ECE 6345

Spring 2015

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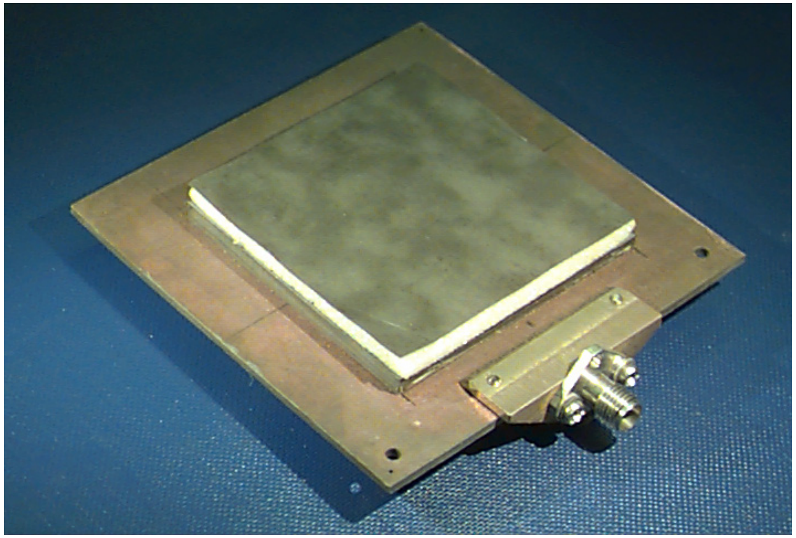
ECE Dept.



Notes 7

Overview

In this set of notes we use reciprocity to calculate the far field for rectangular patch using the electric-current model, assuming an infinite substrate.

- Review of reciprocity to calculate the far field.
- Far field of horizontal electric dipole in the x direction (hex) on top of a grounded the substrate.
- Far field of dominant mode of rectangular patch, using the electric current model and infinite substrate.

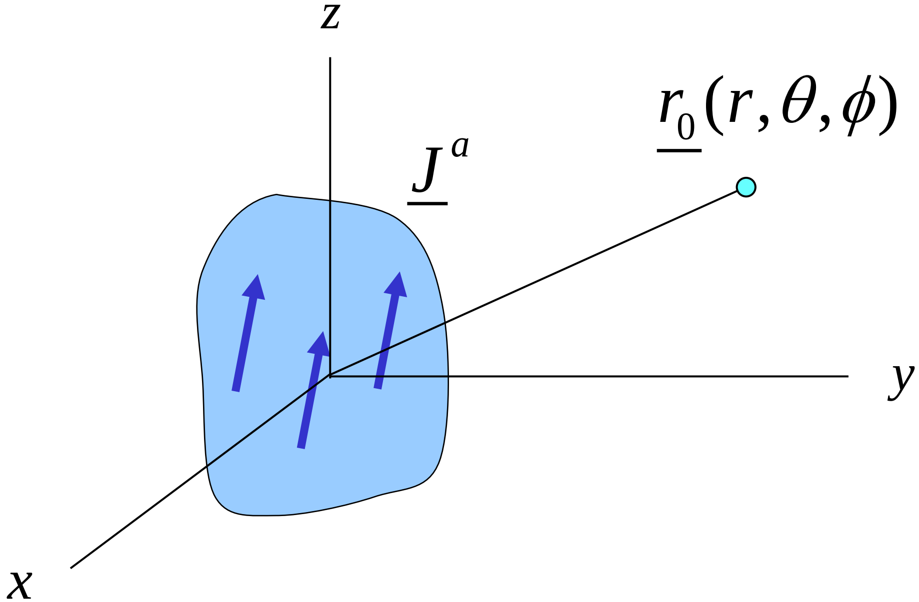
Far-Field

Reciprocity theorem:

$$< a, b > = < b, a >$$

$$\int_V \left(\underline{E}^a \cdot \underline{J}^b - \underline{H}^a \cdot \underline{M}^b \right) dV = \int_V \left(\underline{E}^b \cdot \underline{J}^a - \underline{H}^b \cdot \underline{M}^a \right) dV$$

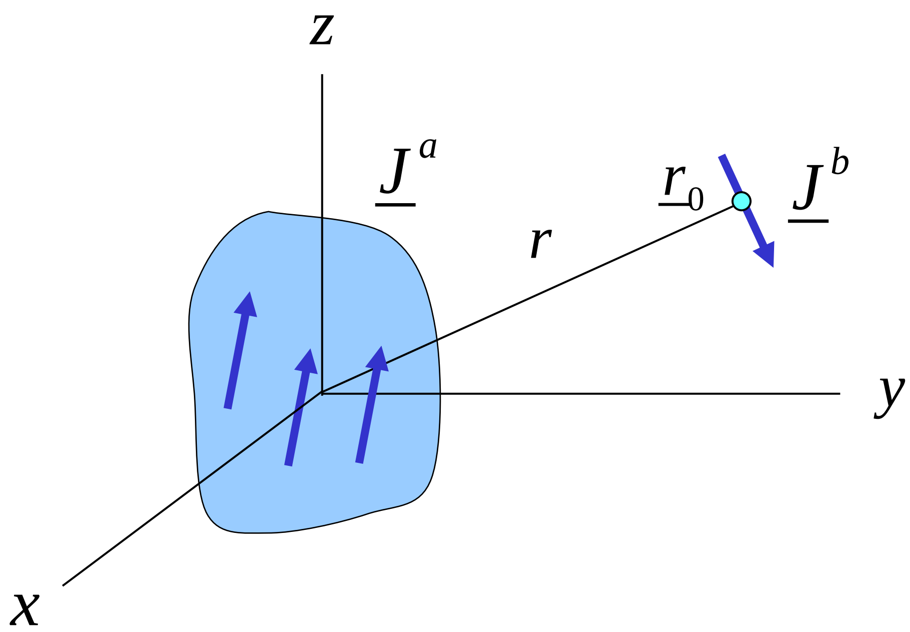
Consider an electric current source:



Far-Field (cont.)

Let

$$\underline{J}^{b} = \underline{\hat{\theta}}\,\delta\left(\underline{r} - \underline{r}_0\right)$$



$$< a,b > = \int_V \left(\underline{E}^{a} \cdot \underline{J}^{b}\right) dV$$

$$= \underline{E}^{a}\left(\underline{r}_0\right) \cdot \underline{\hat{\theta}}$$

$$= E_{\theta}^{rad}\left(r,\theta,\phi\right)$$

Far-Field (cont.)

