ECE 6382

Fall 2023

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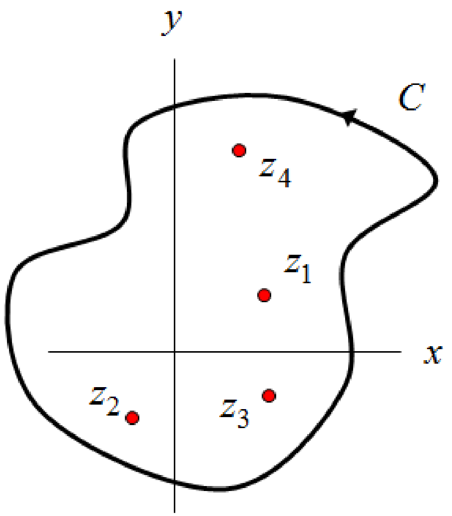


Notes 10

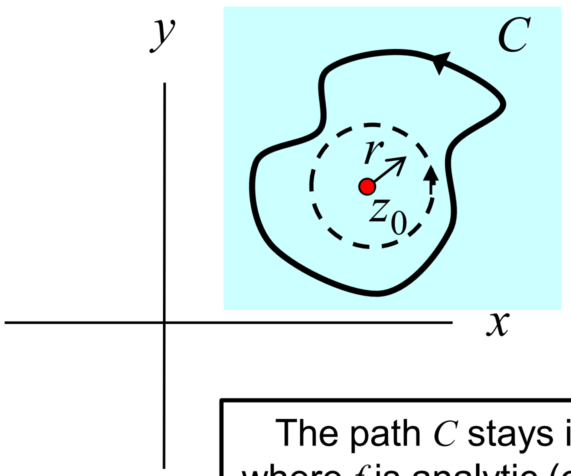
The Residue Theorem and Residue Evaluation

Notes are from D. R. Wilton, Dept. of ECE

The Residue Theorem

Consider a line integral about a path enclosing an isolated singular point:

$$I = \oint_C f(z)\,dz$$



The path C stays in a region where f is analytic (except at z_0).

Expand $f(z)$ in a Laurent series, deform the contour C to a circle of (arbitrary) radius r centered at z_0 that stays inside C, and evaluate the integral:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Let $z - z_0 \equiv re^{i\theta}, \quad dz = rie^{i\theta} d\theta,$

$$\oint_C f(z)\,dz = i \sum_{n=-\infty}^{\infty} a_n r^{n+1} \underbrace{\int_0^{2\pi} e^{i(n+1)\theta} d\theta}_{=0,\ n\neq -1}$$

$$= i\, a_{-1} r^0\, 2\pi$$

$$= 2\pi i\, a_{-1}$$

Note:
For a Laurent series we can integrate term-by-term (switch the order of integration and summation) in the region of convergence, due to uniform convergence.

The Residue Theorem (cont.)

The value a_{-1} corresponding to an isolated singular point z_0 is called the residue of $f(z)$ at z_0.

$$I = \oint_C f(z)\,dz = 2\pi i \operatorname{Res} f(z_0)$$

Note:
The path orientation is assumed counterclockwise.

Laurent series expansion:

$$f(z) = \cdots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

Residue

The Residue Theorem (cont.)

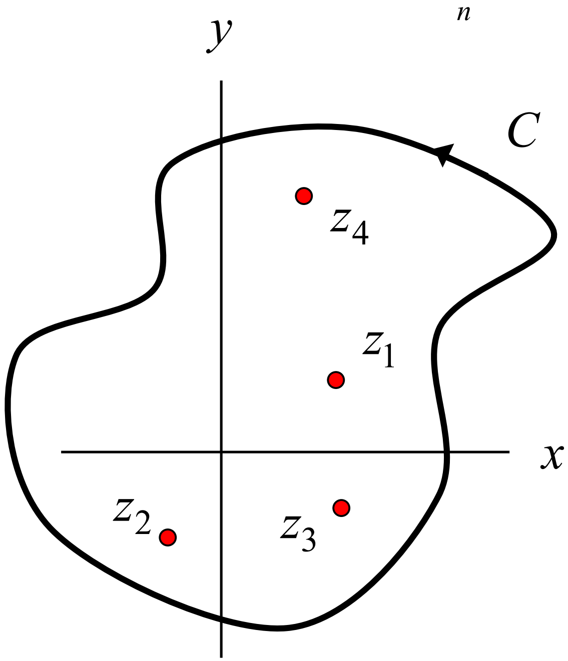
Extend the theorem to multiple isolated singularities:

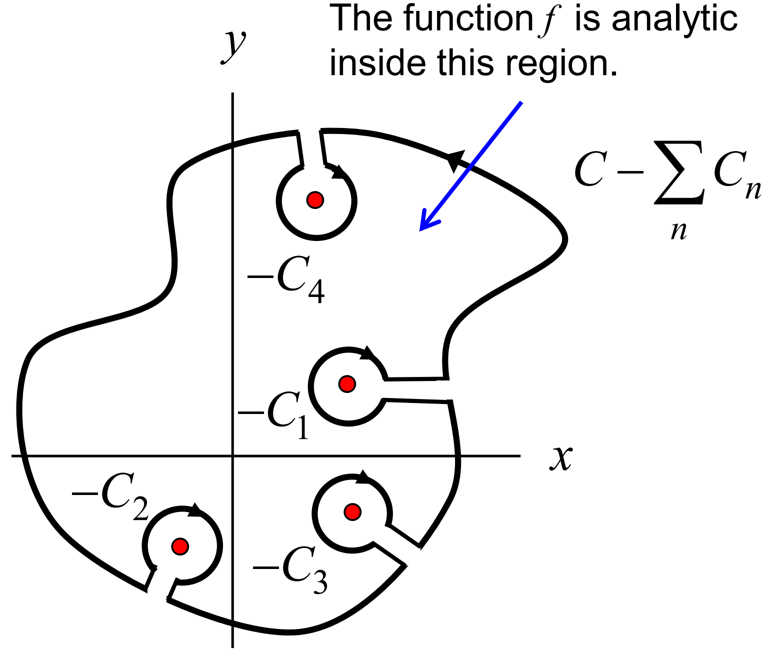
$$\oint_{C-\sum_n C_n} f(z)\,dz = \oint_C f(z)\,dz - \sum_n \oint_{C_n} f(z)\,dz = 0$$

