ECE 6345

Spring 2015

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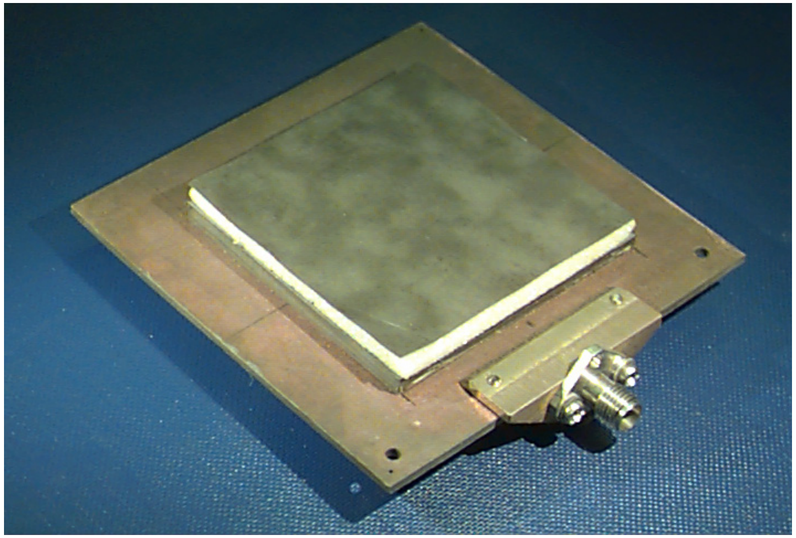


Notes 16

Overview

In this set of notes we calculate the power radiated into space by the circular patch.

This will lead to Q_{sp} of the circular patch.

Radiated Power of Circular Patch

Assumption:

$$E_z\left(a,\phi\right)=\cos\phi$$

From Notes 10 we have:

$$E_\theta^{FF}\left(r,\theta,\phi\right)=\frac{E_0}{\eta_0}(ah)\cos\phi\,\text{tanc}(k_{z1}h)\,Q(\theta)\;2\pi\,J_1'(k_0 a\sin\theta)$$

$$E_\phi^{FF}\left(r,\theta,\phi\right)=-\frac{E_0}{\eta_0}(ah)\sin\phi\,\text{tanc}\left(k_{z1}h\right)P(\theta)\;2\pi J_{inc}(k_o a\sin\theta)$$

where

$$Q(\theta)=1-\Gamma^{TM}(\theta)$$

$$P(\theta)\equiv\cos\theta\left(1-\Gamma^{TE}(\theta)\right)$$

$$J_{inc}(x)\equiv\frac{J_1(x)}{x}$$

Radiated Power of Circular Patch (cont.)

The power density in the far field from the Poynting vector is

$$S_r\left(r,\theta,\phi\right) = \frac{1}{2\eta_0}\left[\left|E_\theta\right|^2 + \left|E_\phi\right|^2\right]$$

$$= \frac{1}{2\eta_0}\frac{\left|E_0\right|^2}{\eta_0^2}\left(a\,h\right)^2 \operatorname{tanc}^2\left(k_{z1}\,h\right)\left(2\pi\right)^2$$

$$\cdot\left[\cos^2\phi\, J_1'^2\left(k_0 a \sin\theta\right)\left|Q(\theta)\right|^2 + \sin^2\phi\, J_{inc}^2(k_o a \sin\theta)\left|P(\theta)\right|^2\right]$$

Next, use $\left|E_0\right| = \dfrac{\omega\,\mu_0}{4\pi\, r} = \dfrac{k_0\,\eta_0}{4\pi\, r}$

Radiated Power of Circular Patch (cont.)