Hence $E_\theta^{rad}\left(r,\theta,\phi\right) = < a,b >$

$$= < b,a >$$

$$< b,a > = \int_V \left(\underline{E}^b \cdot \underline{J}^a\right) dV$$

Assume $r \to \infty$ $\quad \underline{E}^{b,inc} = \underline{E}^{ipw}$ ipw = "incident plane wave"

Hence

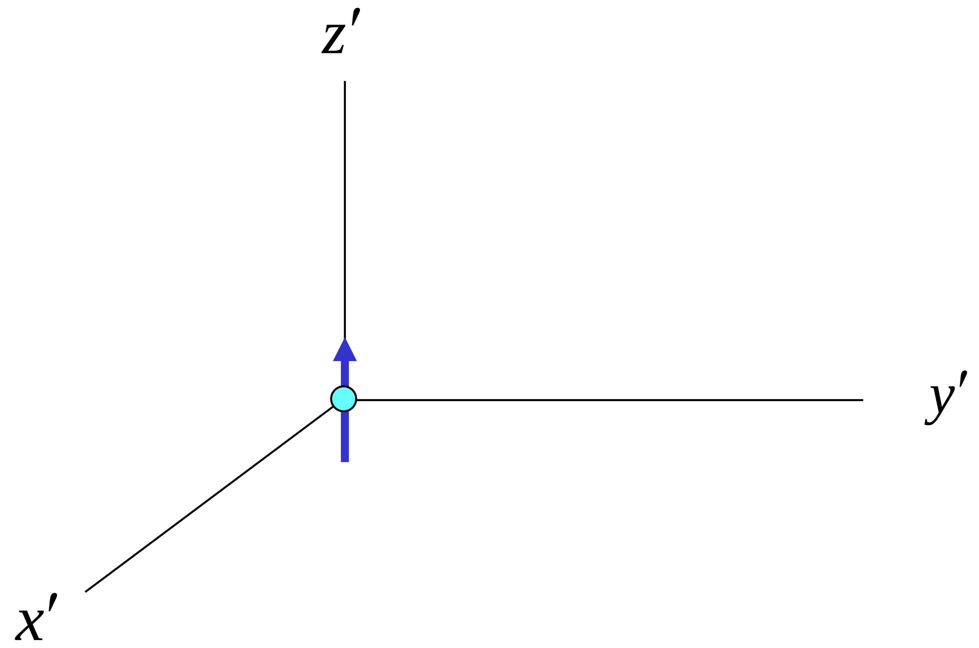
$$E_\theta^{FF}\left(r,\theta,\phi\right) = \int_V \left(\underline{E}^{ipw} \cdot \underline{J}^a\right) dV$$

FF = "far field"

Far-Field (cont.)

Consider a dipole
in free space:

The primed coordinates
denote local coordinates.



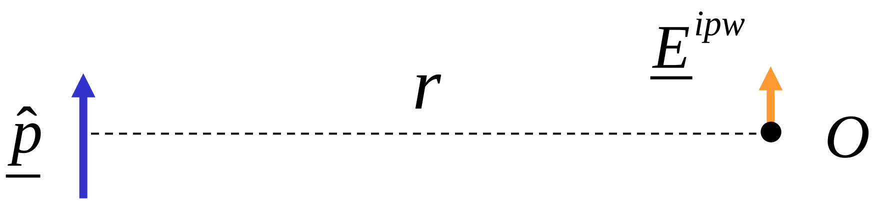
$$\underline{E} \sim \underline{\hat{\theta}'} \left(\frac{j\omega\mu_0}{4\pi r'} \right) \sin\theta' \, e^{-jk_0 r'}$$

At $\theta' = 90^0 \quad \left(\underline{\hat{\theta}'} = -\underline{\hat{z}'} \right)$

$$\underline{E} \sim -\frac{j\omega\mu_0}{4\pi r'} \, e^{-jk_0 r'} \, \underline{\hat{z}'}$$

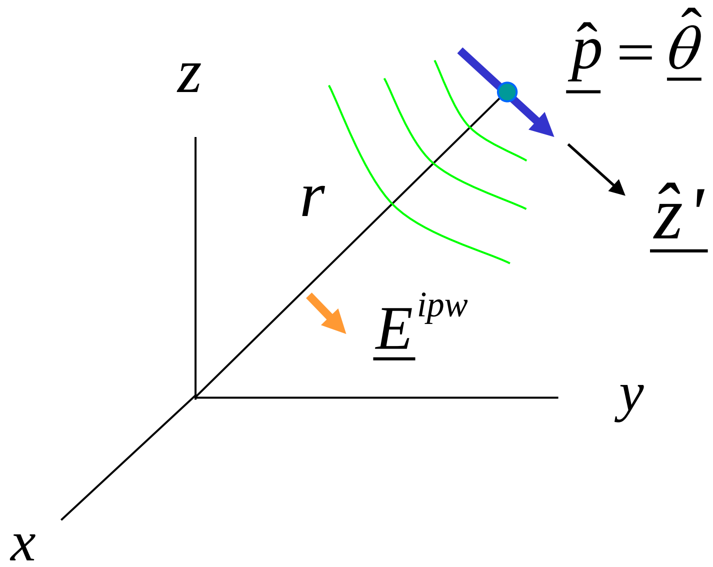
Far-Field (cont.)

In general,



At O : $\underline{E}^{ipw} \sim \underline{\hat{p}} E_0$

where $E_0 = -\dfrac{j\omega\mu_0}{4\pi r} e^{-jk_0 r}$



At (0, 0, 0): $\underline{E}^{ipw} \sim \underline{\hat{\theta}} E_0$

Far-Field (cont.)

