# ECE 6345

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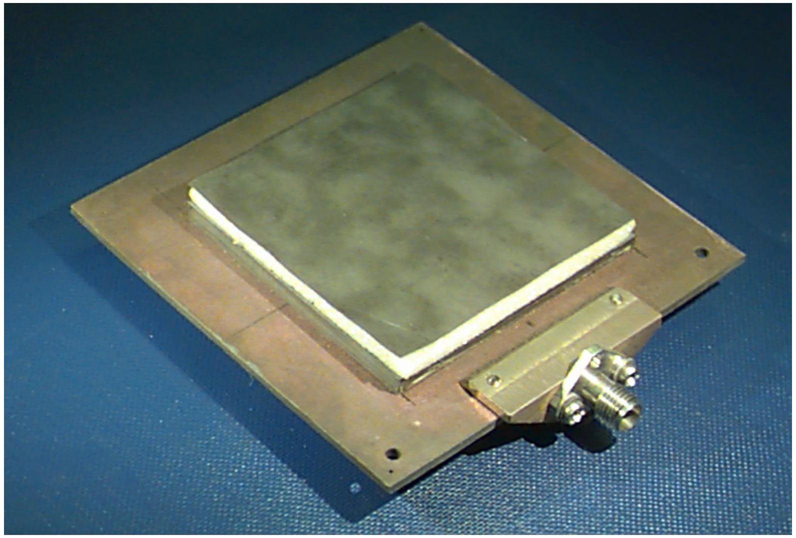

Notes 13

# Overview

In this set of notes we perform the algebra necessary to evaluate the $p$ factor in closed form (assuming a thin substrate) and to simplify the final result.

# Approximation of "*p*"

From Notes 12 we have

$$p = \frac{\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\left[\left|F(\theta)\right|^2\sin^2\phi + \left|G(\theta)\right|^2\cos^2\phi\right]\left|A(\theta,\phi)\right|^2\sin\theta\, d\theta\, d\phi}{\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\left[\left|F(\theta)\right|^2\sin^2\phi + \left|G(\theta)\right|^2\cos^2\phi\right]\left|A(0,0)\right|^2\sin\theta\, d\theta\, d\phi}$$

$$A(\theta,\phi) = \left(\frac{\pi}{2}WL\right)\operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\dfrac{L}{2}\right)}{\left(\dfrac{\pi}{2}\right)^2 - \left(k_x\dfrac{L}{2}\right)^2}\right]$$

$$k_x = k_0\sin\theta\cos\phi$$
$$k_y = k_0\sin\theta\sin\phi$$

# Approximation of "$p$" (cont.)

Assume $h / \lambda_0 \rightarrow 0$

Then we have (see Notes 11):

$$F(\theta) \approx \left(2 j \mu_r \cos\theta\right) k_0 h$$

$$G(\theta) \approx \left(\frac{2j}{\varepsilon_r}\right)\left(n_1^2 - \sin^2\theta\right) k_0 h$$

Also assume that $n_1^2 = \varepsilon_r \mu_r >> \sin^2\theta$

This implies that the patch is fairly small (high permittivity substrate) or that the angles of significant radiation are small.

Then

$$F(\theta) \approx \left(2 j \mu_r \cos\theta\right) k_0 h$$

$$G(\theta) \approx \left(2 j \mu_r\right) k_0 h$$

# Approximation of "*p*" (cont.)

We then have

$$\sin^2\phi\left|F(\theta)\right|^2+\cos^2\phi\left|G(\theta)\right|^2\approx 4\mu_r^2\left(k_0h\right)^2\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)$$

Therefore

$$p\approx\frac{\int_0^{2\pi}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\left|A(\theta,\phi)\right|^2\sin\theta\,d\theta\,d\phi}{\int_0^{2\pi}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\left|A(0,0)\right|^2\sin\theta\,d\theta\,d\phi}$$

Note: The *p* factor is now only a function of the patch dimensions – not the substrate.

# Approximation of "$p$" (cont.)

The patch array factor is

$$A(\theta,\phi)=\left(\frac{\pi}{2}WL\right)\operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

or

$$A(\theta,\phi)=\left(\frac{\pi}{2}\right)^2\left(\frac{2}{\pi}WL\right)\operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

Recall:

$$A(0,0)=\left(\frac{2}{\pi}WL\right)$$

or

$$A(\theta,\phi)=\left(\frac{\pi}{2}\right)^2 A(0,0)\ \operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

# Approximation of "$p$" (cont.)

In the denominator of the $p$ expression we have (after cancelling the $A(0,0)$ term):

$$\int_0^{2\pi}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\sin\theta\,d\theta\,d\phi=\pi\int_0^{\pi/2}\left(\cos^2\theta+1\right)\sin\theta\,d\theta$$

$$=\pi\left(\frac{1}{3}+1\right)$$

$$=\left(\frac{4\pi}{3}\right)$$

# Approximation of "$p$" (cont.)

Hence

$$p \approx \frac{3}{4\pi}\left(\frac{\pi}{2}\right)^4 \int_0^{2\pi}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\mathrm{sinc}^2\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]^2 \sin\theta\, d\theta\, d\phi$$