We then have

$$S_r\left(r,\theta,\phi\right)=\frac{1}{8\eta_0}\left(k_0\,a\right)^2 h^2\,\mathrm{tanc}^2\left(k_{z1}\,h\right)$$

$$\cdot\left[\cos^2\phi\left|Q(\theta)\right|^2 J_1'^2\left(k_o\,a\sin\theta\right)+\sin^2\phi\left|P(\theta)\right|^2 J_{inc}^2\left(k_0\,a\sin\theta\right)\right]\frac{1}{r^2}$$

The space-wave power is

$$P_{sp}=\int_0^{2\pi}\int_0^{\pi/2} S_r\left(r,\theta,\phi\right)\,r^2\sin\theta\,d\theta\,d\phi$$

Performing the ϕ integrals,

$$P_{sp}=\pi\frac{1}{8\eta_0}\left(k_0 a\right)^2 h^2$$

$$\cdot\int_0^{\pi/2}\mathrm{tanc}^2\left(k_{z1}\,h\right)\sin\theta\left[\left|Q(\theta)\right|^2 J_1'^2\left(k_0 a\sin\theta\right)+\left|P(\theta)\right|^2 J_{inc}^2\left(k_0 a\sin\theta\right)\right]d\theta$$

Radiated Power of Circular Patch (cont.)

Define
$$I_c = \int_0^{\pi/2} C(\theta)\, d\theta$$

where

$$C(\theta) = \sin\theta \,\operatorname{tanc}^2\left(k_{z1} h\right)\left[\left|Q(\theta)\right|^2 J_1'^2\left(k_0 a \sin\theta\right) + \left|P(\theta)\right|^2 J_{inc}^2\left(k_0 a \sin\theta\right)\right]$$

We then have

$$P_{sp} = \frac{\pi}{8\eta_0}(k_0 a)^2 h^2 I_c$$

Note: We will get a CAD formula for I_c later.

Calculation of Q_{sp}

The *Q* formula is $Q_{sp} = \omega_0 \left(\frac{U_S}{P_{sp}} \right)$

$$
\begin{aligned}
U_S &= 2U_E \\
&= 2\int_V \frac{1}{4}\varepsilon_0\,\varepsilon_r\left|E_z\right|^2 dV \\
&= \frac{1}{2}\varepsilon_0\varepsilon_r h \int_S \left|E_z\right|^2 dS \\
&= \frac{1}{2}\varepsilon_0\varepsilon_r h \int_0^{2\pi}\int_0^a \left|E_z\right|^2 \rho\, d\rho\, d\phi \\
&= \pi\frac{1}{2}\varepsilon_0\varepsilon_r h \int_0^a \left|E_z\right|^2 \rho\, d\rho
\end{aligned}
$$

The electric field is

$$E_z(\rho,\phi) = \cos\phi \left[\frac{J_1(k\rho)}{J_1(ka)} \right]$$

Note:

$$E_z(a,\phi) = \cos\phi$$

Calculation of Q_{sp} (cont.)

The stored energy is then

$$U_S = \frac{1}{2}\varepsilon_0\varepsilon_r h\pi \frac{1}{J_1^2(ka)}\int_0^a J_1^2(k\rho)\rho\, d\rho$$

Denote

$$I_1 = \int_0^a J_1^2(k\rho)\rho\, d\rho = \left[\frac{1}{2}\rho^2 J_1'^2(k\rho) + \frac{1}{2}\rho^2\left(1-\frac{1}{(k\rho)^2}\right)J_1^2(k\rho)\right]_0^a$$

$$= \frac{a^2}{2}\left[J_1'^2(ka) + \left(1-\frac{1}{(ka)^2}\right)J_1^2(ka)\right]$$

Calculation of Q_{sp} (cont.)

Recall that $k = \dfrac{x'_{11}}{a}$ $\quad x'_{11} = 1.84118$

so $J'_1(ka) = J'_1(x'_{11}) = 0$

Hence $I_1 = \dfrac{a^2}{2}\left(1 - \dfrac{1}{x_{11}^2}\right) J_1^2(x'_{11})$

We then have

$$U_S = \frac{1}{2}\varepsilon_0 \varepsilon_r h \pi \left(\frac{1}{J_1^2(x'_{11})}\right)\left[\left(\frac{a^2}{2}\right)\left(1 - \frac{1}{x'^2_{11}}\right) J_1^2(x'_{11})\right]$$

or

$$U_S = \frac{1}{2}\varepsilon_0 \varepsilon_r h \pi \left[\left(\frac{a^2}{2}\right)\left(1 - \frac{1}{x'^2_{11}}\right)\right]$$

Calculation of Q_{sp} (cont.)