For an arbitrary observation point (x, y, z) we have

$$\underline{E}^{ipw}\left(x,y,z\right) \sim \underline{\hat{\theta}}\, E_0\, \psi(x,y,z)$$

where $$\psi(x,y,z) = e^{+j\left(k_x x + k_y y + k_z z\right)} = e^{+j\underline{k}\cdot\underline{r}}$$

$$k_x = k_0 \sin\theta\cos\phi$$

$$k_y = k_0 \sin\theta\sin\phi$$

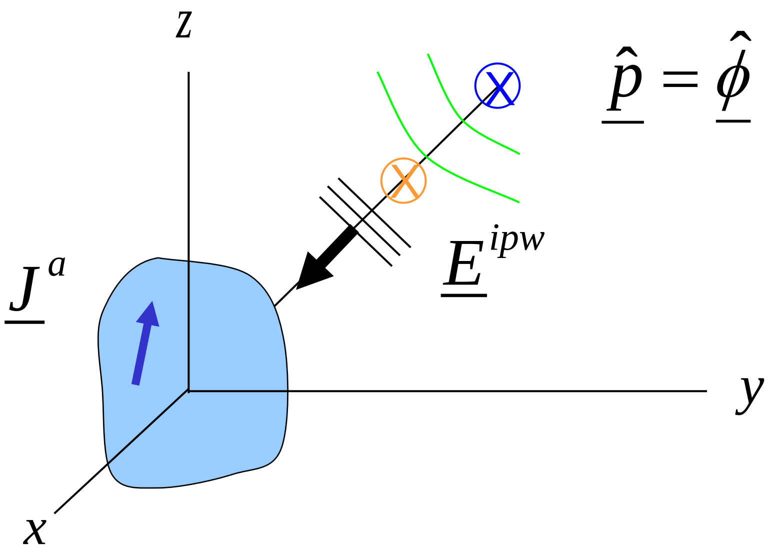
$$k_z = k_0 \cos\theta$$

Far-Field (cont.)

Hence $\underline{E}^{ipw} \sim \underline{\hat{\theta}}\, E_0\, \psi(x,y,z)$

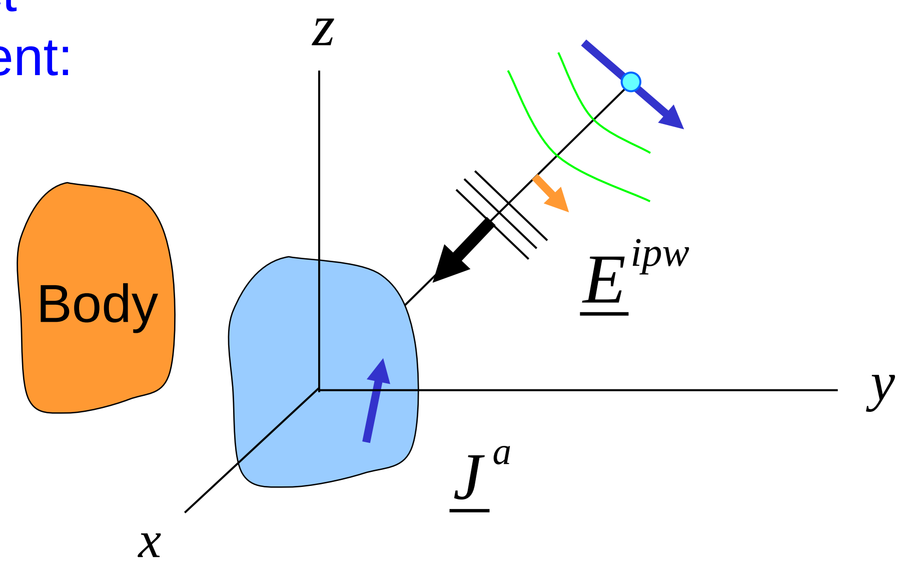
Similarly, if $\underline{\hat{p}} = \underline{\hat{\phi}}$

$$\underline{E}^{ipw} = \underline{\hat{\phi}}\, E_0 \psi\left(x,y,z\right)$$



Far-Field (cont.)

Now consider an object near the radiating current:



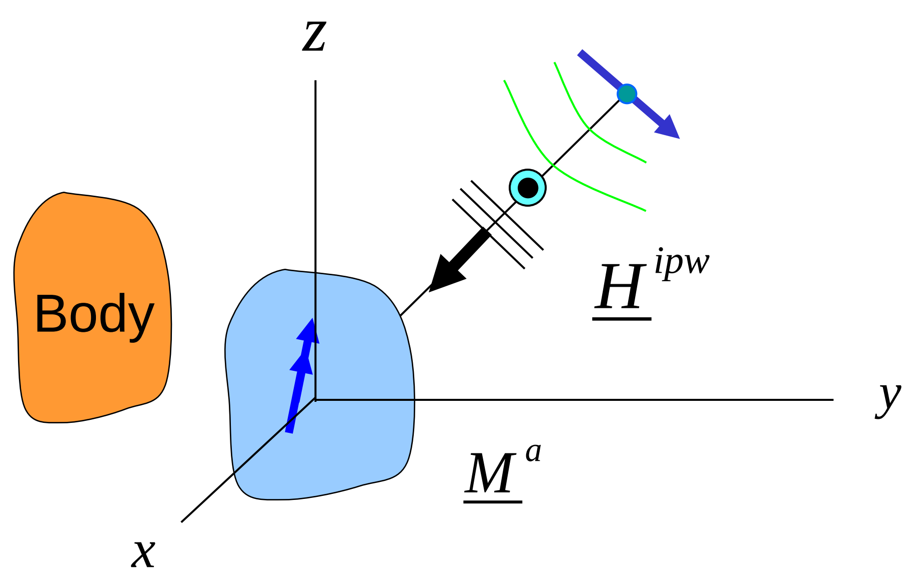
$$E_\theta^{FF}(r,0,\phi) = \int_V \left(\underline{E}^{b} \cdot \underline{J}^{a} \right) dV$$

$$\underline{E}^{b} = \underline{E}^{ipw} + \underline{E}^{scattered}$$

If the "body" is an infinite layered dielectric structure, the scattered field can be calculated exactly.

Far-Field (cont.)