$$\Rightarrow \oint_C f(z)\,dz = \sum_n \oint_{C_n} f(z)\,dz$$

$$= \sum_n 2\pi i \,\mathrm{Res}\, f(z_n)$$ (each small path sees a single isolated singularity)

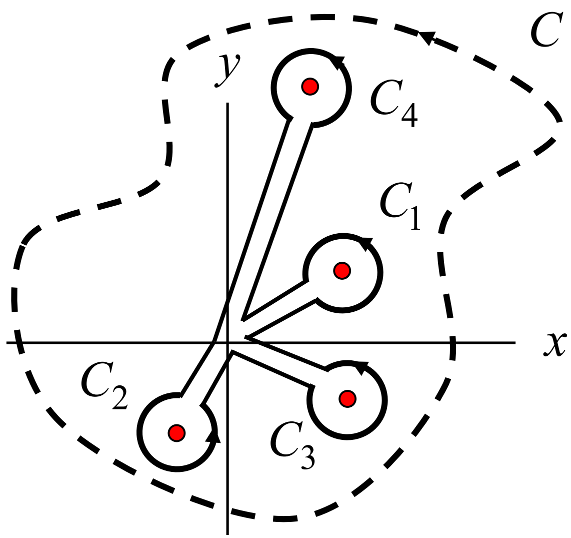
$$= 2\pi i \sum_n \mathrm{Res}\, f(z_n)$$





The Residue Theorem (cont.)

Alternatively, shrink the path, leaving only the singularities encircled:



Isolated singularities at z_n

$$\oint_C f(z)\,dz = \sum_n \oint_{C_n} f(z)\,dz$$

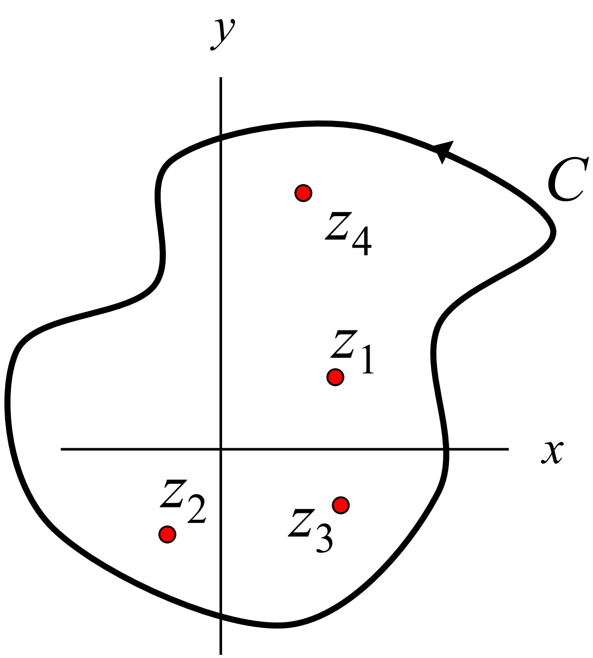
$$= \sum_n 2\pi i \operatorname{Res} f(z_n)$$ (each small path sees a single isolated singularity)

$$= 2\pi i \sum_n \operatorname{Res} f(z_n)$$

The Residue Theorem (cont.)

Summary of Residue Theorem

A function f is an analytic within C except at isolated singularities.



Isolated singularities at z_n

$$\oint_C f(z)\,dz = 2\pi i \sum_n \operatorname{Res} f(z_n)$$

The integral is equal to $2\pi i$ times the sum of the residues at the singularities.

Note: The integral is taken counterclockwise.

The Residue Theorem (cont.)

Note that the residue theorem subsumes several of our earlier results and theorems:

Cauchy's Theorem:

$$\oint_C f(z)\,dz = 0 \text{ if } f(z) \text{ analytic (no isolated singularities) in } C$$

Cauchy Integral Formula:

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) \text{ if } f(z) \text{ analytic in } C$$

(Hence z_0 is the only isolated singularity in C.)

Note : $f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n \Rightarrow \frac{f(z)}{z-z_0} = \sum_{n=0}^{\infty} b_n (z-z_0)^{n-1} \Rightarrow \text{Residue}\left(\frac{f(z)}{z-z_0}\right) = b_0 = f(z_0)$

Evaluating Residues

- Construct Laurent series about each singularity z_n, and then identify the coefficient $a_{-1,n} = \operatorname{Res} f(z_n)$.
  - Sometimes this can be a tedious approach.

