) = -\int_0^{2\pi}\int_{-h}^{0} H_{\phi'}^{pw}(a,\phi',z') \cos\phi' \, a \, dz' d\phi'
$$

$$
H_{\phi'}^{pw} = H_x^{pw}(-\sin\phi') + H_y^{pw}(\cos\phi')
$$

Far Field of Circular Patch (cont.)

Inside the substrate we have (see Notes 9):

$$H_{x,y}^{pw}\left(x',y',z'\right)=H_{x,y}^{pw}\left(0,0,0\right)\,e^{j\left(k_x x'+k_y y'\right)}\left[\sec\left(k_{z1}h\right)\cos\left(k_{z1}(z'+h)\right)\right]$$

$$\left(k_{z1}=k_0 N_1(\theta)\right)$$

The exponent term may be put in cylindrical coordinates as follows:

$$\begin{aligned}k_x x'+k_y y' &=\left(k_0\sin\theta\cos\phi\right)\left(a\cos\phi'\right)+\left(k_0\sin\theta\sin\phi\right)\left(a\sin\phi'\right)\\ &=k_0\,a\sin\theta\left(\cos\phi\cos\phi'+\sin\phi\sin\phi'\right)\\ &=k_0\sin\theta\cos\left(\phi'-\phi\right)\end{aligned}$$

Far Field of Circular Patch (cont.)

Hence

$$H_{\phi'}^{pw} = \sec\left(k_{z1}h\right)\cos k_{z1}\left(z'+h\right)\, e^{j(k_0 a)\sin\theta\cos\left(\phi'-\phi\right)} \cdot\left[-\sin\phi' H_x^{pw}(0,0,0)+\cos\phi' H_y^{pw}(0,0,0)\right]$$

Since the horizontal magnetic field components are modeled as current in the TEN, we have

$$H_{x,y}^{pw}\left(0,0,0\right) = H_{x,y}^{ipw}\left(0,0,0\right)\left(1-\Gamma(\theta)\right)$$

$$p=\theta : \mathrm{TM}, \quad p=\phi : \mathrm{TE}$$

Far Field of Circular Patch (cont.)

TM$_z$ $\left(\underline{\hat{p}} = \underline{\hat{\theta}}\right)$

$$H_x^{ipw}(0,0,0) = \left(-\frac{E_0}{\eta_0}\underline{\hat{\phi}}\right)\cdot\underline{\hat{x}} = \frac{E_0}{\eta_0}(\sin\phi)$$

$$H_y^{ipw}(0,0,0) = \left(-\frac{E_0}{\eta_0}\underline{\hat{\phi}}\right)\cdot\underline{\hat{y}} = \frac{E_0}{\eta_0}(-\cos\phi)$$

TE$_z$ $\left(\underline{\hat{p}} = \underline{\hat{\phi}}\right)$

$$H_x^{ipw}(0,0,0) = \left(\frac{E_0}{\eta_0}\underline{\hat{\theta}}\right)\cdot\underline{\hat{x}} = \frac{E_0}{\eta_0}(\cos\theta\cos\phi)$$

$$H_y^{ipw}(0,0,0) = \left(\frac{E_0}{\eta_0}\underline{\hat{\theta}}\right)\cdot\underline{\hat{y}} = \frac{E_0}{\eta_0}(\cos\theta\sin\phi)$$

Far Field E_θ

TM$_z$ $\left(\underline{\hat{p}} = \underline{\hat{\theta}}\right)$

Substituting for H_x and H_y, we have

$$H_{\phi'}^{pw} = \sec\left(k_{z1}h\right)\cos k_{z1}\left(z'+h\right)e^{j(k_0 a)\sin\theta\cos\left(\phi'-\phi\right)}$$

$$\cdot\left(\frac{E_0}{\eta_0}\right)\left[-\sin\phi'\sin\phi-\cos\phi'\cos\phi\right]\left(1-\Gamma^{TM}(\theta)\right)$$

Note: $\left[\quad\right] = -\cos\left(\phi'-\phi\right)$

Hence, we have

$$E_\theta^{FF}\left(r,\theta,\phi\right) = \int_0^{2\pi}\int_{-h}^{0}\frac{E_0}{\eta_0}\sec\left(k_{z1}h\right)\cos k_{z1}(z'+h)\; e^{j(k_0 a)\sin\theta\cos\left(\phi'-\phi\right)}$$

$$\cdot\left(1-\Gamma^{TM}(\theta)\right)\left(\cos(\phi'-\phi)\right)\cos\phi'\; a\,dz'd\phi'$$

Far Field E_θ (cont.)

For the z' integral we have

$$\sec\left(k_{z1}h\right)\int_{-h}^{0}\cos k_{z1}(z'+h)\,dz' = h\,\mathrm{tanc}(k_{z1}h)$$

so that

$$E_\theta^{FF}\left(r,\theta,\phi\right) = a\left(\frac{E_0}{\eta_0}\right)h\,\mathrm{tanc}(k_{z1}h)\left(1-\Gamma^{TM}(\theta)\right)I_{TM}$$

where

$$I_{TM} \equiv \int_0^{2\pi} e^{jq\cos\left(\phi'-\phi\right)}\cos\left(\phi'-\phi\right)\cos\phi'\,d\phi'$$

$$q \equiv (k_0 a)\sin\theta$$

Let $\phi'' = \phi' - \phi$

$$I_{TM} = \int_{-\phi}^{2\pi-\phi} e^{jq\cos\left(\phi''\right)}\cos\left(\phi''\right)\cos\left(\phi''+\phi\right)d\phi''$$

Far Field E_θ (cont.)

