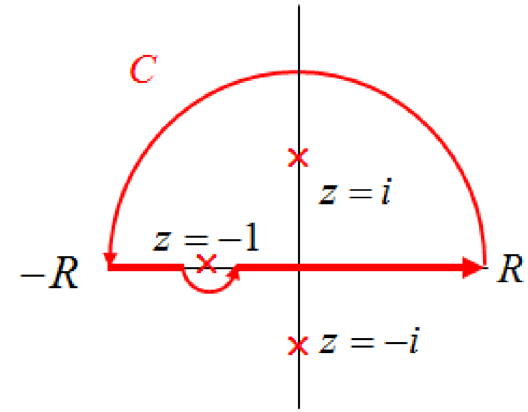
ECE 6382



Fall 2023

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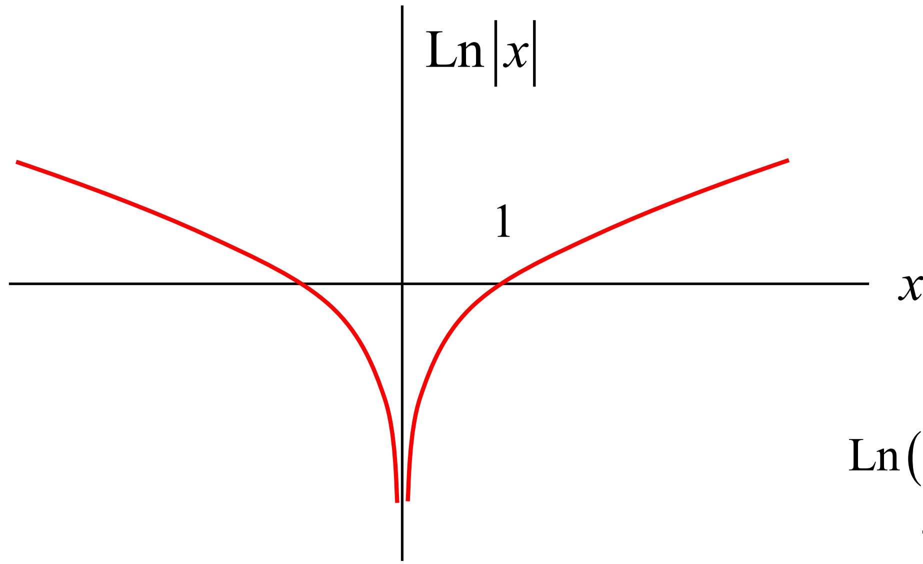
Notes 11

Evaluation of Definite Integrals via the Residue Theorem

Notes are from D. R. Wilton, Dept. of ECE

Review of Singular Integrals



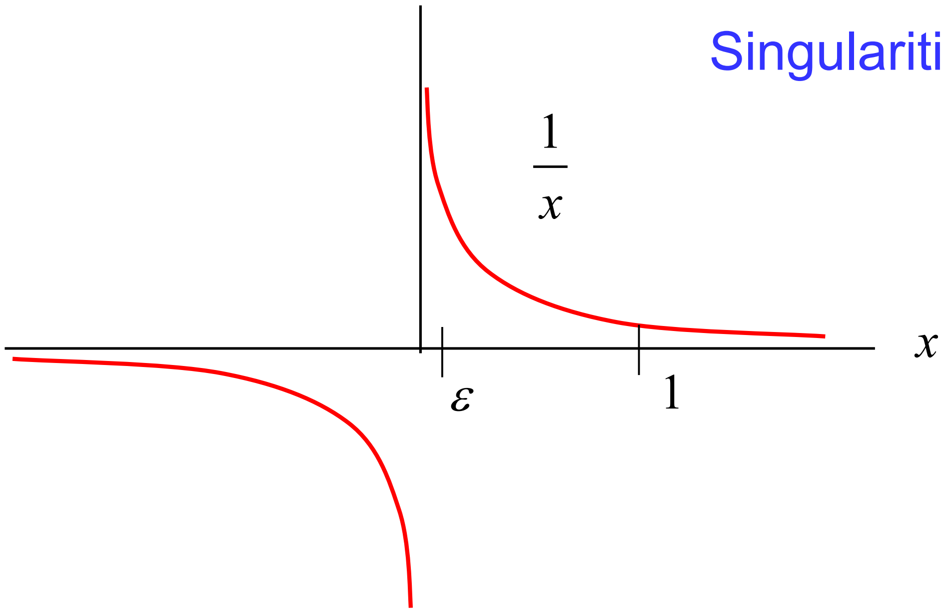
$$\text{Ln}(z) = \ln(r) + i\theta$$
$$-\pi < \theta < \pi$$

- Logarithmic singularities are examples of <u>integrable</u> singularities:

$$\int_0^1 \text{Ln}(x)\, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \text{Ln}(x)\, dx = \lim_{\varepsilon \to 0} \left(x\text{Ln}(x) - x \right)\Big|_{x=\varepsilon}^{1} = -1 \quad \text{since} \quad \lim_{x \to 0} x\text{Ln}(x) = 0$$

Note: There might be numerical trouble if one integrates this function numerically!

Review of Singular Integrals (cont.)



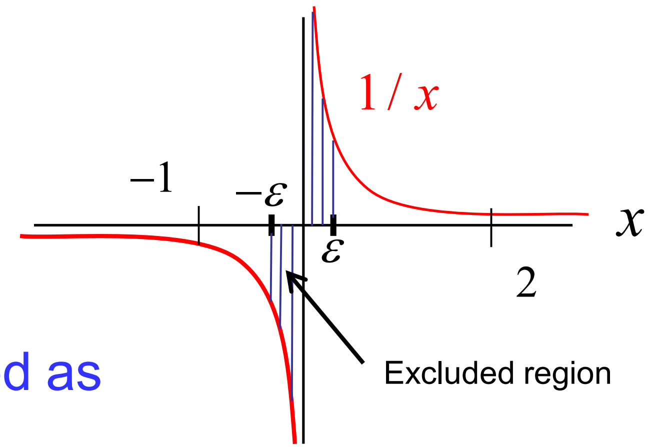
Singularities like $1/x$ are non-integrable.

$$\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to 0} \left(\mathrm{Ln}(x) \right) \Big|_{x=\varepsilon}^{1} = \infty$$

Review of Cauchy Principal Value Integrals

Consider the following integral:

$$I=\int_{-1}^{2}\frac{dx}{x}=\int_{-1}^{0}\frac{dx}{x}+\int_{0}^{2}\frac{dx}{x}=\mathrm{Ln}\left|x\right|_{x=-1}^{0}+\mathrm{Ln}\left|x\right|_{x=0}^{2}$$



A <u>finite</u> result is obtained if the integral interpreted as

$$I=\int_{-1}^{2}\frac{dx}{x}=\lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon}\frac{dx}{x}+\int_{\varepsilon}^{2}\frac{dx}{x}=\lim_{\varepsilon\to 0}\left(\mathrm{Ln}\left|x\right|_{x=-1}^{-\varepsilon}+\mathrm{Ln}\left|x\right|_{x=+\varepsilon}^{2}\right)_{\varepsilon}$$

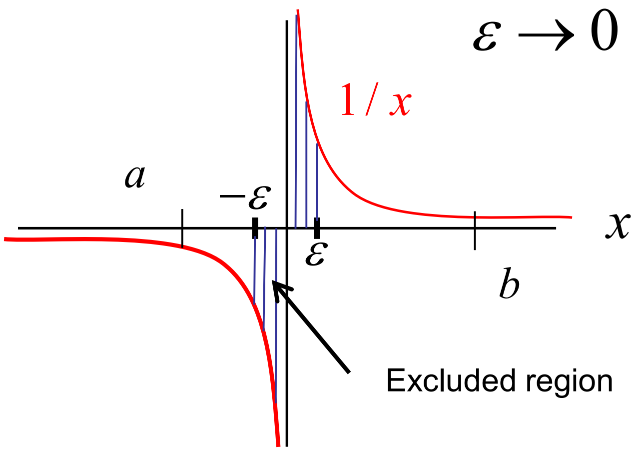
$$=\lim_{\varepsilon\to 0}\left(\cancel{\mathrm{Ln}\varepsilon}-\mathrm{Ln}1+\mathrm{Ln}2-\cancel{\mathrm{Ln}\varepsilon}\right)=\mathrm{Ln}2$$