The formula for Q_{sp} then becomes

$$Q_{sp} = \omega_0 \left(\frac{\frac{1}{2}\varepsilon_0 \varepsilon_r h \pi \left(\frac{a^2}{2}\right)\left(1 - \frac{1}{x_{11}'^2}\right)}{\frac{\pi}{8\eta_0}(k_0 a)^2 h^2 I_c} \right)$$

This may be simplified by using the following expressions to eliminate ω_0 and k_0:

$$k_0 \sqrt{\mu_r \varepsilon_r}\, a = x_{11}'$$

$$\omega_0 \sqrt{\mu_0 \varepsilon_0}\, \sqrt{\mu_r \varepsilon_r}\, a = x_{11}'$$

$\Rightarrow$

$$k_0 a = \frac{x_{11}'}{\sqrt{\mu_r \varepsilon_r}}$$

$$\omega_0 = \frac{x_{11}'}{a \sqrt{\mu_0 \varepsilon_0}\, \sqrt{\mu_r \varepsilon_r}}$$

Calculation of Q_{sp} (cont.)

We then have

$$Q_{sp} = \frac{2}{x_{11}'^2}\left(x_{11}'^2 - 1\right)\left(\frac{1}{k_0 h}\right)\varepsilon_r\left[\frac{1}{I_c}\right]$$

Note that Q_{sp} is proportional to the substrate permittivity and inversely proportional to the substrate thickness.

Calculation of Q_{sp} (cont.)

Summary (exact Q_{sp})

$$Q_{sp} = 2\frac{1}{x_{11}'^2}\left(x_{11}'^2 - 1\right)\left(\frac{1}{I_c}\right)\left(\frac{1}{k_0 h}\right)\varepsilon_r$$

$$I_c \equiv \int_0^{\pi/2} C(\theta)\, d\theta$$

$$x_{11}' = 1.84118$$

$$C(\theta) = \sin\theta \ \operatorname{tanc}^2\left(k_{z1} h\right)\left[\left|Q(\theta)\right|^2 J_1'^2\left(k_0 a \sin\theta\right) + \left|P(\theta)\right|^2 J_{inc}^2\left(k_0 a \sin\theta\right)\right]$$

The p Factor

We can express the Q_{sp} formula in terms of a p factor (which will eventually be approximated in closed form).

Define:

$$C_0(\theta) \equiv C(\theta) \Big|_{a \to 0}$$

$$I_0 \equiv \int_0^{\pi/2} C_0(\theta)\, d\theta$$

The term C_0 ignores the patch array factor.

Also, define:

$$p \equiv \frac{I_c}{I_0} = \frac{\int_0^{\pi/2} C(\theta)\, d\theta}{\int_0^{\pi/2} C_0(\theta)\, d\theta}$$

The p term gives the ratio of the power radiated by the actual patch to the power radiated if we ignore the array factor, and collapse the magnetic current down to a single dipole.

(See the end of the notes for a derivation of the equivalent dipole moment of the circular patch.)

The p Factor (cont.)

Then we have

$$Q_{sp} = 2\frac{1}{x_{11}'^2}\left(x_{11}'^2 - 1\right)\left(\frac{1}{p}\right)\left(\frac{1}{I_0}\right)\left(\frac{1}{k_0 h}\right)\varepsilon_r$$

Note that as $x \to 0$

$$J_1'(x) \to \frac{1}{2}$$

$$J_{inc}(x) \to \frac{1}{2}$$

This allows us to express I_0 in a simpler form without the Bessel functions:

$$I_0 = \int_0^{\pi/2} \sin\theta \, \mathrm{tanc}^2\left(k_0 h \, N_1(\theta)\right)\frac{1}{4}\left[\left|P(\theta)\right|^2 + \left|Q(\theta)\right|^2\right] d\theta$$

The p Factor (cont.)

