ECE 6382

Fall 2023

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Notes 1

Introduction to Complex Variables

Notes are adapted from D. R. Wilton, Dept. of ECE

Some Applications of Complex Variables

- Phasor-domain analysis in physics and engineering
- Laplace and Fourier transforms
- Series expansions (Taylor, Laurent)
- Evaluation of integrals
- Asymptotics (method of steepest descent)
- Conformal Mapping (solution of Laplace's equation)
- Radiation physics (branch cuts, poles)

Complex Arithmetic and Algebra

A complex number z may be thought of simply as an ordered pair of real numbers (x, y) with rules for addition, multiplication, etc.

$$z = x + iy \quad \left(i \; (= j) = \sqrt{-1},\; x = \operatorname{Re} z,\; y = \operatorname{Im} z \right) \quad \Leftrightarrow \quad z = (x, y)$$

$$= r(\cos\theta + i\sin\theta) \quad \text{(from figure)}$$

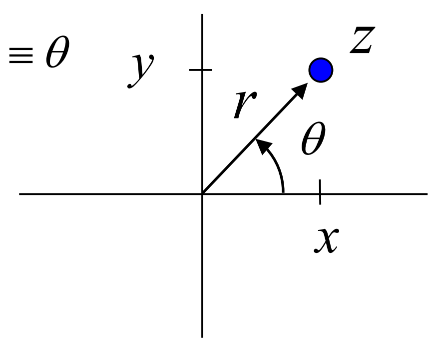
$$= re^{i\theta} \qquad \text{(Euler formula (not yet proven!))}$$

Note: In Euler's formula, the angle θ must be in radians.

z plane

$|z| \equiv r$

$\arg(z) \equiv \theta$



Argand diagram
(polar form)

Note:
We can say that

$$i = \sqrt{-1}$$

But we need to be careful to properly interpret the square root (using the principal branch). This is what the radical sign usually denotes.

$$-\pi < \theta \le \pi$$

Note: Usually we will use i to denote the square root of -1.
However, we will often switch to using *j* when we are doing an engineering example.

Complex Arithmetic and Algebra

Note on phase angle (argument):

The phase angle θ is non-unique. We can add any multiple of 2π (360°) to it. This does not change x and y.

Principal branch:

The most common choice for the "principal branch" is*:

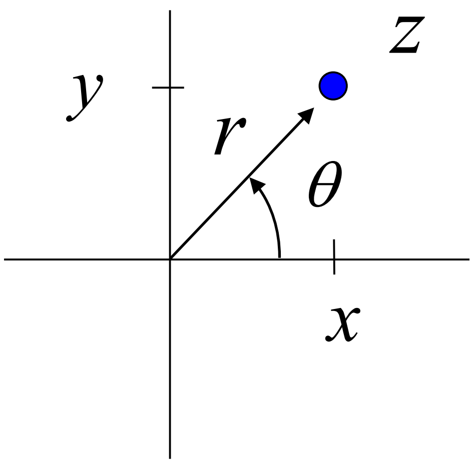
$$-\pi < \theta \le \pi$$

Note:
Adding multiples of 2π to θ will affect some functions, but not others.

Examples: $f(z) = z$ (no effect)

$f(z) = z^{1/2}$ (will effect)

z plane



$$\theta = \theta_p + 2\pi n$$

$$-\pi < \theta_p \le \pi$$

*e.g., the one that Matlab uses

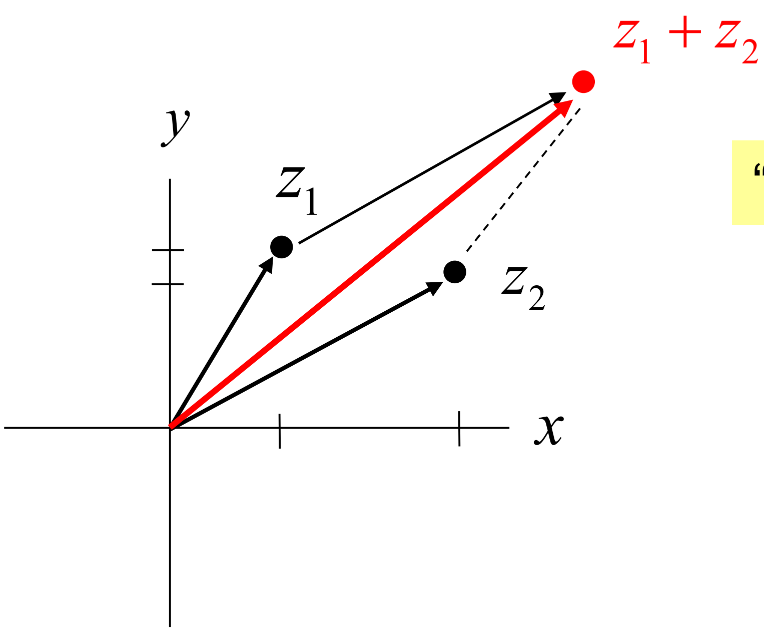
Complex Arithmetic and Algebra (cont.)