The <u>infinite</u> contributions from the two symmetrical shaded parts shown exactly cancel in this limit. Integrals evaluated in this way are said to be (Cauchy) principal value (PV) integrals:

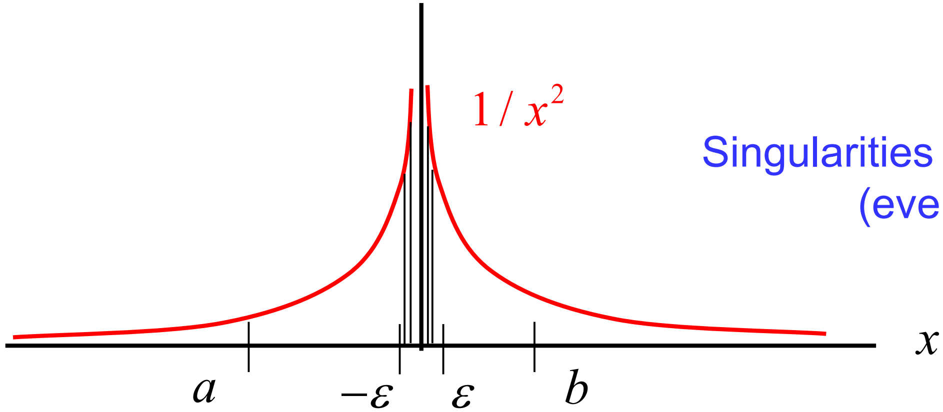
Notation: $I=\mathrm{PV}\int_{-1}^{2}\frac{dx}{x}$ or $=\;⨍_{-1}^{2}\frac{dx}{x}$

Cauchy Principal Value Integrals (cont.)

- $1/x$ singularities are examples of singularities integrable only in the principal value (PV) sense.
- Principal value integrals must not start or end at the singularity, but must pass through them to permit cancellation of infinities



Cauchy Principal Value Integrals (cont.)



Singularities like $1/x^2$ are non- integrable (even in the PV sense).

$$\int_a^{-\varepsilon} \frac{1}{x^2}\,dx + \int_\varepsilon^b \frac{1}{x^2}\,dx = \left(\frac{1}{a} + \frac{1}{\varepsilon}\right) + \left(\frac{1}{\varepsilon} - \frac{1}{b}\right) \to \infty$$

(but note that $\dfrac{\operatorname{sgn}(x)}{x^2} = \begin{cases} \frac{1}{x^2}, & x > 0 \\ -\frac{1}{x^2}, & x < 0 \end{cases}$ has a PV integral)

Singular Integral Examples

Summary of some results:

- $\text{Ln}\,x$ is integrable at $x = 0$
- $1/x^{\alpha}$ is integrable at $x = 0$ for $0 < \alpha < 1$
- $1/x^{\alpha}$ is non-integrable at $x = 0$ for $\alpha \geq 1$
- $f(x)\text{sgn}(x)/|x|^{\alpha}$ has a PV integral at $x = 0$ for $\alpha < 2$ if $f(x)$ is smooth (proof omitted)

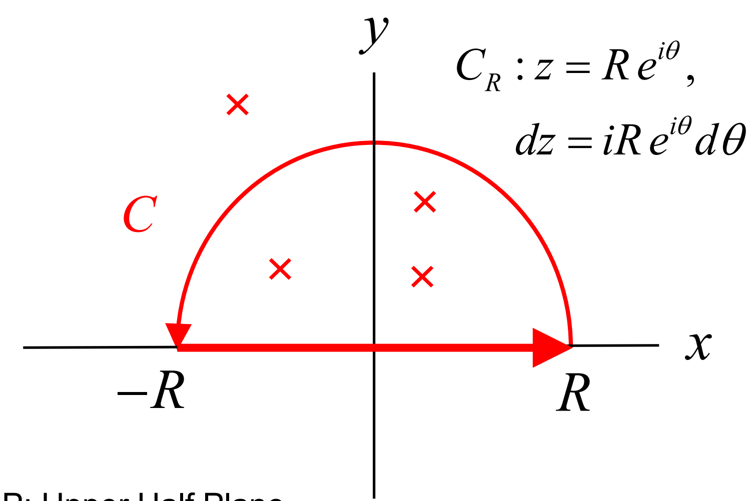
(The above results translate to singularities at a point a via the transformation $x \rightarrow x\text{-}a$.)

Integrals Along the Real Axis

$$I = \int_{-\infty}^{\infty} f(x)\,dx$$

Assumptions:

- f is analytic in the UHP except for a finite number of poles.
- f is $o(1/z)$, i.e. $\lim_{z\to\infty} |z\,f(z)| = 0$ in the UHP.



UHP: Upper Half Plane

On the large semicircle we then have

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz \;=\; \lim_{R\to\infty} \int_0^{\pi} f(Re^{i\theta})\,iRe^{i\theta}\,d\theta \;=0$$

Note:
If the function is analytic in the LHP except for poles, then we would close the contour in the LHP.

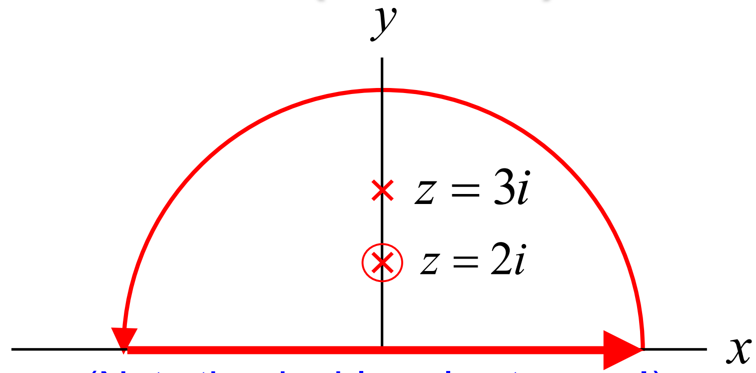
Hence

$$I = \int_{-\infty}^{\infty} f(x)\,dx = I_C = \oint_C f(z)\,dz = 2\pi i \sum \text{residues of } f(z) \text{ in the UHP}$$

Integrals Along the Real Axis (cont.)