For the p factor we have:

$$p = \frac{\displaystyle\int_0^{\pi/2} \sin\theta \,\mathrm{tanc}^2\left(k_0 h\, N_1(\theta)\right)\left[\left|Q(\theta)\right|^2 J_1'^2\left(k_0 a \sin\theta\right) + \left|P(\theta)\right|^2 J_{inc}^2\left(k_0 a \sin\theta\right)\right] d\theta}{\displaystyle\int_0^{\pi/2} \sin\theta \,\mathrm{tanc}^2\left(k_0 h\, N_1(\theta)\right)\frac{1}{4}\left[\left|P(\theta)\right|^2 + \left|Q(\theta)\right|^2\right] d\theta}$$

The term p depends on the patch radius a and the substrate parameters.
(After making some approximations, it will depend only on the patch radius.)

Approximation for a Thin Substrate

For a thin substrate, we have:

$$\mathrm{tanc}^2\left(k_0 h\, N_1(\theta)\right) \approx 1$$

so

$$I_0 \approx \int_0^{\pi/2} \sin\theta \, \frac{1}{4}\left[\left|P(\theta)\right|^2 + \left|Q(\theta)\right|^2\right] d\theta$$

$$p \approx \frac{\displaystyle\int_0^{\pi/2} \sin\theta \left[\left|Q(\theta)\right|^2 J_1'^2\left(k_0 a \sin\theta\right) + \left|P(\theta)\right|^2 J_{inc}^2\left(k_0 a \sin\theta\right)\right] d\theta}{\displaystyle\int_0^{\pi/2} \sin\theta \, \frac{1}{4}\left[\left|P(\theta)\right|^2 + \left|Q(\theta)\right|^2\right] d\theta}$$

Approximation for a Thin Substrate (cont.)

From Notes 9 we also have

$$P(\theta) = \cos\theta\,(1-\Gamma^{TE}(\theta)) = \frac{2\cos\theta}{1 + j\left(\dfrac{\mu_r \cos\theta}{N_1(\theta)}\right)\tan\left(k_0 h\, N_1(\theta)\right)}$$

$$Q(\theta) = 1 - \Gamma^{TM}(\theta) = \frac{2}{1 + j\left(\dfrac{N_1(\theta)\sec\theta}{\varepsilon_r}\right)\tan\left(k_0 h\, N_1(\theta)\right)}$$

For a thin substrate we then have

$$P(\theta) \approx 2\cos\theta$$

$$Q(\theta) \approx 2$$

Approximation for a Thin Substrate (cont.)

For the I_0 term (the denominator of the p function) we then have:

$$I_0 \approx \int_0^{\pi/2} \frac{1}{4} \sin\theta \left[4\left(\cos^2\theta + 1\right) \right] d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left(\cos^2\theta + 1\right) d\theta$$

This yields

$$I_0 \approx \frac{4}{3}$$

Approximation for a Thin Substrate (cont.)

The formula for the p function then becomes

$$p \approx \frac{3}{4} \int_0^{\pi/2} \sin\theta \left[\left|2\right|^2 J_1'^2 \left(k_0 a \sin\theta\right) + \left|2\cos\theta\right|^2 J_{inc}^2 \left(k_0 a \sin\theta\right) \right] d\theta$$

so that

$$p \approx 3 \int_0^{\pi/2} \sin\theta \left[J_1'^2 \left(k_0 a \sin\theta\right) + \cos^2\theta \; J_{inc}^2 \left(k_0 a \sin\theta\right) \right] d\theta$$

The p factor now only depends only on the patch size.

Approximation for a Thin Substrate (cont.)