Addition / subtraction:

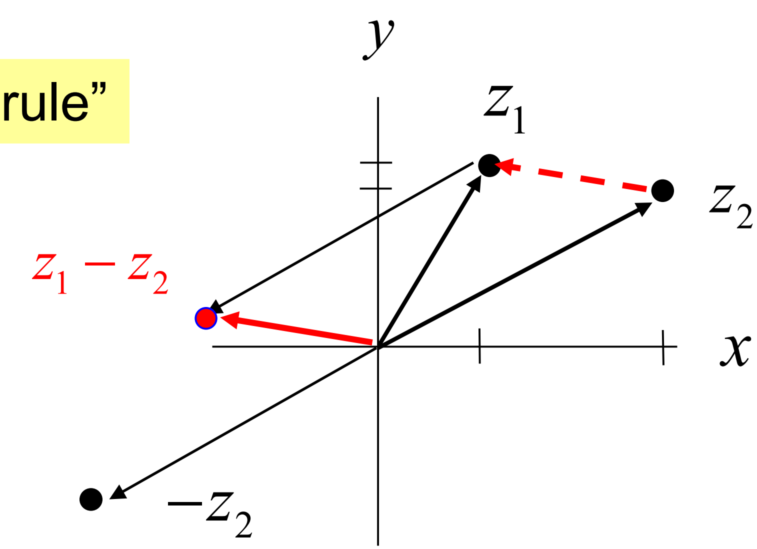
$$z_1 \pm z_2 = \left(x_1 + iy_1\right) \pm \left(x_2 + iy_2\right)$$
$$\equiv \left(x_1 \pm x_2\right) + i\left(y_1 \pm y_2\right)$$

Geometrically, this works the same way and adding and subtracting two-dimensional vectors:

Note:
$z_1 - z_2$ has a vector direction that points from z_2 to z_1.



"tip-to-tail rule"



Complex Arithmetic and Algebra (cont.)

Multiplication:

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$\equiv (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Example: $i^2 = (0+i1)(0+i1) = -1$

Division:

$$z_1 / z_2 = \frac{x_1 + iy_1}{x_2 + iy_2} \times \overbrace{\frac{x_2 - iy_2}{x_2 - iy_2}}^{=1}$$

$$= \frac{x_1x_2 + y_1y_2 + i(y_1x_2 - y_2x_1)}{x_2^2 + y_2^2}$$

$$\Leftrightarrow \quad z_1 / z_2 = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - y_2x_1}{x_2^2 + y_2^2}\right)$$

$$z_1 / z_2 = \left(r_1 e^{i\theta_1}\right) / \left(r_2 e^{i\theta_2}\right) = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}$$

Example: $1/i = -i$

Multiplication and division are easier in polar form!

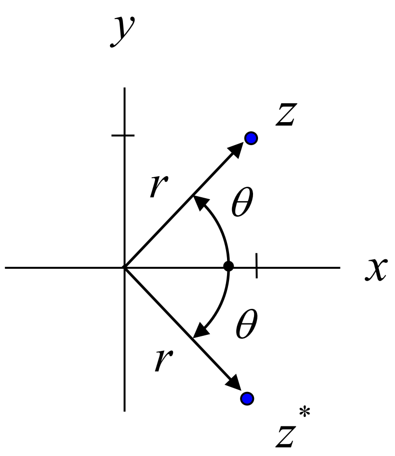
Complex Arithmetic and Algebra (cont.)

Important point:

- We can multiply and divide complex numbers. We cannot divide two-dimensional vectors.

(We can, however, multiply two-dimensional vectors in two different ways, using the dot product and the cross product.)

Complex Arithmetic and Algebra (cont.)