Example:

$$I = \int_0^\infty \frac{x^2\,dx}{\left(x^2+9\right)\left(x^2+4\right)^2}$$



(Note the double pole at $z = 2i$!)

$$I = \int_0^\infty \frac{z^2\,dx}{\left(z^2+9\right)\left(z^2+4\right)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \underbrace{\frac{z^2}{\left(z^2+9\right)\left(z^2+4\right)^2}}_{f(z)=\mathcal{O}\left(z^{-4}\right)} dx = \frac{1}{2} 2\pi i\left[\operatorname{Res} f(3i) + \operatorname{Res} f(2i)\right]$$

$$\operatorname{Res} f(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Big|_{z=z_0}$$

$$= \pi i\left[\lim_{z\to 3i} \frac{(z-3i)z^2}{(z+3i)(z-3i)\left(z^2+4\right)^2} + \lim_{z\to 2i}\frac{d}{dz}\left(\frac{(z-2i)^2 z^2}{\left(z^2+9\right)(z-2i)^2(z+2i)^2}\right)\right]$$

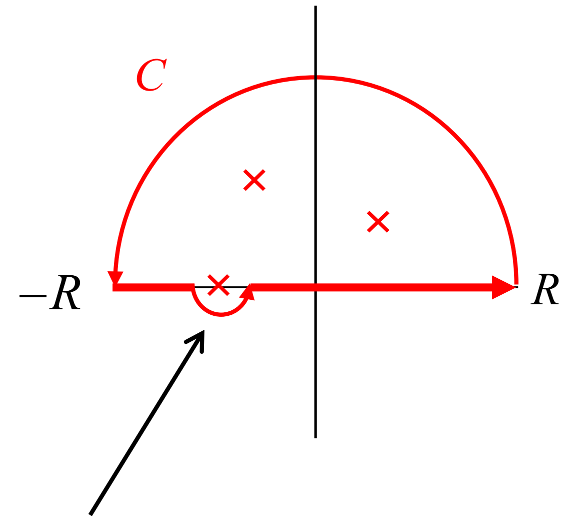
$$= \pi i\left[\frac{3i}{50} + \lim_{z\to 2i}\left(\frac{\left(z^2+9\right)(z+2i)^2 2z - z^2\left(\left(z^2+9\right)2(z+2i)+(z+2i)^2 2z\right)}{\left(z^2+9\right)^2(z+2i)^4}\right)\right]$$

$$= \pi i\left[\frac{3i}{50} - \frac{13i}{200}\right] = \frac{\pi}{200}$$

$$I = \int_0^\infty \frac{x^2\,dx}{\left(x^2+9\right)\left(x^2+4\right)^2} = \frac{\pi}{200}$$

Cauchy Principal Value Integrals

$$I = ⨍_{-\infty}^{\infty} f(x)\,dx$$



A small semicircle of radius ρ is introduced at a simple pole on the real axis. We let $\rho \to 0$.

Assumptions:

- f is analytic in the UHP except for a finite number of poles.
- f is $o(1/z)$, i.e. $\lim_{z\to\infty} |z f(z)| = 0$ in the UHP.

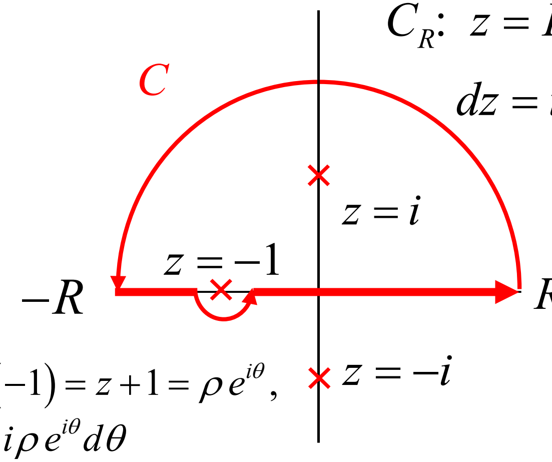
Note:
If the function is analytic in the LHP except for poles, then we would close the contour in the LHP.

❖ It is our choice to detour above or below the pole on the real axis.

Cauchy Principal Value Integrals

Example

Evaluate: $$I = \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)\left(x+1\right)}$$



C_R: $z = R\,e^{i\phi}$, $dz = iR\,e^{i\phi}d\phi$

C_ρ: $z-(-1) = z+1 = \rho\,e^{i\theta}$, $dz = i\rho\,e^{i\theta}d\theta$

Consider the following integral:

$$I_C = \oint_C \frac{dz}{\left(z^2+1\right)\left(z+1\right)} = \lim_{\substack{R\to\infty,\\ \rho\to 0}} \left(\underbrace{\int_{-R}^{-1-\rho} + \int_{-1+\rho}^{R}}_{I} + \int_{C_\rho} + \int_{C_R} \right) \frac{dz}{\left(z^2+1\right)\left(z+1\right)}$$

$$= I + \lim_{\substack{R\to\infty\\ \rho\to 0}} \left(\int_{C_\rho} + \int_{C_R} \right) \frac{dz}{\left(z^2+1\right)\left(z+1\right)} = I + \lim_{\rho\to 0} \int_{\pi}^{2\pi} \frac{i\,\rho e^{i\theta}\,d\theta}{\left[\left(-1+\rho\,e^{i\theta}\right)^2+1\right]\rho e^{i\theta}} + \lim_{R\to\infty} \int_0^{\pi} \frac{iR\,e^{i\phi}d\phi}{\left(R^2 e^{i2\phi}+1\right)\left(R\,e^{i\phi}+1\right)}$$

$$= I + \int_{\pi}^{2\pi} \frac{i\,d\theta}{2} + \lim_{R\to\infty} \int_0^{\pi} \frac{ie^{-i2\phi}d\phi}{R^2} = I + \frac{i\pi}{2} + \lim_{R\to\infty} \int_0^{\pi} \frac{ie^{-i2\phi}d\phi}{R^2} = I + \frac{i\pi}{2} + 0$$

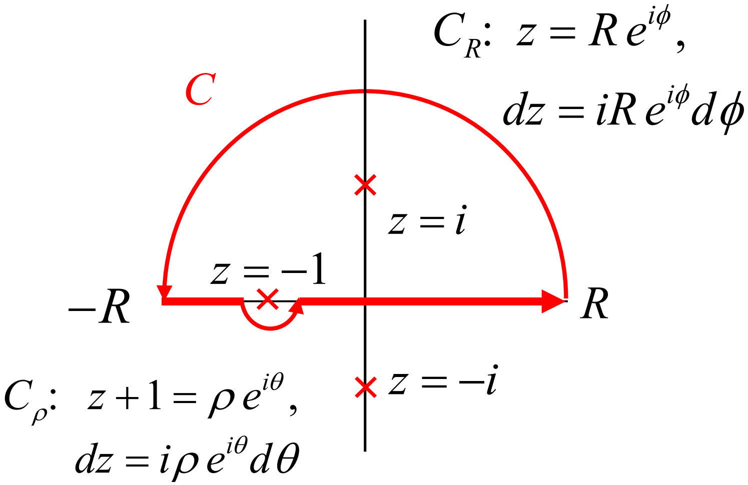
We could have also used the formula from Notes 10 for going halfway around a simple pole on a small semicircle (the residue at -1 is 1/2).

Hence,

$$I_C = I + \frac{i\pi}{2}$$

Cauchy Principal Value Integrals (cont.)

$$I = ⨍_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x+1)}$$



Next, evaluate the integral I_C using the residue theorem:

$$I_C = 2\pi i \left[\operatorname{Res} f(z=i) + \operatorname{Res} f(z=-1) \right] = 2\pi i \left[\lim_{z\to i} \frac{(z-i)}{(z-i)(z+i)(z+1)} + \lim_{z\to -1} \frac{(z+1)}{(z^2+1)(z+1)} \right]$$

$$= 2\pi i \left[\frac{1}{2i(1+i)} + \frac{1}{2} \right] = 2\pi i \left[\frac{1-i}{4i} + \frac{1}{2} \right] = 2\pi i \left[\frac{-1-i}{4} + \frac{1}{2} \right] = \frac{\pi}{2} + i\left(\frac{\pi}{2}\right)$$

Thus:

$$I_C = \frac{\pi}{2} + i\left(\frac{\pi}{2}\right)$$

Recall:

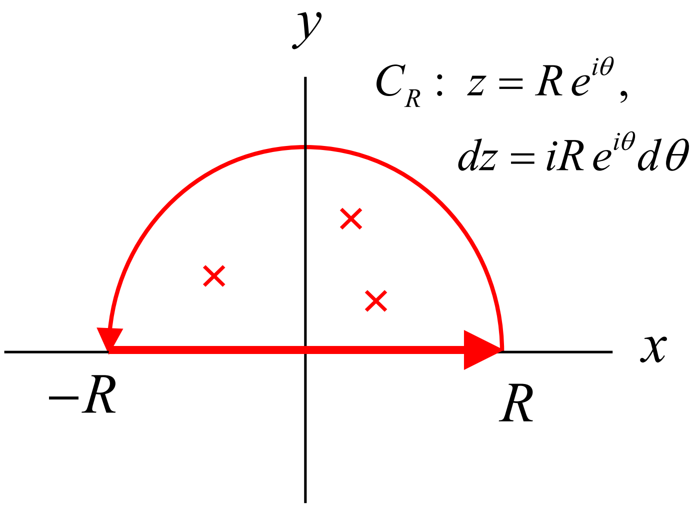
$$I_C = I + \frac{i\pi}{2}$$

Hence:

$$I = \frac{\pi}{2}$$

Fourier Integrals

$$I = \int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx$$



Assume $a > 0$ (close the path in the UHP)

Assumptions:

- f is analytic in the UHP <u>except</u> for a finite number of poles (can easily be extended to handle poles on the real axis via PV integrals),
- $\lim_{z \to \infty} f(z) = 0, \quad 0 \le \arg z \le \pi$ (z in UHP)

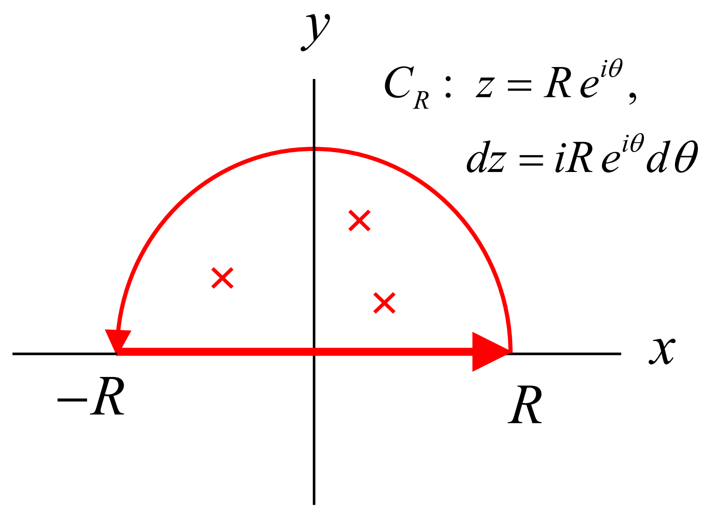
Choosing the contour shown, the contribution from the semicircular arc vanishes as $R \to \infty$ by ***Jordan's lemma***.

(See next slide.)

Jean-Baptiste Joseph Fourier

Fourier Integrals (cont.)

Jordan's lemma



Assume

$$\lim_{z\to\infty} f(z) = 0, \quad 0 \le \arg z \le \pi \ \ (z \text{ in UHP})$$

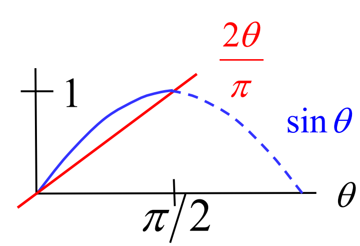
Then $\displaystyle\int_{C_R} f(z)\,e^{iaz}\,dz \ \to \ 0$

Proof:

$$\left|\int_{C_R} f(z)\,e^{iaz}\,dz\right| = \left|\int_0^{\pi} f(Re^{i\theta})\,e^{iaR\cos\theta}\,e^{-aR\sin\theta}\,iRe^{i\theta}\,d\theta\right| \le (2R)\max\left|f(Re^{i\theta})\right| \int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta$$

$$\le \left|(2R)\max\left|f(Re^{i\theta})\right| \int_0^{\pi/2} e^{-\frac{2aR\theta}{\pi}}\,d\theta\right| = \left|(\cancel{2R})\max\left|f(Re^{i\theta})\right| \frac{\pi}{a\,\cancel{2R}}\left(1-e^{-aR}\right)\right| \to 0$$

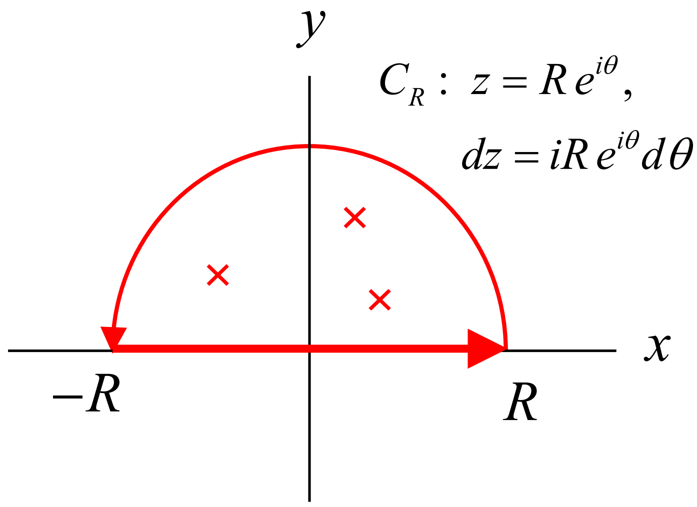
$$0 \le \theta \le \pi/2: \quad \sin\theta \ge \frac{2\theta}{\pi} \ \Rightarrow \ e^{-aR\sin\theta} \le e^{-\frac{2aR\theta}{\pi}}$$



Fourier Integrals (cont.)

$$I = \int_{-\infty}^{\infty} f(x) e^{iax} dx$$

$$\int_{C_R} f(z) e^{iaz} dz \;\rightarrow\; 0$$



$$C_R: \; z = R e^{i\theta}, \quad dz = iR e^{i\theta} d\theta$$

We then have

$$I = I_C = \oint_C f(z) e^{iaz} \, dz = 2\pi i \sum \text{residues of } f(z) e^{iaz} \text{ in the UHP}$$

Questions:

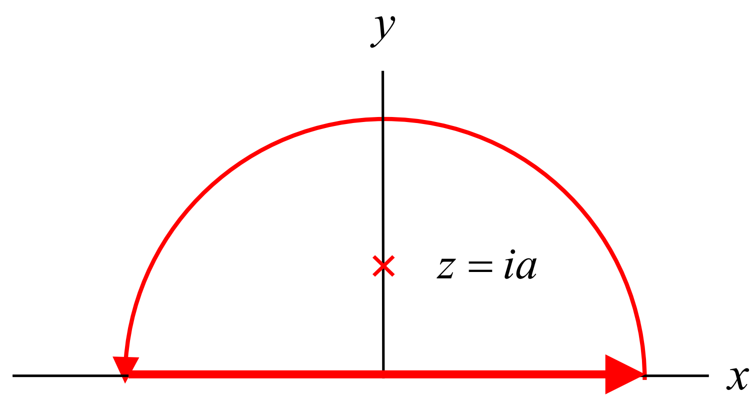
What would change if $a < 0$? If we had a negative sign in the exponential?

Fourier Integrals (cont.)

Example:

$$I = \int_0^\infty \frac{\cos\lambda x}{x^2 + a^2}\, dx, \quad (a, \lambda) > 0.$$

Note : $e^{i\lambda x} = \cos\lambda x + i \sin\lambda x$



Use the symmetries of $\cos(\lambda x)$ and $\sin(\lambda x)$ and the Euler formula, we have:

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2 + a^2}\, dx$$ (imaginary part vanishes by symmetry!)