- For a <u>simple</u> pole at $z = z_0$ we have a simple formula:

$$\lim_{z\to z_0}\left(z-z_0\right)f(z) = \lim_{z\to z_0}\left(z-z_0\right)\left(\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+\cdots\right)$$

$$= a_{-1}$$

$$\operatorname{Res} f(z_0) = \lim_{z\to z_0}\left(z-z_0\right)f(z)$$

Question:
Would this limit exist if it were a higher-order pole?

Evaluating Residues (cont.)

Example:

$$f(z) = \frac{2z}{(z+2)(z^2+1)}$$ has *simple* poles at $z = -2, +i, -i$

$$\operatorname{Res} f(-2) = \lim_{z\to -2} \cancel{(z+2)} \frac{2z}{\cancel{(z+2)}(z^2+1)} = -\frac{4}{5}$$

$$\operatorname{Res} f(i) = \lim_{z\to i} \cancel{(z-i)} \frac{2z}{(z+2)(z+i)\cancel{(z-i)}} = \frac{\cancel{2i}}{(i+2)\cancel{2i}} = \frac{2-i}{5}$$

$$\operatorname{Res} f(-i) = \lim_{z\to -i} \cancel{(z+i)} \frac{2z}{(z+2)(z-i)\cancel{(z+i)}} = \frac{\cancel{(-2i)}}{(-i+2)\cancel{(-2i)}} = \frac{2+i}{5}$$

Evaluating Residues (cont.)

Example:

$$f(z)=\frac{1}{\sin z}$$ has simple poles at $z=n\pi,\ \ n=0,\ \pm1,\ \pm2,\ \cdots$

$$\text{Res}\, f(n\pi) = \lim_{z\to n\pi}(z-n\pi)\times\frac{1}{\sin z} \overset{\text{L'Hospital's rule}}{=} \lim_{z\to n\pi}\frac{1}{\cos z}=(-1)^n$$

Alternatively,

$$\text{Res}\, f(n\pi) = \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin z} = \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin(z-n\pi+n\pi)}$$

$$= \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin(z-n\pi)\cos n\pi+\cos(z-n\pi)\sin n\pi}$$

$$= \lim_{z\to n\pi}\frac{(z-n\pi)}{\sin(z-n\pi)\cos n\pi} = (-1)^n$$ since we already know $\lim_{w\to 0}\frac{w}{\sin w}=1$

$$\text{Res}\, f(n\pi) = (-1)^n$$

Evaluating Residues (cont.)

The previous rule can be specialized to functions of the form f $f(z) = N(z)/D(z)$, where $D(z)$ has a <u>simple</u> zero at z_0 :

$$f(z) = \frac{N(z)}{D(z)}, \quad D(z_0) = 0, \quad N, D \text{ are analytic at } z_0$$

Simple zero : $D(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \ldots \quad (a_1 \neq 0)$

$$\operatorname{Res} f(z_0) = \operatorname{Res}\left(\frac{N(z)}{D(z)}\right)_{z_0} = \lim_{z \to z_0}(z - z_0)\frac{N(z)}{D(z)} = \lim_{z \to z_0}(z - z_0)\frac{N(z)}{D(z) - D(z_0)}$$

zero

$$= \lim_{z \to z_0} \frac{N(z)}{\frac{D(z) - D(z_0)}{(z - z_0)}} = \frac{N(z_0)}{D'(z_0)}$$

Question:
Would this limit exist if it were a higher-order zero?

Hence

$$\operatorname{Res}\left(\frac{N(z)}{D(z)}\right)_{z_0} = \frac{N(z_0)}{D'(z_0)}$$

Note : $D'(z_0) = a_1$

Evaluating Residues (cont.)

Example: $f(z) = \tan z$

$$\tan z = \frac{\sin z}{\cos z} \text{ has simple poles at } z = \frac{(2n+1)\pi}{2},$$

$$\text{for } n = 0, \pm 1, \pm 2, \cdots$$

$$\Rightarrow \operatorname{Res}(\tan z) = \frac{\sin \frac{(2n+1)\pi}{2}}{-\sin \frac{(2n+1)\pi}{2}} = -1$$

$$\operatorname{Res}(\tan z) = -1, \quad z = \frac{(2n+1)\pi}{2}, \; n = 0, \pm 1, \pm 2, \cdots$$

Evaluating Residues (cont.)

Including a Multiplying Function

$$f(z) = A(z)F(z)$$

$F(z)$

$A(z) =$ analytic function at z_0

$F(z) =$ function with simple pole at z_0

$$\operatorname{Res} f(z_0) = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (z - z_0) A(z) F(z) = \lim_{z \to z_0} (z - z_0) A(z) \left(\frac{b_{-1}}{z - z_0} + b_0 + b_1 (z - z_0) + \ldots \right)$$

$$= \lim_{z \to z_0} A(z) \left(b_{-1} + b_0 (z - z_0) + \ldots \right)$$

$$= A(z_0) b_{-1}$$

Hence, we have:

$$\operatorname{Res} f(z_0) = A(z_0) \operatorname{Res} F(z_0)$$

Evaluating Residues (cont.)

For a non-simple pole of finite order m at $z = z_0$:

At the pole, the Laurent series is

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Hence the following series is a Taylor series about the pole :

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \cdots$$

A formula for the coefficent of $(z-z_0)^{m-1}$ is

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Bigg|_{z=z_0}$$

Example $(m = 3)$:

$$(z-z_0)^3 f(z) = a_{-3} + a_{-2}(z-z_0) + a_{-1}(z-z_0)^2 + a_0(z-z_0)^3 + \ldots$$

$$\operatorname{Res} f(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Bigg|_{z=z_0} \quad (m \text{ finite})$$

Evaluating Residues (cont.)