For magnetic current source:

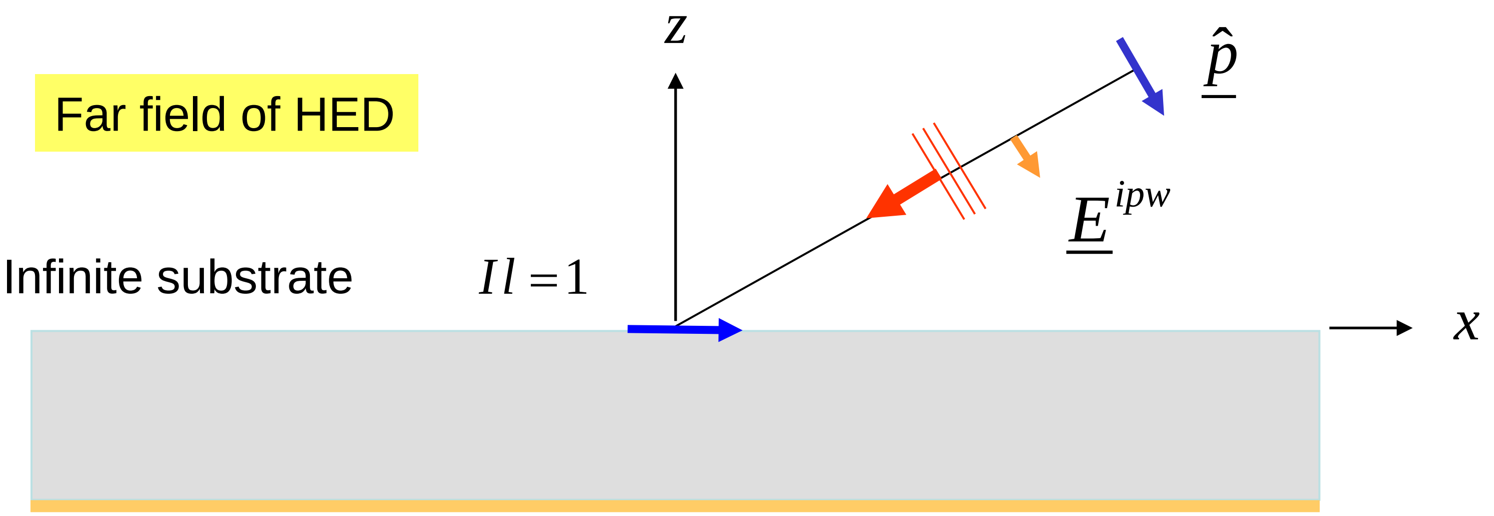


$$E_{\theta}^{FF}(r,\theta,\phi) = -\int_V \left(\underline{H}^{b} \cdot \underline{M}^{a}\right) dV$$

$$\underline{H}^{b} = \underline{H}^{ipw} + \underline{H}^{scattered}$$

where $$\underline{H}^{ipw} = -\frac{1}{\eta_0}\left(\hat{\underline{r}} \times \underline{E}^{ipw}\right)$$

Electric Dipole



$$\underline{J}^{a} = \underline{\hat{x}}(Il)\delta(\underline{r}) = \underline{\hat{x}}\ \delta(\underline{r}) = \underline{\hat{x}}\,\delta(x)\delta(y)\delta(z)$$

$$\begin{aligned} E_p^{hex}(r,\theta,\phi) &= \langle b, a \rangle \\ &= \int_V \left(\underline{E}^b \cdot \underline{J}^a\right) dV \\ &= E_x^b(0,0,0) \\ &= \left[E_x^{ipw}(0,0,0) + E_x^{rpw}(0,0,0)\right] \end{aligned}$$

"hex" = unit-amplitude horizontal electric dipole in the x direction.

rpw = "reflected plane wave"

Electric Dipole (cont.)

For $\underline{\hat{p}} = \underline{\hat{\theta}}$

$$\underline{E}^{ipw} = \underline{\hat{\theta}}\, E_0 e^{+j\underline{k}\cdot\underline{r}}$$

$$E_x^{ipw} = E_0 \cos\theta \cos\phi\, e^{+j\underline{k}\cdot\underline{r}}$$

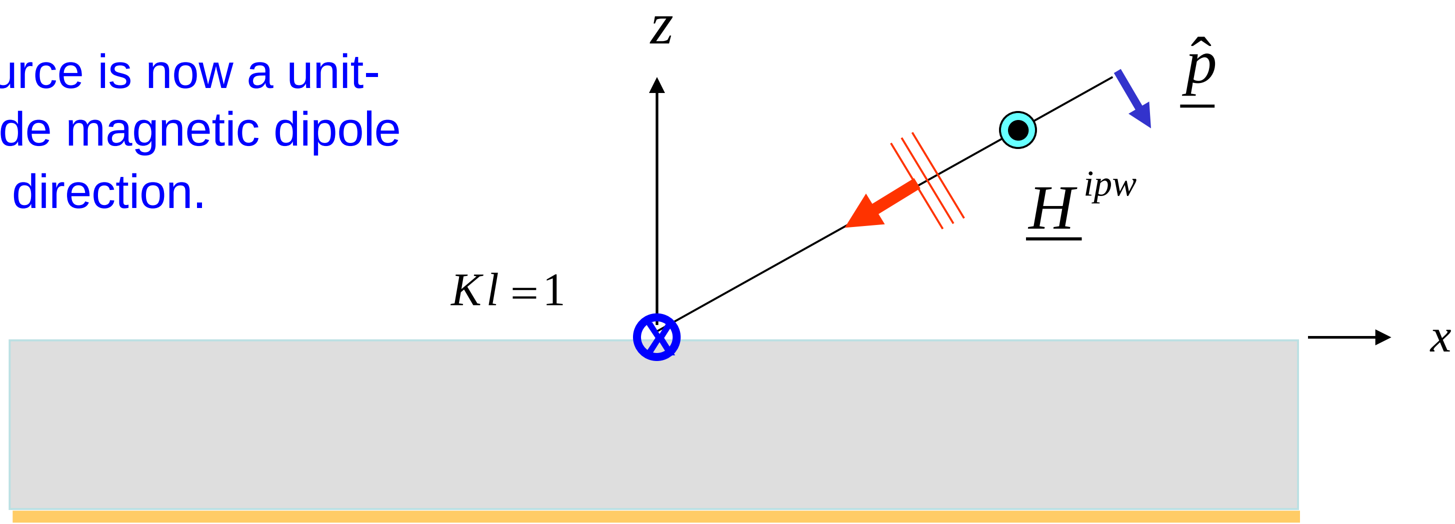
For $\underline{\hat{p}} = \underline{\hat{\phi}}$

$$\underline{E}^{ipw} = \underline{\hat{\phi}}\, E_0 e^{+j\underline{k}\cdot\underline{r}}$$

$$\underline{E}_x^{ipw} = E_0 (-\sin\phi)\, e^{+j\underline{k}\cdot\underline{r}}$$