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i\lambda z}}{z^2 + a^2}\, dz = 2\pi i \ \text{Res}\left(\frac{1}{2}\frac{e^{i\lambda z}}{z^2 + a^2}\right)_{z=ia} = 2\pi i \lim_{z\to ia} \frac{1}{2}\frac{\cancel{(z-ia)}\, e^{i\lambda z}}{(z+ia)\cancel{(z-ia)}}$$

$$= \cancel{2}\pi \cancel{i} \frac{1}{\cancel{2}} \frac{e^{-a\lambda}}{2\cancel{i}a} = \frac{\pi e^{-a\lambda}}{2a}$$

$$I = \int_0^\infty \frac{\cos\lambda x}{x^2 + a^2} = \frac{\pi e^{-a\lambda}}{2a}$$

Rational Functions of sin and cos

Assumptions:

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$$

- f is finite within the interval.
- f is a rational function (ratio of polynomials) of $\sin\theta$, $\cos\theta$.

Let $\underbrace{z = e^{i\theta}}_{\text{unit circle}}, \; dz = ie^{i\theta} d\theta \;\Rightarrow\; d\theta = -i\frac{dz}{z}$

$$\sin\theta = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}$$

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta = -i \oint_{|z|=1} f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{z}$$

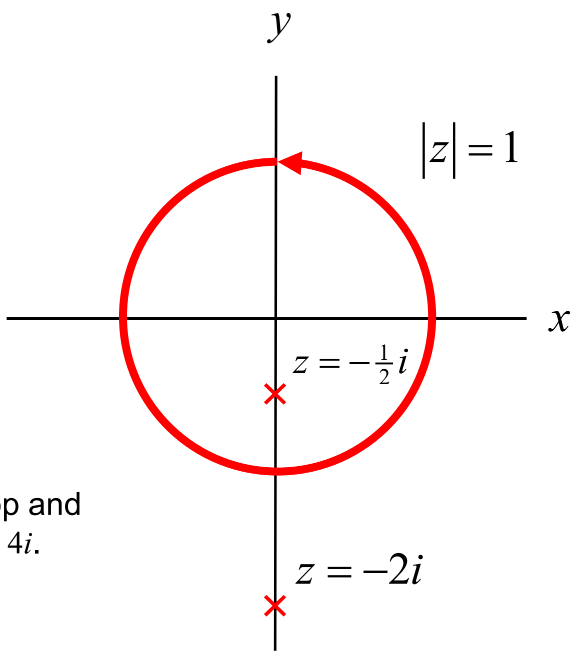
$$= 2\pi \sum \text{Residues of } \frac{1}{z} f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \text{ inside the unit circle}$$

Note : $\frac{1}{z} f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)$ will be a rational function of z

Rational Functions of sin and cos (cont.)

Example:

$$I = \int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta}$$



$$I = \int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta} = \oint_{|z|=1} \frac{1}{\frac{5}{4} + \frac{z - z^{-1}}{2i}} \left(-i \frac{dz}{z} \right)$$

Multiply top and bottom by $4i$.

$$= \oint_{|z|=1} \frac{4i}{5iz + 2z^2 - 2} (-i dz) = \oint_{|z|=1} \frac{4\, dz}{2(z + 2i)(z + \frac{1}{2}i)}$$

$$= \oint_{|z|=1} \frac{2dz}{(z + 2i)(z + \frac{1}{2}i)} = 2\pi i \left[\lim_{z \to -\frac{1}{2}i} \frac{(z + \frac{1}{2}i)\, 2}{(z + 2i)(z + \frac{1}{2}i)} \right] = \frac{8\pi}{3}$$

$$I = \int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta} = \frac{8\pi}{3}$$

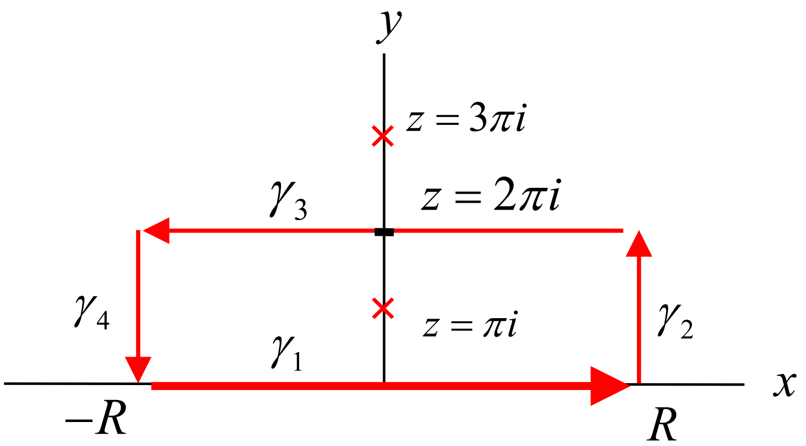
Note:
The poles (roots of the denominator) are found by using the quadratic equation.

Exponential Integrals

❖ There is no general rule for choosing the contour of integration; if the integral can be done by contour integration and the residue theorem, the contour is usually specific to the problem.

Example:

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\, dx \quad , \quad 0 < a < 1$$



Consider the contour integral over the path shown in the figure:

$$I_C = \oint_C \frac{e^{az}}{1+e^{z}}\, dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} \right) \frac{e^{az}}{1+e^{z}}\, dz$$

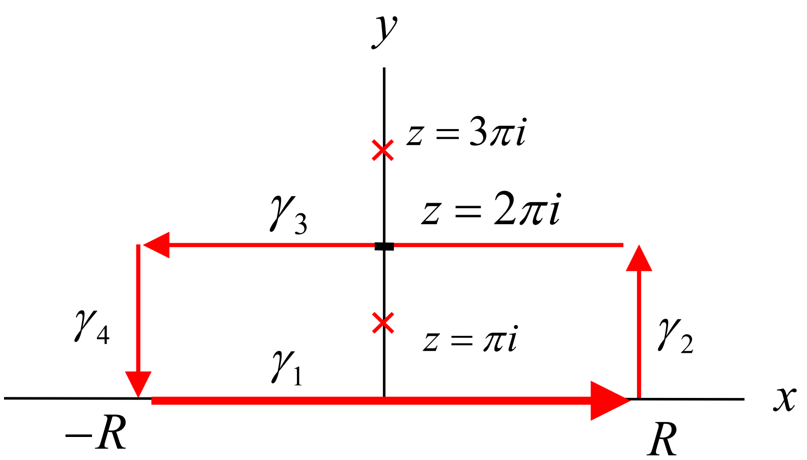
The integrand has simple poles at

$$e^{z} = -1 \quad \Rightarrow \quad z = \pi i + 2\pi n, \quad n = 0, \pm 1, \pm 2, \cdots$$

The contribution from each contour segment in the limit $R \to \infty$ must be separately evaluated (next slide).

Exponential Integrals (cont.)

$$R \to \infty$$



$\gamma_1 : \; z = x, \; dz = dx,$

$$\lim_{R\to\infty} \int_{\gamma_1} \frac{e^{az}}{1+e^{z}}\, dz = I$$

$\gamma_3 : \; z = x + 2\pi i \;, \; dz = dx,$

$$\lim_{R\to\infty} \int_{\gamma_3} \frac{e^{az}}{1+e^{z}}\, dz = \lim_{R\to\infty} e^{ia2\pi} \int_{R}^{-R} \frac{e^{ax}}{1+e^{x}}\, dx = -e^{ia2\pi} I$$

$\gamma_2 : \; z = R + iy \;, \; dz = i\,dy$

$$\left| \int_{\gamma_2} \frac{e^{az}}{1+e^{z}}\, dz \right| = \left| i \int_{0}^{2\pi} \frac{e^{aR} e^{iay}}{1+e^{R} e^{iy}}\, dy \right|$$

$$\left(\left| e^{R} + e^{-iy} \right| > e^{R} - 1 \right)$$

$$\le \int_{0}^{2\pi} \frac{\left| e^{aR} e^{iay} \right|}{\left| 1+e^{R} e^{iy} \right|}\, dy = \int_{0}^{2\pi} \frac{\left| e^{aR} e^{iay} \right|}{\left| e^{iy} \right| \left| e^{R} + e^{-iy} \right|}\, dy \le \int_{0}^{2\pi} \frac{e^{aR}}{e^{R}-1}\, dy = \int_{0}^{2\pi} \frac{e^{(a-1)R}}{1-e^{-R}}\, dy \to 0, \quad a < 1$$