A test to find the order m of a pole:

For a non-simple pole of finite order m at $z = z_0$:

$$L = \lim_{z \to z_0} \left(z - z_0\right)^p f(z) = \begin{cases} a_{-m,} \; p = m \\ 0, \; p > m \\ \infty, \; p < m \end{cases}$$

Proof:

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

where $a_{-m} \neq 0$

$$(z-z_0)^p f(z) = a_{-m}(z-z_0)^{p-m} + a_{-m+1}(z-z_0)^{p-m+1} + \cdots + a_{-1}(z-z_0)^{p-1} + a_0(z-z_0)^p + \cdots$$

Evaluating Residues (cont.)

$$\operatorname{Res} f(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Big|_{z=z_0} \qquad (m \text{ finite})$$

Example:

$$f(z) = \frac{2z^2+z}{(z+2)^3}$$ has a pole of order 3 at $z=-2$

$$\Rightarrow \operatorname{Res} f(-2) = \frac{1}{2!}\frac{d^2}{dz^2}\left[\cancel{(z+2)^3}\frac{2z^2+z}{\cancel{(z+2)^3}}\right]\Bigg|_{z=-2}$$

$$= \frac{1}{2!}\frac{d^2}{dz^2}\left[2z^2+z\right]\Big|_{z=-2} = \frac{4}{2!} = 2$$

$$\operatorname{Res} f(-2) = 2$$

Evaluating Residues (cont.)

Example:

$f(z)=\dfrac{1}{\sin^2 z}$ has poles of order 2 at $z=n\pi,\ n=0,\pm1,\pm2,\cdots$

$$\Rightarrow \operatorname{Res} f(n\pi) = \frac{1}{1!}\frac{d}{dz}\left[\frac{(z-n\pi)^2}{\sin^2 z}\right]\Bigg|_{z=n\pi} = \left[\frac{2(z-n\pi)\sin^2 z-2(z-n\pi)^2\sin z\cos z}{\sin^4 z}\right]_{z=n\pi}$$

$$= \left[\frac{2(z-n\pi)\sin z-2(z-n\pi)^2\cos z}{\sin^3 z}\right]_{z=n\pi}$$

After <u>three</u> applications of L'Hospital's rule:

$$\operatorname{Res} f(n\pi)=0$$

Evaluating Residues (cont.)

Alternative calculation:

$$f(z) = \frac{1}{\sin^2 z} = \frac{1}{\sin^2(z - n\pi + n\pi)} = \frac{1}{\sin^2(z - n\pi)} = \frac{1}{\sin^2 u} \quad (u = z - n\pi)$$

$$= \frac{1}{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right)^2} = \frac{1}{u^2\left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \cdots\right)^2}$$

Geometric Series

$$= \frac{1}{u^2}\left[1 + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!}\cdots\right) + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!}\cdots\right)^2 + \cdots\right]^2$$

$$= \frac{1}{u^2}\left[1 + \frac{u^2}{3!} + \left(-\frac{1}{5!} + \frac{1}{(3!)^2}\right)u^4 + \cdots\right]^2 = \frac{1}{u^2} + \frac{2}{3!} + \cdots$$

Note:
A simple shift does not affect the residue (we have the same coefficients in the Laurent series)

missing $\frac{a_{-1}}{u}$ term $\Rightarrow$ $\text{Residue} = 0$

$$\operatorname{Res} f(n\pi) = 0$$

Evaluating Residues (cont.)

Example:

$$f(z) = e^{1/z}$$ (essential singularity at $z = 0$)

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \ldots$$

Residue: $a_{-1} = 1$

$$\operatorname{Res} f(0) = 1$$

Evaluating Residues (cont.)

Summary of Residue Formulas

$$\operatorname{Res} f(z_0) = \lim_{z \to z_0} \left(z - z_0 \right) f(z)$$ Simple pole

$$\operatorname{Res}\left(\frac{N(z)}{D(z)}\right) = \frac{N(z_0)}{D'(z_0)}$$ if $D(z_0) = 0$ (simple zero) Simple pole

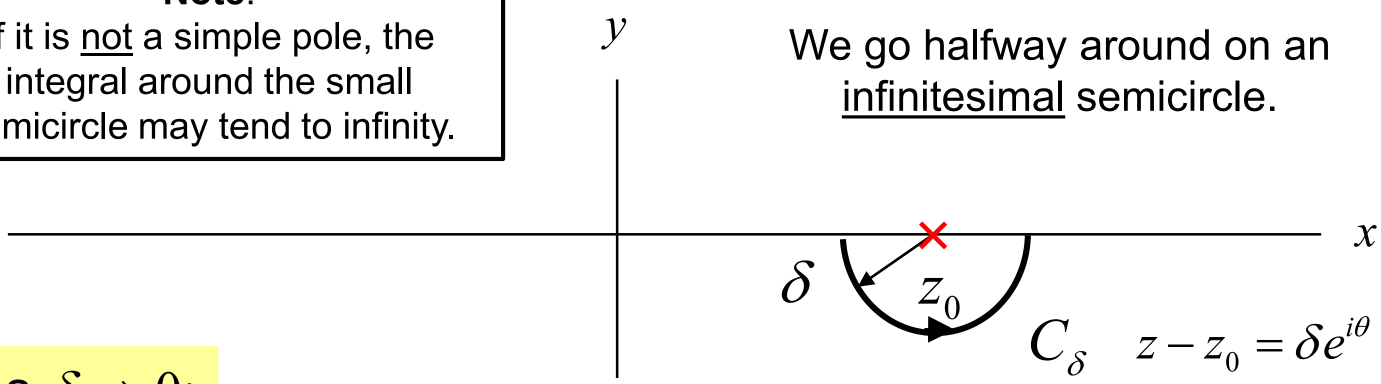
$$\operatorname{Res} f(z_0) = \lim_{z \to z_0} \frac{1}{\left(m-1\right)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$$ Pole of order m

Evaluating Residues (cont.)