Using $\int_0^{2\pi}\int_0^{\pi/2}(\quad)\,d\theta\,d\phi = 4\int_0^{\pi/2}\int_0^{\pi/2}(\quad)\,d\theta\,d\phi$ and factoring out a $(\pi/2)^{-4}$, we have

$$p \approx \frac{3}{\pi}\int_0^{\pi/2}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\mathrm{sinc}^2\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{1-\left(\frac{2}{\pi}\left(k_x\frac{L}{2}\right)\right)^2}\right]^2 \sin\theta\, d\theta\, d\phi$$

# Approximation of "*p*" (cont.)

Next, use [Abromowitz & Stegun]

$$\frac{\sin x}{x} \approx 1 + a_2 x^2 + a_4 x^4$$

$$\cos x \approx 1 + b_2 x^2 + b_4 x^4$$

$$x = k_y \frac{W}{2}$$

$$x = k_x \frac{L}{2}$$

for $0 \le x \le \dfrac{\pi}{2}$

where

$$a_2 = -0.16605$$
$$a_4 = 0.00761$$
$$b_2 = -0.49670$$
$$b_4 = 0.03705$$

Note: These are not Taylor series, but are approximations that are more uniformly accurate over the entire range.

# Approximation of "$p$" (cont.)

The cosine term may thus be approximated as

$$\frac{\cos x}{1-\left(\frac{2}{\pi}x\right)^2} \approx \left[1+b_2x^2+b_4x^4\right]\left[1+\left(\frac{2}{\pi}x\right)^2+\left(\frac{2}{\pi}x\right)^4\right]$$

where we have used use a Taylor series for $\left[1-\left(\frac{2}{\pi}x\right)^2\right]^{-1}$

We then have (keeping terms up to $x^4$)

$$\frac{\cos x}{1-\left(\frac{2}{\pi}x\right)^2} \approx 1+x^2\left[b_2+\frac{4}{\pi^2}\right]+x^4\left(b_4+b_2\frac{4}{\pi^2}+\frac{16}{\pi^4}\right)$$

# Approximation of “p” (cont.)

Define
$$c_2 = b_2 + \frac{4}{\pi^2}$$
$$c_4 = b_4 + b_2\frac{4}{\pi^2} + \frac{16}{\pi^4}$$

The numerical values are
$$c_2 = -0.0914153$$
$$c_4 = 7.884\times 10^{-7} \approx 0$$

Then
$$\frac{\cos x}{1-\left(\frac{2}{\pi}x\right)^2} \approx 1 + c_2 x^2 + \cancel{c_4} x^4$$

# Approximation of "$p$" (cont.)

We then have

$$p \approx \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \left( \cos^2\theta \sin^2\phi + \cos^2\phi \right)$$

$$\cdot \left[ 1 + a_2 \left( \frac{k_0 W}{2} \sin\theta \sin\phi \right)^2 + a_4 \left( \frac{k_0 W}{2} \sin\theta \sin\phi \right)^4 \right]^2$$

$$\cdot \left[ 1 + c_2 \left( \frac{k_0 L}{2} \sin\theta \cos\phi \right)^2 \right]^2 \sin\theta \; d\theta \, d\phi$$

# Approximation of "*p*" (cont.)

Neglect the following terms:

$$a_4^2,\ a_4 a_2,\ c_2^2$$

We then have

$$p \approx \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \left(\cos^2\theta \sin^2\phi + \cos^2\phi\right)$$

$$\cdot \left[ 1 + 2a_2 \left( \frac{k_0 W}{2} \sin\theta \sin\phi \right)^2 + \left(a_2^2 + 2a_4\right) \left( \frac{k_0 W}{2} \sin\theta \sin\phi \right)^4 \right]$$

$$\cdot \left[ 1 + 2c_2 \left( \frac{k_0 L}{2} \sin\theta \cos\phi \right)^2 \right] \sin\theta \, d\theta \, d\phi$$

# Approximation of "$p$" (cont.)

Next, we also neglect the following terms:

$$a_4 c_2, a_2^2 c_2$$

We then have

$$p \approx \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \left( \cos^2\theta \sin^2\phi + \cos^2\phi \right) \cdot \left[ 1 + 2a_2 \left( \frac{k_0 W}{2} \sin\theta \sin\phi \right)^2 + \left( a_2^2 + 2a_4 \right) \left( \frac{k_0 W}{2} \sin\theta \sin\phi \right)^4 + 2c_2 \left( \frac{k_0 L}{2} \sin\theta \cos\phi \right)^2 + 4a_2 c_2 \left( \frac{k_0 W}{2} \sin\theta \sin\phi \right)^2 \left( \frac{k_0 L}{2} \sin\theta \cos\phi \right)^2 \right] \cdot \sin\theta \, d\theta \, d\phi$$