Conjugation:

$$z^* \equiv (x - iy)$$

Magnitude:

$$|z| = \sqrt{z\,z^*}$$

To see this:

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{(x+iy)(x-iy)} = \sqrt{z\,z^*}$$

or

$$|z| = r = \sqrt{\left(re^{i\theta}\right)\left(re^{-i\theta}\right)} = \sqrt{z\,z^*}$$

Euler's Formula

Leonhard Euler

Recall :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Define extension to a complex variable $(x \to z = x + iy)$ **:**

$$e^z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$ **(converges for all z)**

$$\Rightarrow \quad e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\Rightarrow \quad \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ Also : $$\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$$

Note: The variable θ here is usually taken to be real, but it does not have to be.

More generally,

$$\boxed{e^{iz} = \cos z + i\sin z} \qquad \boxed{e^{-iz} = \cos z - i\sin z}$$

$$\Rightarrow \quad \boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}} \qquad \boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}$$

$$\Rightarrow \quad \cos(iz) = \frac{e^z + e^{-z}}{2} = \cosh z, \quad \sin(iz) = -\frac{e^z - e^{-z}}{2i} = i\frac{e^z - e^{-z}}{2} = i\sinh z$$

Application to Trigonometric Identities

Many trigonometric identities follow from a simple application of Euler's formula :

- $$e^{i2\theta} = \cos 2\theta + i\sin 2\theta$$

On the other hand,

$$e^{i2\theta} = \left(e^{i\theta} \right)^2 = \left(\cos\theta + i\sin\theta\right)^2 = \cos^2\theta - \sin^2\theta + i\left(2\cos\theta\sin\theta\right)$$

Equating real and imaginary parts of the two expressions yields *two* identities :

$$\boxed{\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\cos\theta\sin\theta \end{aligned}}$$

- $$e^{i(\theta_1 \pm \theta_2)} = \cos\left(\theta_1 \pm \theta_2\right) + i\sin\left(\theta_1 \pm \theta_2\right)$$

On the other hand,

$$\begin{aligned} e^{i(\theta_1 \pm \theta_2)} &= e^{i\theta_1} \; e^{\pm i\theta_2} \\ &= \left(\cos\theta_1 + i\sin\theta_1\right)\left(\cos\theta_2 \pm i\sin\theta_2\right) \\ &= \left(\cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2\right) + i\left(\sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2\right) \end{aligned}$$

Equating real and imaginary parts yields :

$$\boxed{\begin{aligned} \cos\left(\theta_1 \pm \theta_2\right) &= \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 \\ \sin\left(\theta_1 \pm \theta_2\right) &= \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2 \end{aligned}}$$

DeMoivre's Theorem

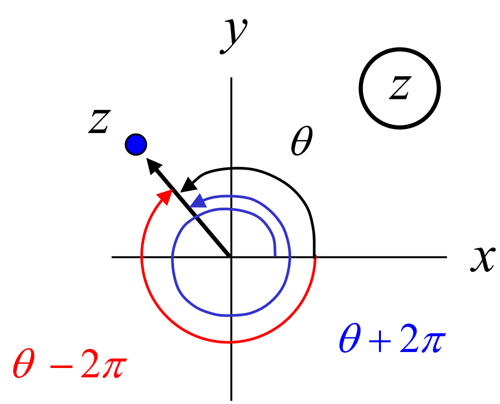
- $z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n\left(\cos n\theta + i\sin n\theta\right)$ **(DeMoivre's Theorem)**

- **Note that for n an integer, the result is *independent* of how θ is measured**

$$\left(re^{i(\theta_p+2\pi k)}\right)^n = r^n e^{i(n\theta_p+2\pi kn)} = r^n\left[\cos\left(n\theta_p+2\pi kn\right)+i\sin\left(n\theta_p+2\pi kn\right)\right] \quad \textbf{(}k \textbf{ an integer)}$$

$$= r^n\left(\cos n\theta_p + i\sin n\theta_p\right)$$

$$= z^n$$



Abraham de Moivre

Roots of a Complex Number

$$w = z^{\frac{1}{n}}$$

❖ In this case the results depend on how θ is measured.

$$z^{\frac{1}{n}} = \left(re^{i(\theta_p + 2\pi k)} \right)^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta_p}{n} + 2\pi \frac{k}{n}\right)} = r^{\frac{1}{n}} \left[\cos\left(\frac{\theta_p}{n} + \frac{2\pi k}{n} \right) + i \sin\left(\frac{\theta}{n} + \frac{2\pi k}{n} \right) \right], \quad k = \overbrace{0, 1, 2, \cdots n-1}^{n \text{ roots}}$$