Extension for simple poles (going halfway around)

Note:
If it is not a simple pole, the integral around the small semicircle may tend to infinity.

We go halfway around on an infinitesimal semicircle.



As $\delta \to 0$:

$$\int_{C_\delta} f(z)\,dz \underset{\delta \to 0}{\to} \pi i \,\mathrm{Res}\, f(z_0)$$

Proof:

Residue term: $\displaystyle \int_{C_\delta} \frac{a_{-1}}{z-z_0}\,dz = a_{-1}\int_{C_\delta} \frac{dz}{z-z_0} = a_{-1}\int_{-\pi}^{0} \frac{i\delta e^{i\theta}}{\delta e^{i\theta}}\,d\theta = a_{-1}\int_{-\pi}^{0} i\,d\theta = a_{-1}(\pi i)$

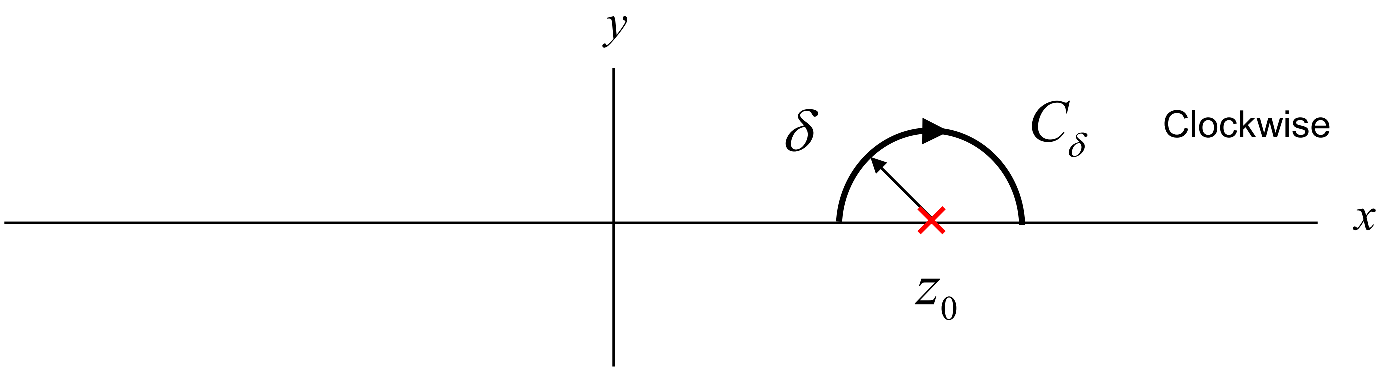
For $n > -1$: $\displaystyle \int_{C_\delta} a_n (z-z_0)^n\,dz = i a_n \delta^{n+1} \int_{-\pi}^{0} e^{i(n+1)\theta}\,d\theta \to 0$

Note: For $n < -1$ the integral does not converge!

Evaluating Residues (cont.)

Extension for simple poles (going halfway around)

Similarly, if we go in the opposite direction:



As $\delta \to 0$:

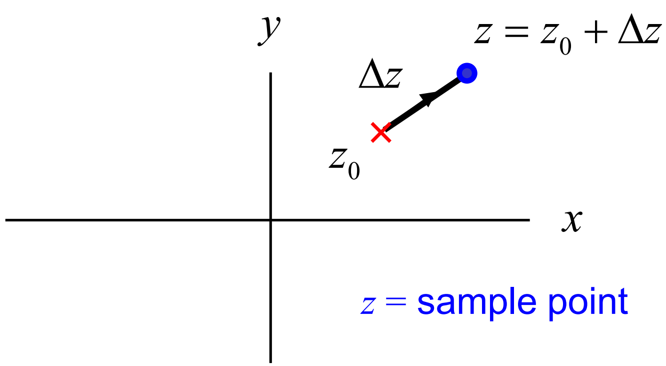
$$\int_{C_\delta} f(z)\,dz \underset{\delta \to 0}{\to} -\pi i \,\mathrm{Res}\, f(z_0)$$

Numerical Evaluation of Residues

Here we assume a simple pole at z_0.

$$\text{Res}\, f(z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Let $\Delta z \equiv z - z_0 \quad (z = z_0 + \Delta z)$



Therefore, we have

$$\text{Res}\, f(z_0) \approx \Delta z\, f(z_0 + \Delta z)$$

Examine the error using a Laurent series:

$$f(z) = f(z_0 + \Delta z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots$$

$$= \frac{a_{-1}}{\Delta z} + a_0 + a_1 (\Delta z) + a_2 (\Delta z)^2 + \ldots$$

$$\Rightarrow \quad \Delta z\, f(z_0 + \Delta z) = a_{-1} + \underbrace{a_0 (\Delta z) + a_1 (\Delta z)^2 + \ldots}_{\text{Error}}$$