We have that

$$\cos\left(\phi''+\phi\right)=\cos\phi''\cos\phi-\sin\phi''\sin\phi$$

and $\displaystyle\int_{-\phi}^{2\pi-\phi}(\quad)\,d\phi''=\int_0^{2\pi}(\quad)\,d\phi''$

so that

$$I_{TM}\equiv\int_0^{2\pi}e^{jq\cos(\phi'')}\cos\left(\phi''\right)\cos\left(\phi''+\phi\right)d\phi''$$

$$=\cos\phi\int_0^{2\pi}e^{jq\cos\phi''}\cos^2\phi''\,d\phi''$$

$$-\sin\phi\int_0^{2\pi}e^{jq\cos\phi''}\sin\phi''\cos\phi''\,d\phi''$$

Integrates to zero (odd function)

Now use $\cos^2\phi''=1-\sin^2\phi''$

Far Field E_θ (cont.)

$$I_{TM} = \cos\phi \int_0^{2\pi} e^{jq\cos\phi''} d\phi'' - \cos\phi \int_0^{2\pi} e^{jq\cos\phi''} \sin^2\phi''\, d\phi''$$

Now we use the following identity:

$$\int_0^{2\pi} e^{jq\cos\phi''} \sin^{2n}\phi'' d\phi'' = J_n(q) \left[\frac{2^{n+1}\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{q^n} \right]$$

$$n = 0, 1, 2 \ldots$$

where

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

Far Field E_θ (cont.)

Hence

$$\int_0^{2\pi} e^{jq\cos\phi''}\,d\phi'' = 2\pi\, J_0(q) \qquad (n=0)$$

and

$$\int_0^{2\pi} e^{jq\cos\phi''}\sin^2\phi''\,d\phi'' = 2\pi\left(\frac{J_1(q)}{q}\right) \qquad (n=1)$$

and thus

$$I_{TM} = \cos\phi\,(2\pi)\left[J_0(q) - \frac{J_1(q)}{q}\right]$$

Next, use

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$$

so that

$$J_1'(x) = J_0(x) - \frac{J_1(x)}{x}$$

Far Field E_θ (cont.)

Hence

$$I_{TM} = 2\pi \cos\phi \, J_1'(q)$$

The far field is then

$$E_\theta^{FF}\left(r,\theta,\phi\right) = \frac{E_0}{\eta_0}(ah)\,\mathrm{tanc}(k_{z1}h)\,Q(\theta) \cdot 2\pi \cos\phi \, J_1'(k_0 a \sin\theta)$$

where $$Q(\theta) = 1 - \Gamma^{TM}(\theta)$$

Far Field E_ϕ

TE$_z$ $\left(\underline{\hat{p}} = \underline{\hat{\phi}}\right)$

Performing similar steps, we have

$$H_{\phi'}^{pw} = \sec\left(k_{z1}h\right)\cos k_{z1}\left(z'-h\right)\left(1-\Gamma^{TE}\left(\theta\right)\right)e^{j(k_0 a)\sin\theta\cos\left(\phi'-\phi\right)}$$
$$\cdot\left(\frac{E_0}{\eta_0}\right)\left[-\sin\phi'\cos\theta\cos\phi+\cos\phi'\cos\theta\sin\phi\right]$$

Using reciprocity and performing the integration in z'', we have

$$E_\phi^{FF}\left(r,\theta,\phi\right) = a\left(\frac{E_0}{\eta_0}\right)(h)\,\mathrm{tanc}(k_{z1}h)\left(1-\Gamma^{TE}\left(\theta\right)\right)$$
$$\cdot\cos\theta\int_0^{2\pi} e^{jq\cos\left(\phi'-\phi\right)}\sin\left(\phi'-\phi\right)\cos\phi'\,d\phi'$$

Far Field E_ϕ (cont.)

Evaluating the integral, we have

$$I_{TE} \equiv \int_0^{2\pi} e^{jq\cos(\phi'-\phi)} \sin(\phi'-\phi)\cos\phi' d\phi'$$

$$= \int_0^{2\pi} e^{jq\cos\phi''} \sin\phi'' \cos\left(\phi''+\phi\right) d\phi''$$

$$= \int_0^{2\pi} e^{jq\cos\phi''} \sin\phi'' \left[\cos\phi''\cos\phi - \sin\phi''\sin\phi\right] d\phi''$$

$$= \cos\phi \underline{\int_0^{2\pi} e^{jq\cos\phi''} \sin\phi'' \cos\phi'' d\phi''} - \sin\phi \int_0^{2\pi} e^{jq\cos\phi''} \sin^2\phi'' \, d\phi''$$

integrates to zero
(odd function)

$$= -\sin\phi \, 2\pi \left(\frac{J_1(q)}{q} \right)$$

Far Field E_ϕ (cont.)

Hence

$$E_\phi^{FF}\left(r,\theta,\phi\right) = -\frac{E_0}{\eta_0}(ah)\,\mathrm{tanc}\left(k_{z1}h\right)\sin\phi\, 2\pi\left(\frac{J_1\left(k_0 a\sin\theta\right)}{k_0 a\sin\theta}\right)P(\theta)$$

where

$$P(\theta) \equiv \cos\theta\left(1-\Gamma^{TE}(\theta)\right)$$