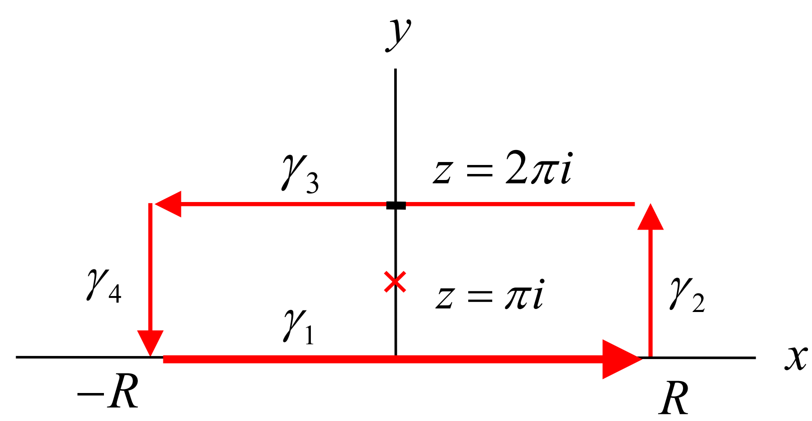
Exponential Integrals (cont.)

$$\gamma_4 :\ z = -R + iy\ ,\ \ dz = i\,dy,$$

$$\left|\int_{\gamma_4} \frac{e^{az}}{1+e^{z}}\,dz\right| = \left| i\int_{2\pi}^{0} \frac{e^{-aR}e^{iay}}{1+e^{-R}e^{iy}}\,dy \right|$$

$$\leq \int_{0}^{2\pi} \frac{e^{-aR}}{1-e^{-R}}\,dy \to 0, \quad a > 0$$

$$\left(\left|1+e^{-R}e^{iy}\right| > 1-e^{-R}\right)$$



Alternatively, $f(z) = \dfrac{g(z)}{h(z)},\ h(z_0) = 0$

$$\Rightarrow \text{Res } f(z = \pi i) = \lim_{z\to z_0} \frac{g(z_0)}{h'(z_0)} = \lim_{z\to \pi i} \frac{e^{az}}{\frac{d}{dz}\left(1+e^{z}\right)}$$

$$= \frac{e^{a\pi i}}{e^{\pi i}} = \frac{e^{a\pi i}}{-1} = -e^{a\pi i}$$

Hence

$$\left(1-e^{ia2\pi}\right)I = 2\pi i \text{ Res } f\left(z = i\pi\right)$$

$$\text{Res } f\left(z=i\pi\right) = \lim_{z\to i\pi}\left(z-i\pi\right)\frac{e^{az}}{1+e^{z}} = \lim_{z\to i\pi}\left(z-i\pi\right)\frac{e^{az}}{1+e^{z-i\pi+i\pi}} = \lim_{z\to i\pi}\left(z-i\pi\right)\frac{e^{az}}{1-e^{z-i\pi}}$$

$$= \lim_{z\to i\pi} \frac{\left(z-i\pi\right)e^{az}}{\cancel{1}-\left[\cancel{1}+\left(z-i\pi\right)+\frac{1}{2}\left(z-i\pi\right)^2+\cdots\right]} = \lim_{z\to i\pi}\frac{e^{az}}{-1-\frac{1}{2}\left(z-i\pi\right)-\cdots} = -e^{ia\pi}$$

Exponential Integrals (cont.)

We thus have

$$\left(1-e^{ia2\pi}\right)I = -2\pi i e^{ia\pi}$$

Therefore

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{-2\pi i\, e^{ia\pi}}{1-e^{ia2\pi}} = \frac{-2\pi i\, \cancel{e^{ia\pi}}}{\cancel{e^{ia\pi}}\left(e^{-ia\pi}-e^{ia\pi}\right)} = \frac{\pi}{\sin a\pi}, \quad 0<a<1$$

Hence

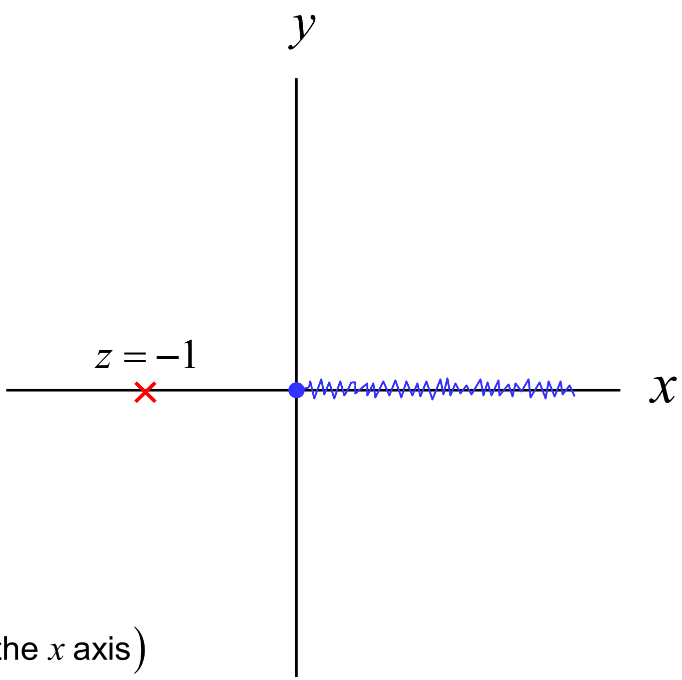
$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin a\pi}, \quad 0<a<1$$

Integration Around a Branch Cut

- A given contour of integration is chosen: usually problem specific, usually must not cross a branch cut.
- We take advantage of the fact that the integrand changes across the branch cut.
- Usually an evaluation of the contribution from the branch point is required.

Example:

$$I = \int_0^\infty \frac{x^{-k}}{x+1}\,dx, \quad 0 < k < 1$$



Assume the branch $0 \le \theta < 2\pi$

$(x^{-k} =$ positive real)

$(x^{-k} = \left(e^{\ln x}\right)^{-k} = e^{-k\ln x} = e^{-k(\ln r + i\theta)} = e^{-k(\ln r)})$ $(\theta = 0$ on the x axis$)$

Note: We choose the branch cut on the positive real axis (the axis of integration).

Integration Around a Branch Cut (cont.)

$$I = \int_0^\infty \frac{x^{-k}}{x+1}\, dx, \quad 0 < k < 1$$

First, note the integral exists since the integral of the asymptotic forms of the integrand at both limits exists:

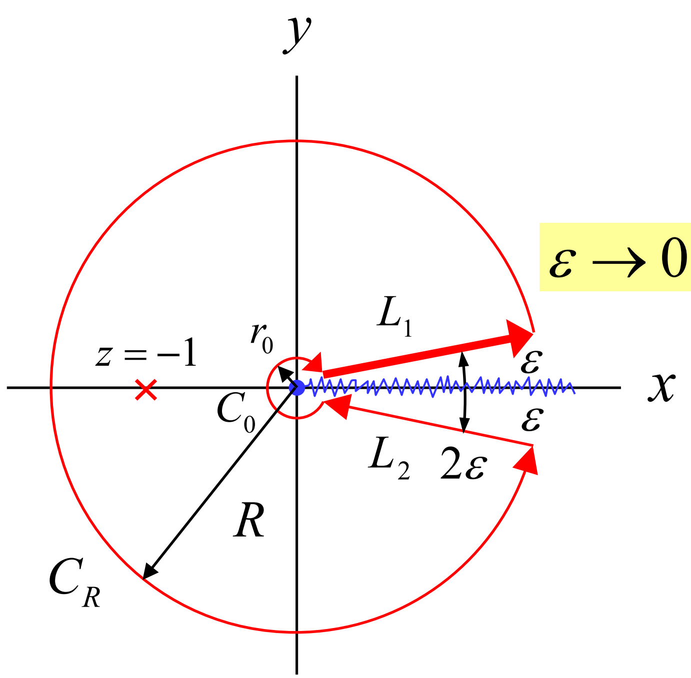
$$\frac{x^{-k}}{x+1} \xrightarrow[x\to 0]{} x^{-k} \text{ which is integrable at } x = 0,\ k < 1$$

$$\frac{x^{-k}}{x+1} \xrightarrow[x\to \infty]{} x^{-k-1} \text{ which is integrable at } x = \infty,\ k > 0$$