↑ Error

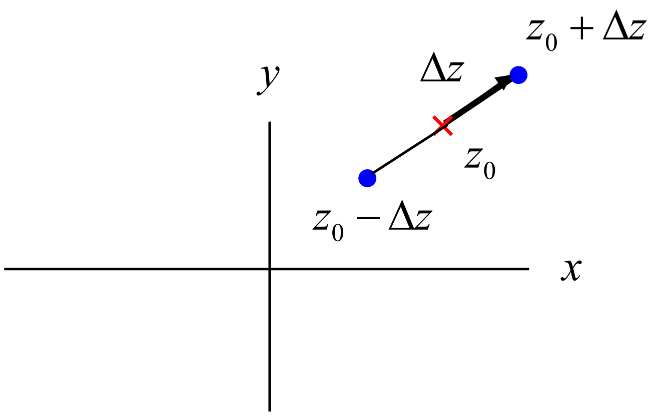
$$\text{Error} \propto |\Delta z|$$

Numerical Evaluation of Residues (cont.)

We can improve this by using <u>two</u> sample points and averaging:

$$\operatorname{Res} f(z_0) \approx \left(\frac{\Delta z_1 f(z_0 + \Delta z_1) + \Delta z_2 f(z_0 + \Delta z_2)}{2} \right)$$

Choose $\Delta z_1 = \Delta z, \; \Delta z_2 = -\Delta z$



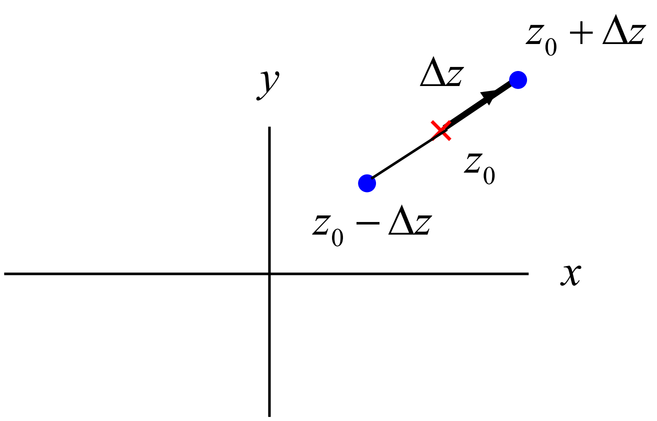
$$\operatorname{Res} f(z_0) \approx \Delta z \left(\frac{f(z_0 + \Delta z) - f(z_0 - \Delta z)}{2} \right)$$

This is a "central-difference" formula.

Numerical Evaluation of Residues (cont.)

Examine the error using a Laurent series:

$$\operatorname{Res} f(z_0) \approx \Delta z \left(\frac{f(z_0 + \Delta z) - f(z_0 - \Delta z)}{2} \right)$$



$$f(z_0 + \Delta z) = \frac{a_{-1}}{\Delta z} + a_0 + a_1 \Delta z + a_2 (\Delta z)^2 + \ldots$$

$$f(z_0 - \Delta z) = \frac{a_{-1}}{-\Delta z} + a_0 + a_1 (-\Delta z) + a_2 (-\Delta z)^2 + \ldots$$

$$\Rightarrow \quad \Delta z \left(\frac{f(z_0 + \Delta z) - f(z_0 - \Delta z)}{2} \right) = a_{-1} + \underbrace{a_1 (\Delta z)^2 + \ldots}_{\text{Error}}$$

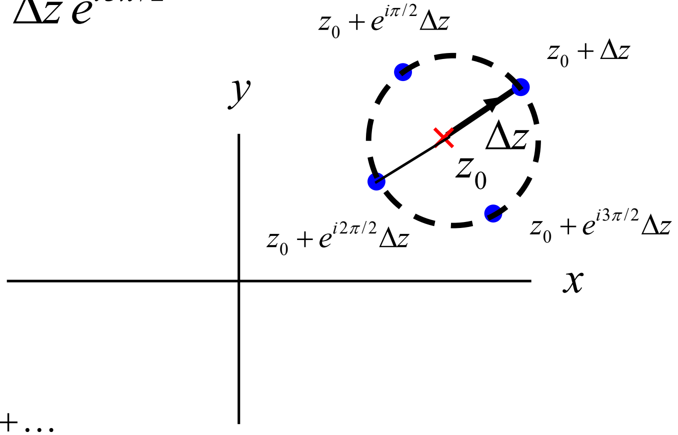
$$\text{Error} \propto |\Delta z|^2$$

Numerical Evaluation of Residues (cont.)

We can improve this even more by using four sample points:

$$\operatorname{Res} f(z_0) \approx \frac{\Delta z\, f(z_0+\Delta z) + \Delta z\, e^{i\pi/2} f(z_0 + e^{i\pi/2}\Delta z) + \Delta z\, e^{i2\pi/2} f(z_0 + e^{i2\pi/2}\Delta z) + \Delta z\, e^{i3\pi/2} f(z_0 + e^{i3\pi/2}\Delta z)}{4}$$

$$\Delta z_1 = \Delta z, \quad \Delta z_2 = \Delta z\, e^{i\pi/2}, \quad \Delta z_3 = \Delta z\, e^{i2\pi/2}, \quad \Delta z_4 = \Delta z\, e^{i3\pi/2}$$