Magnetic Dipole

The source is now a unit-amplitude magnetic dipole in the y direction.



$$\underline{M}^a = \underline{\hat{y}}(Kl)\delta(\underline{r}) = \underline{\hat{y}}\delta(\underline{r}) = \underline{\hat{y}}\delta(x)\delta(y)\delta(z)$$

$$\begin{aligned} E_p^{hmy}(r,\theta,\phi) &= \langle b, a \rangle \\ &= -\int_V \left(\underline{H}^b \cdot \underline{M}^a \right) dV \\ &= -H_y^b(0,0,0) \\ &= -\left[H_y^{ipw}(0,0,0) + H_y^{rpw}(0,0,0) \right] \end{aligned}$$

Magnetic Dipole (cont.)

$$\underline{\hat{p}} = \underline{\hat{\theta}} \quad \Rightarrow \quad \underline{E}^{ipw} = \underline{\hat{\theta}} E_0 e^{+j\underline{k}\cdot\underline{r}}$$

$$\underline{H}^{ipw} = -\underline{\hat{\phi}} \left(\frac{E_0}{\eta_0} \right) e^{+j\underline{k}\cdot\underline{r}}$$

$$H_y^{ipw} = \left(\frac{E_0}{\eta_0} \right) \left(-\cos\phi \right) e^{+j\underline{k}\cdot\underline{r}}$$

$$\underline{\hat{p}} = \underline{\hat{\phi}} \quad \Rightarrow \quad \underline{E}^{ipw} = \underline{\hat{\phi}} E_0 e^{+j\underline{k}\cdot\underline{r}}$$

$$\underline{H}^{ipw} = +\underline{\hat{\theta}} \left(\frac{E_0}{\eta_0} \right) e^{+j\underline{k}\cdot\underline{r}}$$

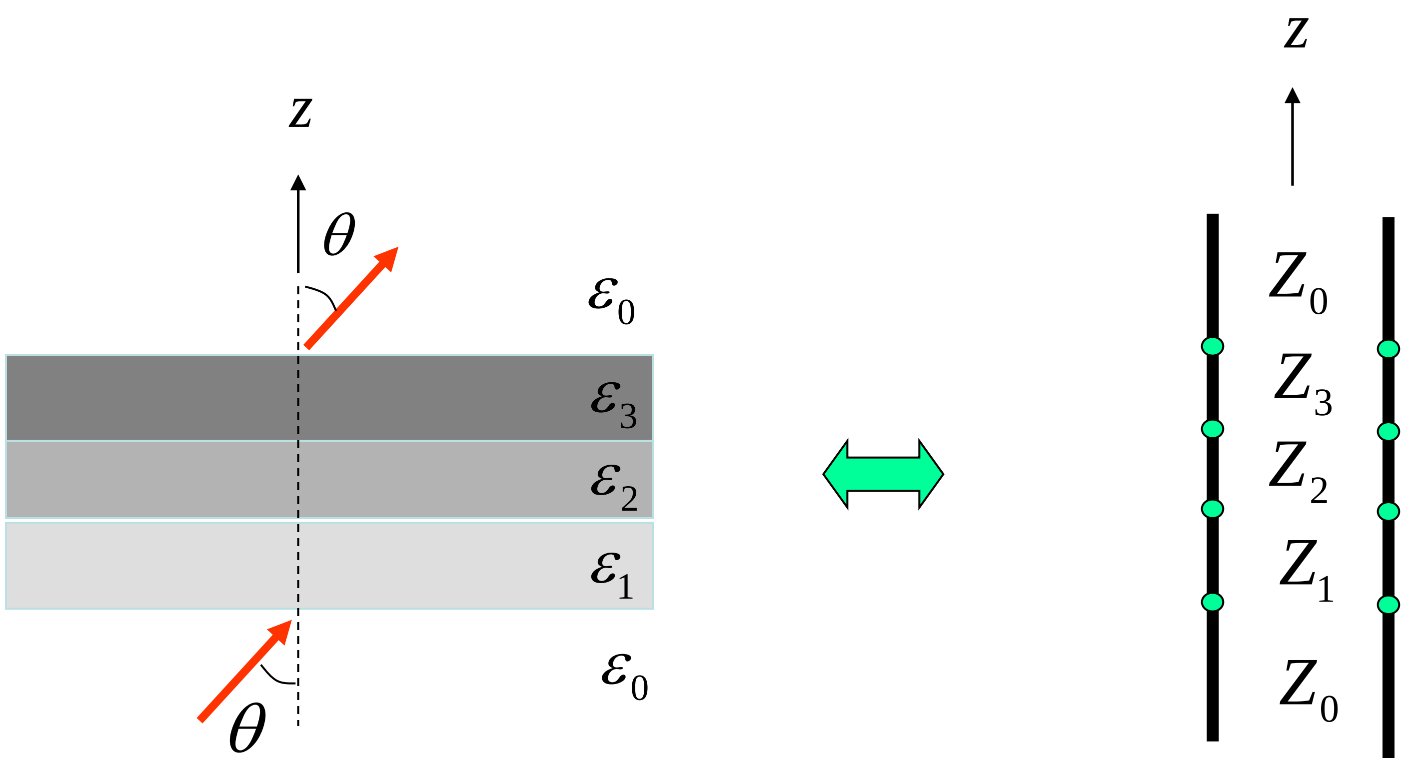
$$H_y^{ipw} = \left(\frac{E_0}{\eta_0} \right) \left(+\cos\theta \, \sin\phi \right) e^{+j\underline{k}\cdot\underline{r}}$$