Integration Around a Branch Cut

$$I = \int_0^\infty \frac{x^{-k}}{x+1}\,dx, \quad 0 < k < 1$$

Note: $x^{-k} = \dfrac{1}{x^k} = \dfrac{1}{r^k} \quad \left(x = re^{i0}\right)$ (principal branch)



We'll evaluate the integral using the contour shown.

$$0 < \theta < 2\pi$$

For either circle, use:

$$z = re^{i\theta} \quad \left(r = r_0 \text{ or } R\right)$$

$$z^{-k} = e^{-k\ln z} = e^{-k(\ln r + i\theta)} = e^{-k\ln r} e^{-ik\theta} = \left(e^{\ln r}\right)^{-k} e^{-ik\theta} = r^{-k} e^{-ik\theta}, \; 0 < \theta < 2\pi$$

$$z^{-k} = r^{-k} e^{-ik\theta}, \; 0 < \theta < 2\pi$$

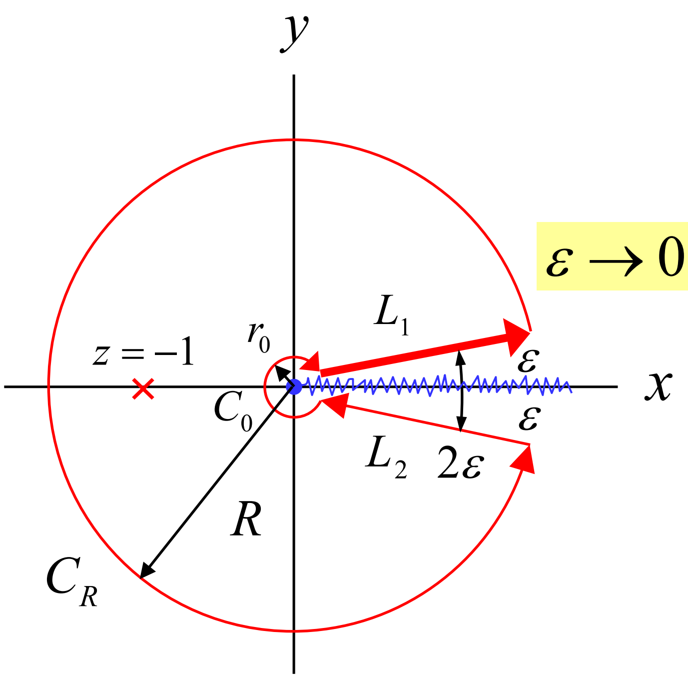
Integration Around a Branch Cut (cont.)

Now consider the various contributions to the contour integral:

$$\oint_{L_1+C_R+L_2+C_0} f(z)\,dz = 2\pi i \operatorname{Res} f(-1),$$

where $f(z) = \dfrac{z^{-k}}{z+1}$

$$z^{-k} = r^{-k} e^{-ik\theta},\ 0 < \theta < 2\pi$$



$$C_0:\ z = r_0 e^{i\theta},\ dz = i r_0 e^{i\theta} d\theta,\ \ z^{-k} = r_0^{-k} e^{-ik\theta}$$

$$\int_{C_0} f(z)\,dz = \lim_{\substack{r_0\to 0,\\ \varepsilon\to 0}} \int_{2\pi-\varepsilon}^{\varepsilon} \frac{r_0^{-k} e^{-ik\theta}\, i r_0 e^{i\theta} d\theta}{r_0 e^{i\theta}+1} = \lim_{r_0\to 0} i r_0^{1-k} \int_{2\pi}^{0} e^{i\theta - ik\theta} d\theta \to 0$$

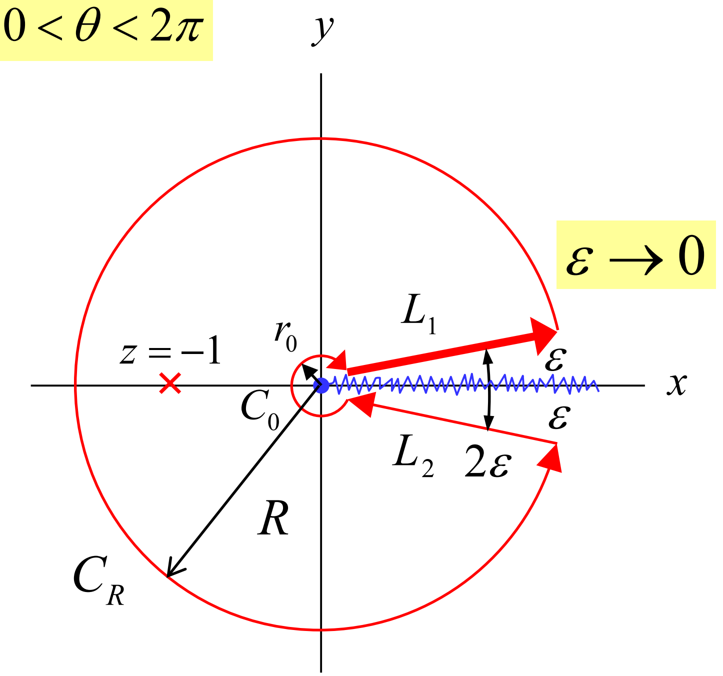
$$C_R:\ z = Re^{i\theta},\ dz = iRe^{i\theta} d\phi,\ \ z^{-k} = R^{-k} e^{-ik\theta}$$

$$\int_{C_R} f(z)\,dz = \lim_{\substack{R\to\infty,\\ \varepsilon\to 0}} \int_{\varepsilon}^{2\pi-\varepsilon} \frac{R^{-k} e^{-ik\theta}\, iRe^{i\theta} d\phi}{Re^{i\theta}+1} = \lim_{R\to\infty} iR^{-k} \int_{0}^{2\pi} e^{-ik\theta} d\phi \to 0$$

Integration Around a Branch Cut (cont.)

$$f(z)=\frac{z^{-k}}{z+1}$$

$$z^{-k}=r^{-k}e^{-ik\theta},\ 0<\theta<2\pi$$



For the path L_1:

$$L_1:\ z=re^{i\varepsilon},\ dz=e^{i\varepsilon}dr,\quad z^{-k}=r^{-k}e^{-ik\varepsilon}$$

$$\int_{L_1} f(z)\,dz=\lim_{\substack{R\to\infty\\ r_0\to 0\\ \varepsilon\to 0}} e^{i(\varepsilon-k\varepsilon)}\int_{r_0}^{R}\frac{r^{-k}\,dr}{re^{i\varepsilon}+1}=\int_0^{\infty}\frac{r^{-k}\,dr}{r+1}=I$$

For the path L_2:

$$L_2:\ z=re^{i(2\pi-\varepsilon)}=re^{-i\varepsilon},\ dz=e^{-i\varepsilon}dr,\quad z^{-k}=r^{-k}e^{-ik(2\pi-\varepsilon)}$$

$$\int_{L_2} f(z)\,dz=\lim_{\substack{R\to\infty\\ r_0\to 0\\ \varepsilon\to 0}} e^{-i[\varepsilon+k(2\pi-\varepsilon)]}\int_{R}^{r_0}\frac{r^{-k}\,dr}{re^{-i\varepsilon}+1}=-e^{-i2\pi k}\int_0^{\infty}\frac{r^{-k}\,dr}{r+1}=-e^{-i2\pi k}I$$

Integration Around a Branch Cut (cont.)