Examine the error using a Laurent series:

$$f(z_0+\Delta z) = \frac{a_{-1}}{\Delta z} + a_0 + a_1(\Delta z) + a_2(\Delta z)^2 + a_3(\Delta z)^3 + \ldots$$

$$f(z_0+\Delta z e^{i\pi/2}) = \frac{a_{-1}}{\Delta z e^{i\pi/2}} + a_0 + a_1\left(\Delta z e^{i\pi/2}\right) + a_2\left(\Delta z e^{i\pi/2}\right)^2 + a_3\left(\Delta z e^{i\pi/2}\right)^3 + \ldots$$

$$f(z_0+\Delta z e^{i2\pi/2}) = \frac{a_{-1}}{\Delta z e^{i2\pi/2}} + a_0 + a_1\left(\Delta z e^{i2\pi/2}\right) + a_2\left(\Delta z e^{i2\pi/2}\right)^2 + a_3\left(\Delta z e^{i2\pi/2}\right)^3 + \ldots$$

$$f(z_0+\Delta z e^{i3\pi/2}) = \frac{a_{-1}}{\Delta z e^{i3\pi/2}} + a_0 + a_1\left(\Delta z e^{i3\pi/2}\right) + a_2\left(\Delta z e^{i3\pi/2}\right)^2 + a_3\left(\Delta z e^{i3\pi/2}\right)^3 + \ldots$$

We see that

$$\Delta z\left(\frac{f(z_0+\Delta z) + e^{i\pi/2} f(z_0 + e^{i\pi/2}\Delta z) + e^{i2\pi/2} f(z_0 + e^{i2\pi/2}\Delta z) + e^{i3\pi/2} f(z_0 + e^{i3\pi/2}\Delta z)}{4}\right) = a_{-1} + a_3(\Delta z)^4 + \ldots$$

$$\text{Error} \propto |\Delta z|^4$$

Numerical Evaluation of Residues (cont.)

A little more detail:

$$f(z_0+\Delta z) = \frac{a_{-1}}{\Delta z} + a_0 + a_1\Delta z + a_2(\Delta z)^2 + a_3(\Delta z)^3 + \dots$$

$$e^{i\pi/2} f(z_0+\Delta z e^{i\pi/2}) = \frac{a_{-1}}{\Delta z} + a_0 e^{i\pi/2} + a_1\Delta z\left(e^{i\pi/2}\right)^2 + a_2(\Delta z)^2\left(e^{i\pi/2}\right)^3 + a_3(\Delta z)^3\left(e^{i\pi/2}\right)^4 + \dots$$

$$e^{i2\pi/2} f(z_0+\Delta z e^{i2\pi/2}) = \frac{a_{-1}}{\Delta z} + a_0 e^{i2\pi/2} + a_1\Delta z\left(e^{i2\pi/2}\right)^2 + a_2(\Delta z)^2\left(e^{i2\pi/2}\right)^3 + a_3(\Delta z)^3\left(e^{i2\pi/2}\right)^4 + \dots$$

$$e^{i3\pi/2} f(z_0+\Delta z e^{i3\pi/2}) = \frac{a_{-1}}{\Delta z} + a_0 e^{i3\pi/2} + a_1\Delta z\left(e^{i3\pi/2}\right)^2 + a_2(\Delta z)^2\left(e^{i3\pi/2}\right)^3 + a_3(\Delta z)^3\left(e^{i3\pi/2}\right)^4 + \dots$$

Simplifying:

$$f(z_0+\Delta z) = \frac{a_{-1}}{\Delta z} + a_0(1) + a_1(\Delta z)(1) + a_2(\Delta z)^2(1) + a_3(\Delta z)^3(1) + \dots$$

$$e^{i\pi/2} f(z_0+\Delta z e^{i\pi/2}) = \frac{a_{-1}}{\Delta z} + a_0(i) + a_1(\Delta z)(-1) + a_2(\Delta z)^2(-i) + a_3(\Delta z)^3(1) + \dots$$

$$e^{i2\pi/2} f(z_0+\Delta z e^{i2\pi/2}) = \frac{a_{-1}}{\Delta z} + a_0(-1) + a_1(\Delta z)(1) + a_2(\Delta z)^2(-1) + a_3(\Delta z)^3(1) + \dots$$

$$e^{i3\pi/2} f(z_0+\Delta z e^{i3\pi/2}) = \frac{a_{-1}}{\Delta z} + a_0(-i) + a_1(\Delta z)(-1) + a_2(\Delta z)^2(i) + a_3(\Delta z)^3(1) + \dots$$

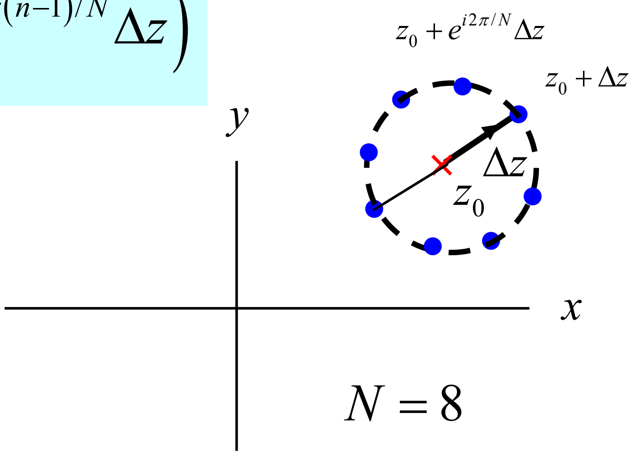
Numerical Evaluation of Residues (cont.)