Summary (approximate Q_{sp})

$$Q_{sp} = 2\frac{1}{x_{11}'^2}\left(x_{11}'^2 - 1\right)\left(\frac{1}{p}\right)\left(\frac{1}{I_0}\right)\left(\frac{1}{k_0 h}\right)\varepsilon_r$$

$$x_{11}' = 1.84118$$

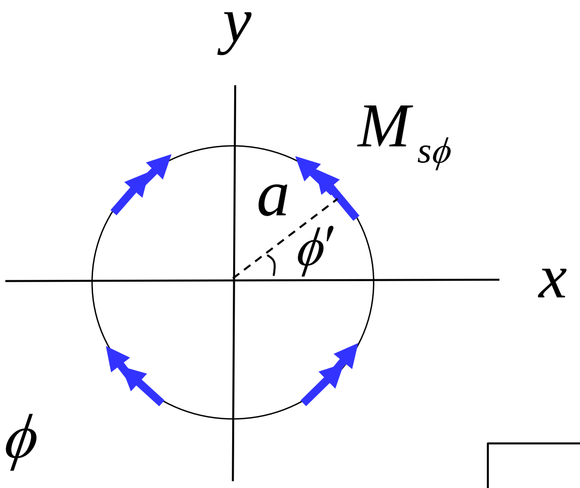
$$I_0 \approx \frac{4}{3}$$

$$p \approx 3\int_0^{\pi/2} \sin\theta \left[J_1'^2\left(k_0 a \sin\theta\right) + \cos^2\theta \, J_{inc}^2\left(k_0 a \sin\theta\right)\right] d\theta$$

Equivalent Dipole Moment of Circular Patch

Consider an equivalent magnetic dipole that models the patch:

Patch

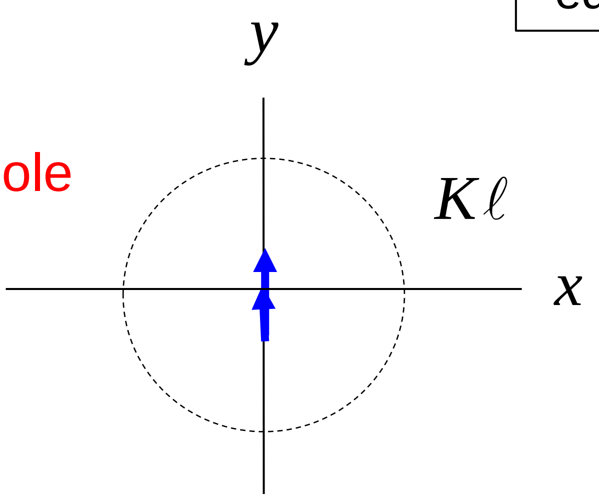


$$M_{s\phi} = \cos\phi$$

$h / \lambda_0 \ll 1$

As $a \to 0$ the magnetic current sheet approaches an equivalent magnetic dipole.

Equivalent dipole



Equivalent Dipole Moment of Circular Patch (cont.)

The dipole moment of the equivalent magnetic dipole is calculated:

$$Kl = \int_S \underline{M}_s \cdot \underline{\hat{y}}\, dS = h \int_0^{2\pi} M_{s\phi} \cos\phi \, a \, d\phi = h \int_0^{2\pi} \cos\phi \cos\phi \, a d\phi =$$

$$ha \int_0^{2\pi} \cos^2\phi \, a d\phi$$

This yields $Kl = \pi a h$

Equivalent Dipole Moment of Circular Patch (cont.)

We can therefore physically interpret the p factor as follows:

$$p = \frac{P_{rad}^{patch}}{P_{rad}^{dip}}$$

where

P_{rad}^{patch} = power radiated by circular patch

P_{rad}^{dip} = power radiated by magnetic dipole of equal moment

$$\left(Kl = \pi a h \right)$$

CAD Formula for p

In the next set of notes we will obtain approximate closed-form CAD expression for p.