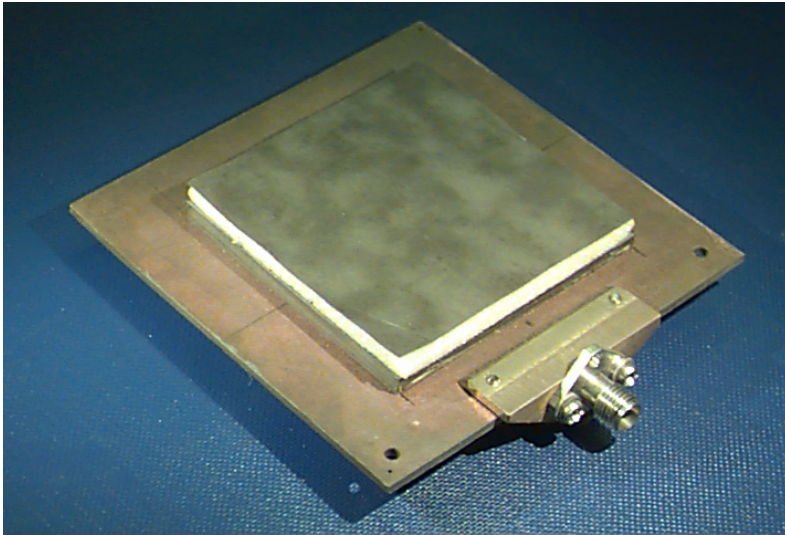


ECE 6345

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Notes 17

Approximation of “ p ” (cont.)

In this set of notes we approximate the p factor for the circular patch to obtain an approximate closed-form CAD formula for it.

Approximation of “ p ” (cont.)

From Notes 16 we have:

$$Q_{sp} = 2 \frac{1}{x'_{11}} (x'^2_{11} - 1) \left(\frac{1}{p} \right) \left(\frac{1}{I_0} \right) \left(\frac{1}{k_0 h} \right) \epsilon_r$$

$$x'_{11} = 1.84118 \quad I_0 \approx \frac{4}{3}$$

$$p \approx 3 \int_0^{\pi/2} \sin \theta \left[J_1'^2(k_0 a \sin \theta) + \cos^2 \theta J_{inc}^2(k_0 a \sin \theta) \right] d\theta$$

The goal is to approximate this integral for the p factor in closed form.

Approximation of “ p ” (cont.)

$$p \approx 3 \int_0^{\pi/2} \sin \theta \left[J_1'^2(k_0 a \sin \theta) + \cos^2 \theta J_{inc}^2(k_0 a \sin \theta) \right] d\theta$$

Using

$$J_1'(x) = J_0(x) - \frac{J_1(x)}{x}$$

we have

$$p \approx 3 \int_0^{\pi/2} \sin \theta \left[\left(J_0(x) - \frac{J_1(x)}{x} \right)^2 + \cos^2 \theta \left(\frac{J_1(x)}{x} \right)^2 \right] d\theta$$

where $x = k_0 a \sin \theta$

We next approximate the Bessel function terms appearing in this expression.

Approximation of “ p ” (cont.)

From Abramowitz and Stegun, we have the following approximations:

$$J_0(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10} + a_{12}x^{12}$$

$$a_0 = 1.0$$

$$a_2 = -0.24999997$$

$$a_4 = 0.015624948$$

$$a_6 = -4.340008 \times 10^{-4}$$

$$a_8 = 6.77456 \times 10^{-6}$$

$$a_{10} = -6.6799 \times 10^{-8}$$

$$a_{12} = 3.951 \times 10^{-10}$$

$$\text{error} < 5 \times 10^{-8}, \quad 0 < x \leq +3$$

We keep (a_0, a_2, a_4, a_6) error < 0.001 for $x \leq x'_{11} = 1.8412$

Approximation of “ p ” (cont.)

Also, from Abramowitz and Stegun, we have:

$$\frac{J_1(x)}{x} = b_0 + b_2x^2 + b_4x^4 + b_6x^6 + b_8x^8 + b_{10}x^{10} + b_{12}x^{12}$$

$$b_0 = 0.5$$

$$b_2 = -0.062499983$$

$$b_4 = 0.0026041448$$

$$b_6 = -5.424265 \times 10^{-5}$$

$$b_8 = 6.75688 \times 10^{-7}$$

$$b_{10} = -5.3788 \times 10^{-9}$$

$$b_{12} = 2.087 \times 10^{-11}$$


$$\text{error} < 1.3 \times 10^{-3}, \quad -3 \leq x \leq 3$$

We keep (b_0, b_2, b_4, b_6) error < 0.001 for $x \leq x'_{11} = 1.8412$

Approximation of “ p ” (cont.)