Example :

$$(-8i)^{\frac{1}{3}} = \left(8e^{-i\frac{\pi}{2} + i2\pi k} \right)^{\frac{1}{3}} = 2e^{-i\frac{\pi}{6} + i\frac{2\pi k}{3}} = 2\left[\cos\left(-\frac{\pi}{6} + \frac{2\pi k}{3} \right) + i \sin\left(-\frac{\pi}{6} + \frac{2\pi k}{3} \right) \right], \quad k = 0, 1, 2$$

$$k = 0: \ (-8i)^{\frac{1}{3}} = 2\left[\cos\left(-\frac{\pi}{6} \right) + i \sin\left(-\frac{\pi}{6} \right) \right] = 2\left[\cos(-30°) + i \sin(-30°) \right] = \cancel{2}\left[\frac{\sqrt{3}}{\cancel{2}} - i\frac{1}{\cancel{2}} \right] = \sqrt{3} - i,$$

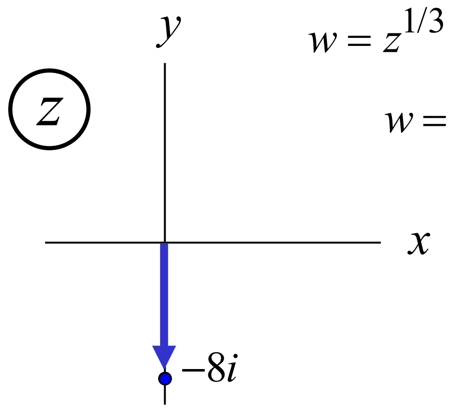
$$k = 1: \ (-8i)^{\frac{1}{3}} = 2\left[\cos\left(-\frac{\pi}{6} + \frac{2\pi}{3} \right) + i \sin\left(-\frac{\pi}{6} + \frac{2\pi}{3} \right) \right] = 2\left[\cos(90°) + i \sin(90°) \right] = 2i,$$

$$k = 2: \ (-8i)^{\frac{1}{3}} = 2\left[\cos\left(-\frac{\pi}{6} + \frac{4\pi}{3} \right) + i \sin\left(-\frac{\pi}{6} + \frac{4\pi}{3} \right) \right] = 2\left[\cos(210°) + i \sin(210°) \right] = -\sqrt{3} - i,$$

Roots of a Complex Number (cont.)

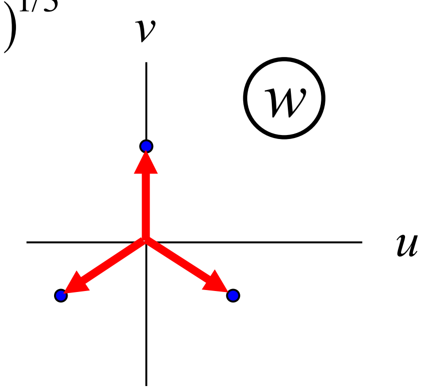
Example (cont.)

- $$(-8i)^{\frac{1}{3}} = \begin{cases} \sqrt{3} - i, \\ 2i, \\ -\sqrt{3} - i \end{cases}$$



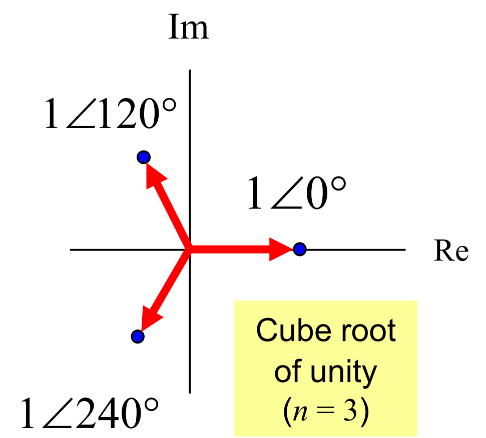
$$w = z^{1/3} = (-8i)^{1/3}$$

$$w = u + iv$$



Note that the n th root of z can also be expressed in terms of the n *th root of unity* :

$$z^{\frac{1}{n}} = \left(r e^{i(\theta_p + 2\pi k)} \right)^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta_p}{n} + 2\pi\frac{k}{n}\right)} = \underbrace{r^{\frac{1}{n}} e^{i\frac{\theta_p}{n}}}_{\text{"principal branch"}} \; \underbrace{e^{i2\pi\frac{k}{n}}}_{n\text{th root of unity}}$$



where $$\underbrace{(1)^{\frac{1}{n}}}_{n\text{th root of unity}} \equiv \left(e^{i2\pi k} \right)^{\frac{1}{n}} = e^{i2\pi\frac{k}{n}} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \qquad k = 0, 1, \cdots, n-1$$