Transverse Equivalent Network (TEN)

$$E_x(x,y,z) = \psi_t(x,y)\, V(z)$$

$$H_y(x,y,z) = \psi_t(x,y)\, I(z)$$

$$\psi(x,y) = e^{j(k_x x + k_y y)}$$

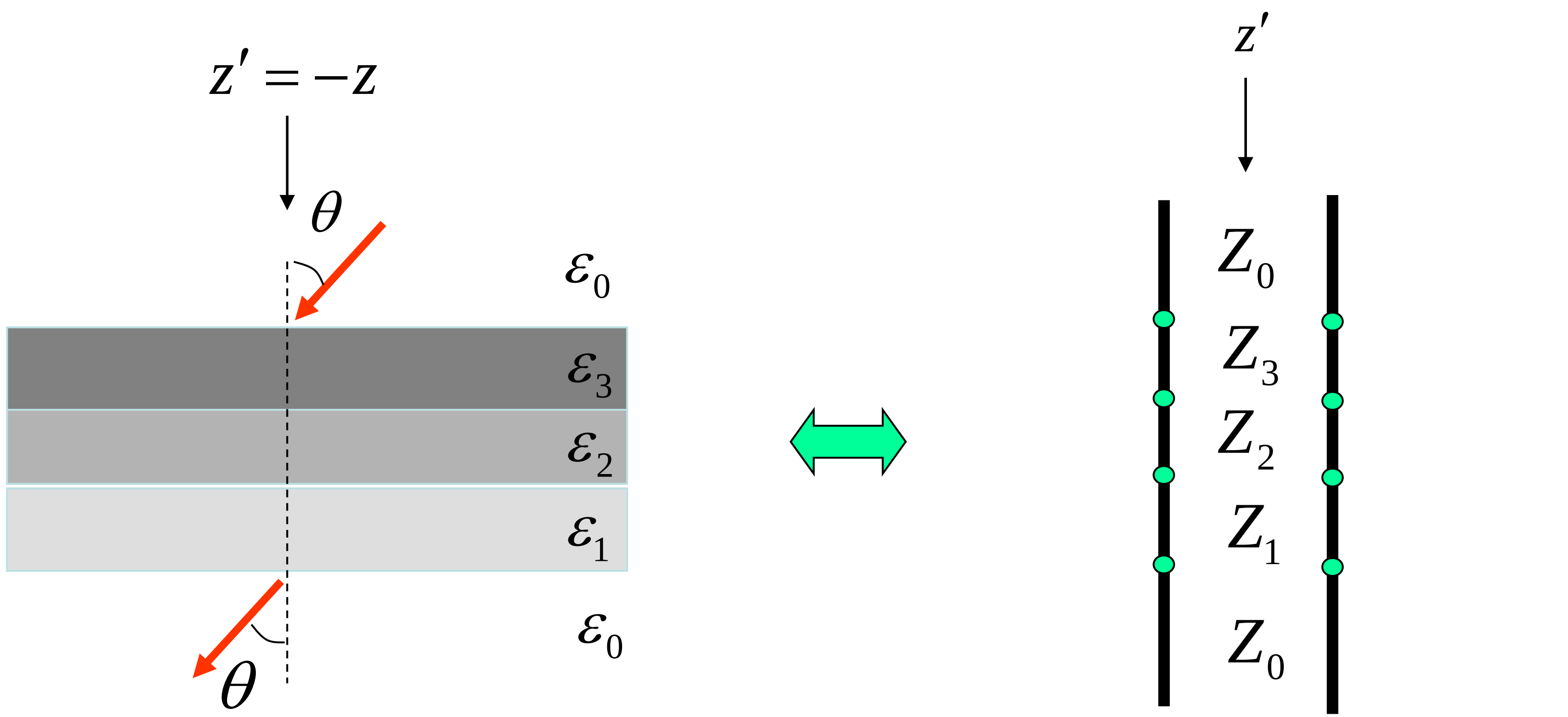


TEN (cont.)

For a wave traveling in the $z' = -z$ direction, we use:

$$E_x(x,y,z') = \psi(x,y)\, V(z')$$
$$-H_y(x,y,z') = \psi(x,y)\, I(z')$$

This is the situation for our incident wave.



TEN (cont.)

$$Z_i^{TM} = \frac{k_{zi}}{\omega \varepsilon_i}$$

$$Z_i^{TE} = \frac{\omega \mu_i}{k_{zi}}$$

where

$$k_{zi} = \sqrt{k_i^2 - k_x^2 - k_y^2}$$

and

$$k_{zi} = k_i \cos \theta_i$$

or

$$k_{zi} = k_i \sqrt{1 - \sin^2 \theta_i}$$

TEN (cont.)

From Snell's law, $(1)\sin\theta = n_i \sin\theta_i$ $\quad \left(n_i = \sqrt{\mu_{ri}\varepsilon_{ri}}\right)$

Hence,

$$
\begin{aligned}
k_{zi} &= k_i\sqrt{1-\sin^2\theta_i} \\
&= k_i\sqrt{1-\left(\frac{\sin\theta}{n_i}\right)^2} \\
&= k_0 n_i\sqrt{1-\frac{\sin^2\theta}{n_i^{\,2}}} \\
&= k_0\sqrt{n_i^{\,2}-\sin^2\theta}
\end{aligned}
$$

Define: $N_i(\theta) \equiv \sqrt{n_i^2-\sin^2\theta}$ Then $k_{zi} = k_0 N_i(\theta)$

TEN (cont.)

Also,

$$Z_i^{TM} = \frac{k_{zi}}{\omega \varepsilon_i} = \frac{k_0 N_i(\theta)}{\omega \varepsilon_i} = \frac{k_0 N_i(\theta)}{\omega \varepsilon_0 \varepsilon_{ri}} = \eta_0 \frac{N_i(\theta)}{\varepsilon_{ri}}$$

$$Z_i^{TE} = \frac{\omega \mu_i}{k_{zi}} = \frac{\omega \mu_i}{k_0 N_i(\theta)} = \frac{\omega \mu_0 \mu_{ri}}{k_0 N_i(\theta)} = \eta_0 \frac{\mu_{ri}}{N_i(\theta)}$$