# Approximation of "$p$" (cont.)

Expanding, we have

$$
\begin{aligned}
p = \frac{3}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \Big\{ & \sin^2\phi\cos^2\theta\sin\theta + \cos^2\phi\sin\theta \\
& + 2a_2\left(\frac{k_0W}{2}\right)^2 \sin^4\phi\sin^3\theta\cos^2\theta + 2a_2\left(\frac{k_0W}{2}\right)^2 \sin^2\phi\cos^2\phi\sin^3\theta \\
& + \left(a_2^2+2a_4\right)\left(\frac{k_0W}{2}\right)^4 \sin^6\phi\sin^5\theta\cos^2\theta + \left(a_2^2+2a_4\right)\left(\frac{k_0W}{2}\right)^4 \sin^4\phi\cos^2\phi\sin^5\theta \\
& + 2c_2\left(\frac{k_0L}{2}\right)^2 \cos^2\phi\sin^2\phi\cos^2\theta\sin^3\theta + 2c_2\left(\frac{k_0L}{2}\right)^2 \cos^4\phi\sin^3\theta \\
& + 4a_2c_2\left(\frac{k_0W}{2}\right)^2\left(\frac{k_0L}{2}\right)^2 \sin^4\phi\cos^2\phi\sin^5\theta\cos^2\theta + 4a_2c_2\left(\frac{k_0W}{2}\right)^2\left(\frac{k_0L}{2}\right)^2 \cos^4\phi\sin^2\phi\sin^5\theta \Big\}\, d\theta\, d\phi
\end{aligned}
$$

# Approximation of "*p*" (cont.)

All of the $\phi$ integrals may now be done in closed form:

$$\int_0^{\frac{\pi}{2}} \sin^2\phi\, d\phi = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^2\phi \cos^2\phi\, d\phi = \frac{\pi}{16}$$

$$\int_0^{\frac{\pi}{2}} \cos^2\phi\, d\phi = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^6\phi\, d\phi = \frac{5\pi}{32}$$

$$\int_0^{\frac{\pi}{2}} \sin^4\phi\, d\phi = \frac{3\pi}{16}$$

$$\int_0^{\frac{\pi}{2}} \sin^2\phi \cos^4\phi\, d\phi = \frac{\pi}{32}$$

$$\int_0^{\frac{\pi}{4}} \cos^4\phi\, d\phi = \frac{3\pi}{16}$$

$$\int_0^{\frac{\pi}{2}} \sin^4\phi \cos^2\phi\, d\phi = \frac{\pi}{32}$$

# Approximation of "$p$" (cont.)

All of the $\theta$ integrals may also be done in closed form:

$$\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{8}{15}$$

$$\int_0^{\frac{\pi}{2}} \sin^3\theta \, d\theta = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta \, d\theta = \frac{2}{15}$$

$$\int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta = \frac{1}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^2\theta \, d\theta = \frac{8}{105}$$

# Approximation of "$p$" (cont.)

This yields

$$p = \frac{3}{\pi}$$

$$\cdot\Bigg\{ \frac{\pi}{4}\frac{1}{3} + \frac{\pi}{4}1 + 2a_2\left(\frac{k_0 W}{2}\right)^2 \frac{3\pi}{16}\frac{2}{15} + 2a_2\left(\frac{k_0 W}{2}\right)^2 \frac{\pi}{16}\frac{2}{3}$$

$$+\left(a_2^2 + 2a_4\right)\left(\frac{k_0 W}{2}\right)^4 \frac{5\pi}{32}\frac{8}{105} + \left(a_2^2 + 2a_4\right)\left(\frac{k_0 W}{2}\right)^4 \frac{\pi}{32}\frac{8}{15}$$

$$+2c_2\left(\frac{k_0 L}{2}\right)^2 \frac{\pi}{16}\frac{2}{15} + 2c_2\left(\frac{k_0 L}{2}\right)^2 \frac{3\pi}{16}\frac{2}{3}$$

$$+4a_2c_2\left(\frac{k_0 W}{2}\right)^2\left(\frac{k_0 L}{2}\right)^2 \frac{\pi}{32}\frac{8}{105} + 4a_2c_2\left(\frac{k_0 W}{2}\right)^2\left(\frac{k_0 L}{2}\right)^2 \frac{\pi}{32}\frac{8}{15} \Bigg\}$$

# Approximation of "$p$" (cont.)

Simplifying, we obtain

$$p = 1 + \frac{a_2}{10}\left(k_0 W\right)^2 + \left(a_2^2 + 2a_4\right)\left(\frac{3}{560}\right)\left(k_0 W\right)^4 + c_2\left(\frac{1}{5}\right)\left(k_0 L\right)^2 + a_2 c_2\left(\frac{1}{70}\right)\left(k_0 W\right)^2\left(k_0 L\right)^2$$

$$a_2 = -0.16605$$
$$a_4 = 0.00761$$
$$c_2 = -0.0914153$$