Define $c_n = a_n - b_n$

We then have

$$p = 3 \int_0^{\pi/2} \sin \theta \left(\sum_{n=0}^3 c_{2n} A^{2n} \sin^{2n} \theta \right)^2 d\theta + 3 \int_0^{\pi/2} (\sin \theta - \sin^3 \theta) \left(\sum_{n=0}^3 b_{2n} A^{2n} \sin^{2n} \theta \right)^2 d\theta$$


where

This comes from $\sin \theta \cos^2 \theta$.

$$A = k_0 a$$

Approximation of “ p ” (cont.)

In order to evaluate the integrals that appear, we use

$$\begin{aligned} \left(\sum_{n=0}^3 b_{2n} A^{2n} \sin^{2n} \theta \right)^2 &= \left(\sum_{n=0}^3 b_{2n} A^{2n} \sin^{2n} \theta \right) \left(\sum_{m=0}^3 b_{2m} A^{2m} \sin^{2m} \theta \right) \\ &= \sum_{n=0}^3 \sum_{m=0}^3 b_{2n} b_{2m} A^{2n+2m} \sin^{2n+2m} \theta \end{aligned}$$

Similarly,

$$\left(\sum_{n=0}^3 c_{2n} A^{2n} \sin^{2n} \theta \right)^2 = \sum_{n=0}^3 \sum_{m=0}^3 c_{2n} c_{2m} A^{2n+2m} \sin^{2n+2m} \theta$$

Approximation of “ p ” (cont.)

Hence

$$p = 3 \sum_{n=0}^3 \sum_{m=0}^3 c_{2n} c_{2m} A^{2(n+m)} \int_0^{\pi/2} \sin^{2(n+m)+1} \theta \, d\theta$$
$$+ 3 \sum_{n=0}^3 \sum_{m=0}^3 b_{2n} b_{2m} A^{2(n+m)} \left\{ \int_0^{\pi/2} \sin^{2(n+m)+1} \theta \, d\theta - \int_0^{\pi/2} \sin^{2(n+m)+3} \theta \, d\theta \right\}$$

Approximation of “ p ” (cont.)

Define $S_n = \int_0^{\pi/2} \sin^n \theta \, d\theta$

General formula: $S_{2m+1} = \frac{(2^m m!)^2}{(2m+1)!} \quad m \geq 1$

Values:

$$S_0 = \pi / 2$$

$$S_1 = 1.0$$

$$S_3 = 2/3$$

$$S_5 = 8/15$$

$$S_7 = 16/35$$

$$S_9 = 128/315$$

$$S_{11} = 256/693$$

$$S_{13} = 1024/3003$$

$$S_{15} = 2048/6435$$

Approximation of “ p ” (cont.)

We then have

$$p = 3 \sum_{n=0}^3 \sum_{m=0}^3 c_{2n} c_{2m} A^{2(n+m)} S_{2(n+m)+1} + 3 \sum_{n=0}^3 \sum_{m=0}^3 b_{2n} b_{2m} A^{2(n+m)} (S_{2(n+m)+1} - S_{2(n+m)+3})$$

Collecting the terms of the series, we have

$$p = \sum_{k=0}^6 A^{2k} e_{2k}$$

where

$$e_0 = 3c_0^2 S_1 + 3b_0^2 (S_1 - S_3)$$

$$e_2 = 6c_0 c_2 S_3 + 6b_0 b_2 (S_3 - S_5)$$

$$e_4 = 3(c_2^2 + 2c_0 c_4) S_5 + 3(b_2^2 + 2b_0 b_4) (S_5 - S_7)$$

$$e_6 = 3(2c_0 c_6 + 2c_2 c_4) S_7 + 3(2b_0 b_6 + 2b_2 b_4) (S_7 - S_9)$$

$$e_8 = 3(c_4^2 + 2c_2 c_6) S_9 + 3(b_4^2 + 2b_2 b_6) (S_9 - S_{11})$$

$$e_{10} = 6c_4 c_6 S_{11} + 6b_4 b_6 (S_{11} - S_{13})$$

$$e_{12} = 3c_6^2 S_{13} + 3b_6^2 (S_{13} - S_{15})$$

Approximation of “ p ” (cont.)

Hence we have

$$p \approx \sum_{k=0}^6 (k_0 a)^{2k} e_{2k}$$

where

$$e_0 = 1$$

$$e_2 = -0.400000$$

$$e_4 = 0.0785710$$

$$e_6 = -7.27509 \times 10^{-3}$$

$$e_8 = 3.81786 \times 10^{-4}$$

$$e_{10} = -1.09839 \times 10^{-5}$$

$$e_{12} = 1.47731 \times 10^{-7}$$