General

$$Z_i^{TM} = \frac{\eta_0}{\varepsilon_{ri}} N_i(\theta)$$

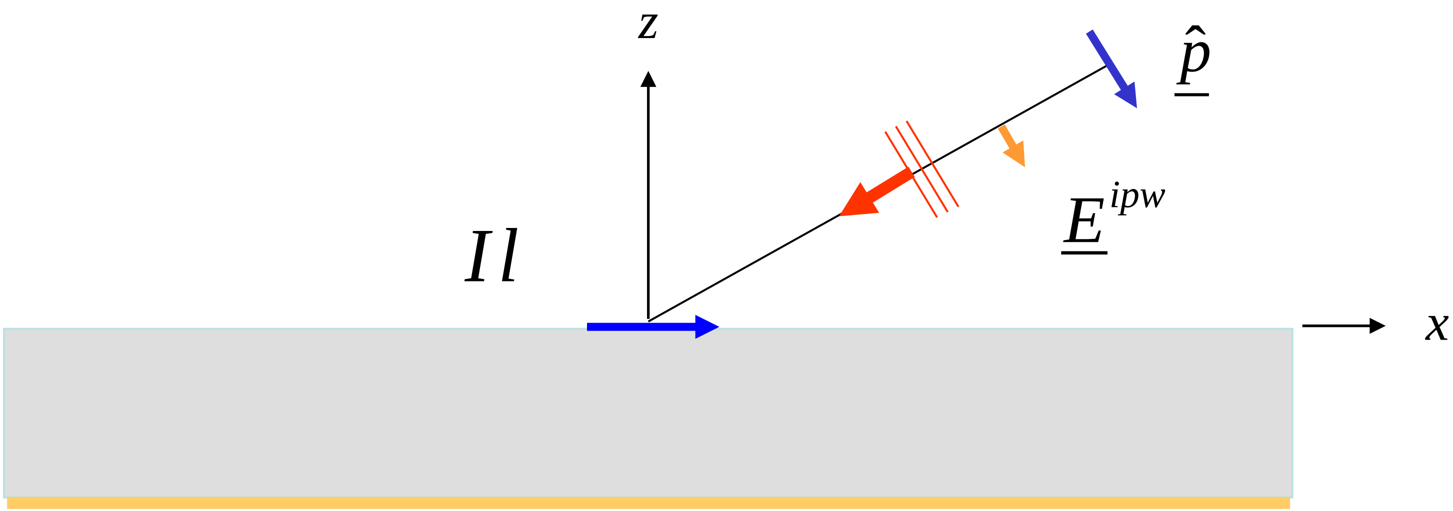
$$Z_i^{TE} = \frac{\eta_0 \mu_{ri}}{N_i(\theta)}$$

Free-space

$$Z_0^{TM} = \eta_0 \cos\theta$$

$$Z_0^{TE} = \frac{\eta_0}{\cos\theta}$$

Electric Dipole



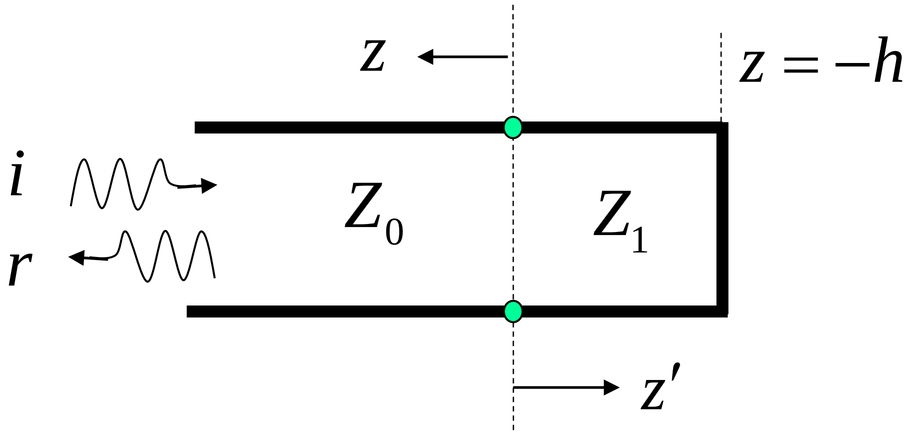
$$E_p^{hex}(r,\theta,\phi)=\left[E_x^{ipw}(0,0,0)+E_x^{rpw}(0,0,0)\right]$$

$$p=\theta: \quad E_x^{ipw}=E_0\cos\theta\cos\phi$$

$$p=\phi: \quad E_x^{ipw}=E_0(-\sin\phi)$$

$$E_0=\frac{-j\omega\mu_0}{4\pi r}e^{-jk_0 r}$$

Electric Dipole: TEN



$$E_x \leftrightarrow V(z)$$

$$V^r(0) = V^i(0)\,\Gamma$$

so

$$E_x^{rpw}(0,0,0) = E_x^{ipw}(0,0,0)\;\Gamma$$

Electric Dipole: TEN (cont.)

Hence $E_p^{hex}\left(r,\theta,\phi\right) = E_x^{ipw}\left(0,0,0\right)\,(1+\Gamma)$

$$p = \theta : \qquad TM_z \qquad \Gamma = \Gamma^{TM}$$

$$p = \phi : \qquad TE_z \qquad \Gamma = \Gamma^{TE}$$

We then have

$$E_\theta^{hex}\left(r,\theta,\phi\right) = E_0 \cos\theta \cos\phi \left[1+\Gamma^{TM}\right]$$
$$E_\phi^{hex}\left(r,\theta,\phi\right) = E_0 \left(-\sin\phi\right)\left[1+\Gamma^{TE}\right]$$

Electric Dipole: TEN (cont.)

For the TM reflection coefficient we have

$$\Gamma^{TM} = \frac{Z_{in}^{TM} - Z_0^{TM}}{Z_{in}^{TM} + Z_0^{TM}}$$

where

$$Z_{in}^{TM} = jZ_1^{TM} \tan(k_{z1}h)$$

so

$$Z_{in}^{TM} = j\left(\frac{\eta_0 N_1(\theta)}{\varepsilon_r}\right)\tan\left(k_0 N_1(\theta) h\right)$$

For the air region, we have

$$Z_0^{TM} = \eta_0 \frac{N_0(\theta)}{1} = \eta_0 \sqrt{1-\sin^2\theta} = \eta_0 \cos\theta$$

Electric Dipole: TEN (cont.)

Note that $$1+\Gamma^{TM} = \frac{2Z_{in}^{TM}}{Z_{in}^{TM} + Z_0^{TM}}$$

After simplifying, we obtain the following results.

$$1+\Gamma^{TM}\left(\theta\right) = \frac{2}{1 - j\left(\dfrac{\varepsilon_r \cos(\theta)}{N_1(\theta)}\right)\cot\left(k_0 h N_1(\theta)\right)}$$

Similarly,

$$1+\Gamma^{TE}\left(\theta\right) = \frac{2}{1 - j\left(\dfrac{N_1(\theta)\sec\theta}{\mu_r}\right)\cot\left(k_0 h N_1(\theta)\right)}$$

Electric Dipole: Final Results

Hence, we have the following final results

$$E_{\theta}^{hex}(r,\theta,\phi) = E_0 \cos\phi \; G(\theta)$$

$$E_{\phi}^{hex}(r,\theta,\phi) = E_0 \left(-\sin\phi\right) \; F(\theta)$$

$$G(\theta) = \cos\theta\left(1+\Gamma^{TM}(\theta)\right)$$

$$F(\theta) = 1+\Gamma^{TE}(\theta)$$

$$E_0 = \left(\frac{-j\omega\mu_0}{4\pi r}\right)e^{-jk_0 r}$$

Electric Dipole: Final Results (cont.)