In general, we can use N sample points

$$\Delta z_n = \Delta z\, e^{i2\pi(n-1)/N},\; n = 1, 2 \ldots N$$

$$\operatorname{Res} f(z_0) \approx \frac{1}{N}\sum_{n=1}^{N}\left(\Delta z\, e^{i2\pi(n-1)/N}\right) f\left(z_0 + e^{i2\pi(n-1)/N}\Delta z\right)$$

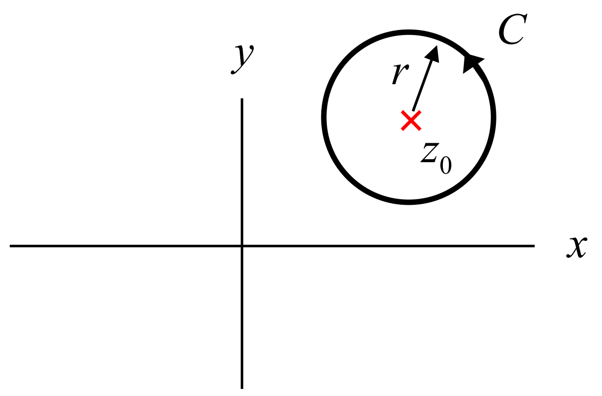
$$\text{Error} \propto |\Delta z|^N$$



Numerical Evaluation of Residues (cont.)

We can also numerically integrate around a simple pole:

$$\text{Res}\, f(z_0) = \frac{1}{2\pi i}\oint_C f(z)\,dz$$

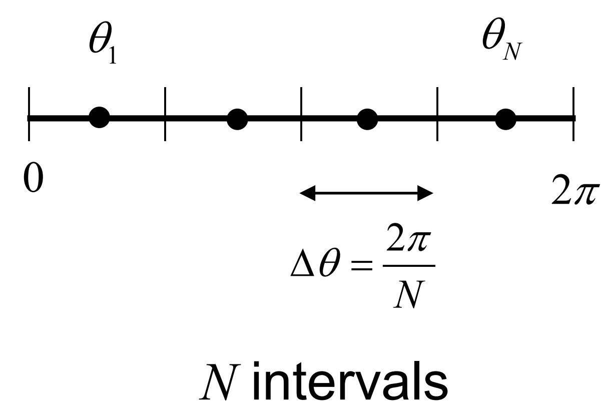


Choose a small circle of radius r:

$$z = z_0 + re^{i\theta}$$

$$dz = ire^{i\theta}d\theta$$

$$\text{Res}\, f(z_0) = \frac{r}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)e^{i\theta}\,d\theta$$



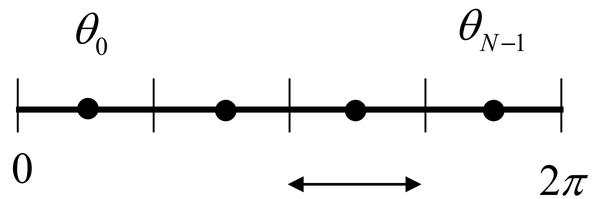
Use the midpoint rule of integration:

Sample points: $\theta_n = \frac{2\pi}{N}\left(n - \frac{1}{2}\right), \quad n = 1, 2 \ldots N$

Numerical Evaluation of Residues (cont.)

We then have

$$\operatorname{Res} f(z_0) = \frac{r}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)e^{i\theta}\,d\theta$$



$$\Delta\theta = \frac{2\pi}{N}$$

$$\theta_n = \frac{2\pi}{N}\left(n - \frac{1}{2}\right), \quad n = 1,2\ldots N$$

⇩ Sample

$$\operatorname{Res} f(z_0) \approx \frac{r}{2\pi}\sum_{n=1}^{N} f\left(z_0 + re^{i2\pi(n-1/2)/N}\right)e^{i2\pi(n-1/2)/N}\left(\frac{2\pi}{N}\right)$$

or

$$\operatorname{Res} f(z_0) \approx \frac{re^{-i\pi/N}}{N}\sum_{n=1}^{N} f\left(z_0 + r\left(e^{i2\pi n/N}e^{-i\pi/N}\right)\right)e^{i2\pi n/N}$$

or

$$\operatorname{Res} f(z_0) \approx \frac{\Delta z_0}{N}\sum_{n=1}^{N} f\left(z_0 + \Delta z_0\, e^{i2\pi n/N}\right)e^{i2\pi n/N}$$

where $\Delta z_0 \equiv re^{-i\pi/N}$

Numerical Evaluation of Residues (cont.)

We then have

$$\operatorname{Res} f(z_0) \approx \frac{1}{N} \sum_{n=1}^{N} \left(\Delta z_0 \, e^{i 2\pi n / N} \right) f\left(z_0 + \Delta z_0 \, e^{i 2\pi n / N} \right)$$

or

$$\operatorname{Res} f(z_0) \approx \frac{1}{N} \sum_{n=1}^{N} \left(\Delta z \, e^{i 2\pi (n-1) / N} \right) f\left(z_0 + \Delta z \, e^{i 2\pi (n-1) / N} \right)$$

(We have the same set of sample points if n is replaced by n-1.)

This is the same result that we obtained from the sampling method!

Note:
The midpoint rule is exceptionally accurate when applied to a smooth periodic function, integrating over a period*.

*J. A. C. Wiedeman, "Numerical Integration of Periodic Functions: A Few Examples," The Mathematical Association of America, vol. 109, Jan. 2002, pp. 21-36.