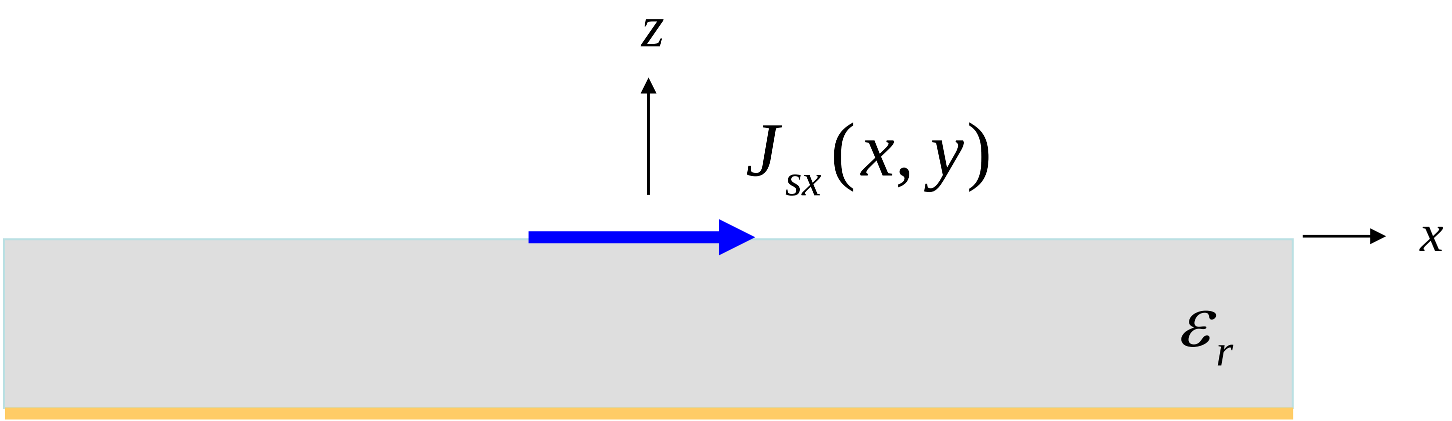
Results for a *y*-directed electric dipole:

$$E_{\theta}^{hey}(r,\theta,\phi) = E_0 \sin\phi \; G(\theta)$$

$$E_{\phi}^{hey}(r,\theta,\phi) = E_0 \cos\phi \; F(\theta)$$

We don’t need this for modeling the (1,0) mode, however.

Far Field of Patch Current



$$E_p^{patch}(r,\theta,\phi) = < b, a >$$

$$= \int_S J_{sx}(x,y)\, E_x^{pw}(x,y,z)\, dS$$

$$= E_x^{pw}(0,0,0) \int_S J_{sx}(x,y)\, e^{j(k_x x + k_y y)} dS$$

$$= E_x^{pw}(0,0,0)\ \tilde{J}_{sx}(k_x, k_y)$$

Far Field of Patch Current (cont.)

Hence

$$E_p^{patch}\left(r,\theta,\phi\right)=E_p^{hex}\left(r,\theta,\phi\right)\tilde{J}_{sx}(k_x,k_y)$$

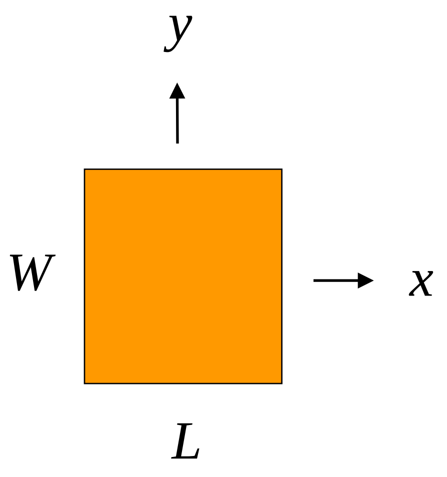
$$k_x=k_0\sin\theta\cos\phi$$

$$k_y=k_0\sin\theta\sin\phi$$

hex = horizontal electric dipole in the *x* direction.

Assume:

$$J_{sx}^{(1,0)}\left(x,y\right)=A_{10}^{J}\cos\left(\frac{\pi x}{L}\right)$$



Far Field of Patch Current (cont.)

For this patch current we have the following Fourier transform:

$$\tilde{J}_{sx}^{(1,0)}\left(k_x,k_y\right) = A_{10}^{J}\left(\frac{\pi}{2}WL\right)\operatorname{sinc}\left[k_y\frac{W}{2}\right]\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

Summary

$$E_p^{patch}\left(r,\theta,\phi\right) = E_p^{hex}\left(r,\theta,\phi\right)\tilde{J}_{sx}^{(1,0)}(k_x,k_y)$$

$\left(p=\theta \text{ or } \phi\right)$

$$k_x = k_0 \sin\theta\cos\phi$$

$$k_y = k_0 \sin\theta\sin\phi$$

$$\tilde{J}_{sx}^{(1,0)}\left(k_x,k_y\right) = A_{10}^J\left(\frac{\pi}{2}WL\right)\operatorname{sinc}\left[k_y\frac{W}{2}\right]\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

$$E_\theta^{hex}(r,\theta,\phi) = E_0\cos\phi\; G(\theta)$$

$$E_\phi^{hex}(r,\theta,\phi) = E_0\left(-\sin\phi\right)\; F(\theta)$$

$$G(\theta) = \cos\theta\left(1+\Gamma^{TM}(\theta)\right)$$

$$F(\theta) = 1+\Gamma^{TE}(\theta)$$

$$E_0 = \left(\frac{-j\omega\mu_0}{4\pi r}\right)e^{-jk_0 r}$$

Assumption: $J_{sx}^{(1,0)}\left(x,y\right) = A_{10}^J\cos\left(\frac{\pi x}{L}\right)$