Hence

$$\left(1-e^{-i2\pi k}\right)I = 2\pi i \ \text{Res} \ f\left(z=-1\right)$$

$$= 2\pi i \lim_{z\to -1} \cancel{(z+1)} \frac{z^{-k}}{\cancel{z+1}}$$

$$= 2\pi i \left(-1\right)^{-k}$$

$$= 2\pi i \left(e^{i\pi}\right)^{-k}$$

Note: $\text{Arg}\ (\text{-}1) = \pi$ here ($0 < \theta < 2\pi$).

$$= 2\pi i \, e^{-ik\pi}$$

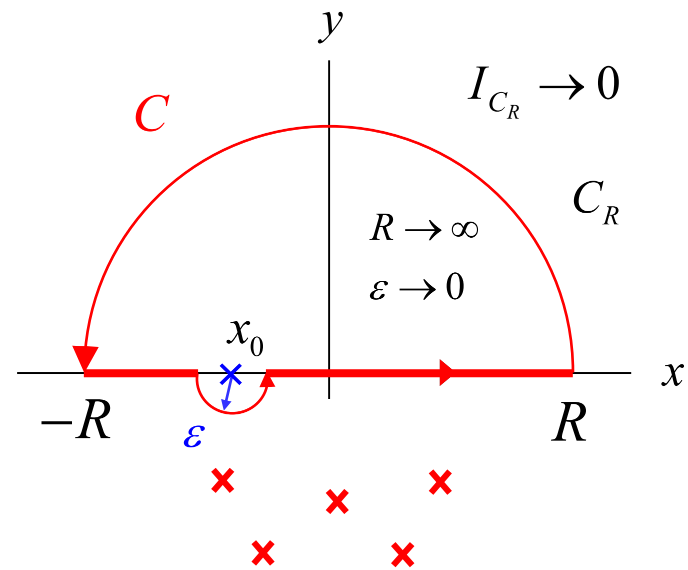
Therefore, we have

$$I = \frac{2\pi i\, e^{-ik\pi}}{\left(1-e^{-i2\pi k}\right)} = \frac{2\pi i\, \cancel{e^{-ik\pi}}}{\cancel{e^{-ik\pi}}\left(e^{ik\pi}-e^{-i\pi k}\right)} = \frac{\pi}{\sin k\pi}, \quad 0 < k < 1$$

Hence

$$I = \int_0^\infty \frac{x^{-k}}{x+1}\, dx = \frac{\pi}{\sin k\pi}, \quad 0 < k < 1$$

Hilbert Transforms



Assumptions:

$f(z) = u + iv$ is analytic in the UHP (including the real axis)

$f(z) \to 0$, z in the UHP

UHP: Upper Half Plane

Consider:

CPV $\qquad\qquad$ $\pi i\, f(x_0)$

$$\oint_C \frac{f(z)}{z-x_0}\,dz = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\left[\left(\int_{-R}^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{R}\right)\frac{f(x)}{x-x_0}\,dx + \int_{\pi}^{2\pi}\frac{f\left(x_0+\varepsilon e^{i\theta}\right)}{\cancel{\varepsilon e^{i\theta}}}\, i\,\cancel{\varepsilon e^{i\theta}}\, d\theta\right] = 2\pi i\, f(x_0)$$

$$\Rightarrow \fint_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx + \pi i\, f(x_0) = 2\pi i\, f(x_0)$$

$$\Rightarrow f(x_0) = \frac{1}{\pi i}\fint_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx$$

$$\Rightarrow u(x_0) + iv(x_0) = \frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{v(x)}{x-x_0}\,dx - \frac{i}{\pi}\fint_{-\infty}^{\infty}\frac{u(x)}{x-x_0}\,dx$$

Let $z - x_0 = \varepsilon e^{i\theta}$

(From the residue theorem or the Cauchy integral theorem.)

Hilbert Transforms (cont.)

$$f(x_0) = u(x_0) + iv(x_0) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{v(x)}{x - x_0} dx - \frac{i}{\pi} \fint_{-\infty}^{\infty} \frac{u(x)}{x - x_0} dx$$

Equate the real and imaginary parts:

$$u(x_0) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{v(x)}{x - x_0} dx$$

$$v(x_0) = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{u(x)}{x - x_0} dx$$

Next, relabel: $x \rightarrow x', \; x_0 \rightarrow x$

Hilbert Transforms (cont.)

Hence, we have

$$u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(x')}{x'-x} dx'$$

$$v(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x')}{x'-x} dx'$$

David Hilbert

$\Rightarrow$ $u(x)$, $v(x)$ are *Hilbert Transforms* of one another

Assumptions:

$f(z) = u + iv$ is analytic in the UHP (including the real axis)

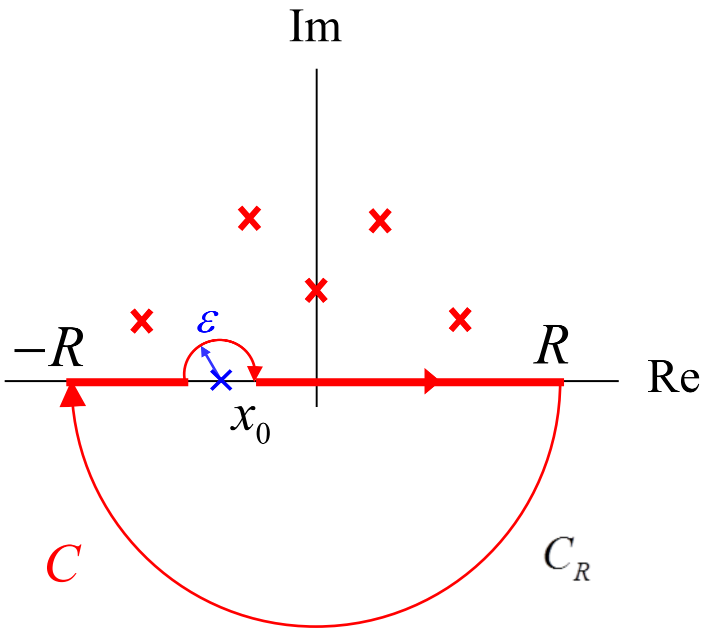
$f(z) \rightarrow 0$, z in the UHP

Hilbert Transforms (cont.)

We can also repeat the derivation assuming that the function is analytic in the <u>lower</u> half plane.

$$u(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(x')}{x'-x} dx'$$

$$v(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x')}{x'-x} dx'$$

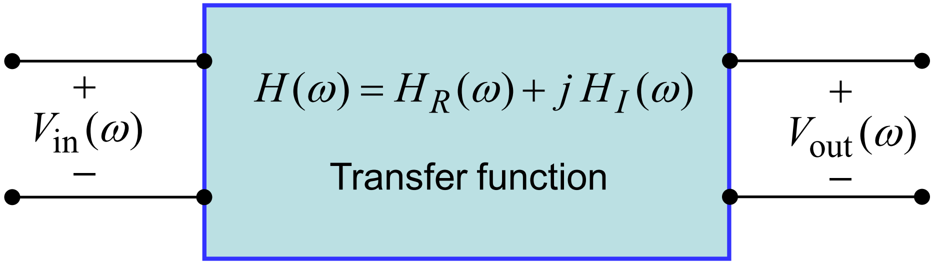


Assumptions :

$f(z) = u + iv$ is analytic in the LHP (including the real axis)

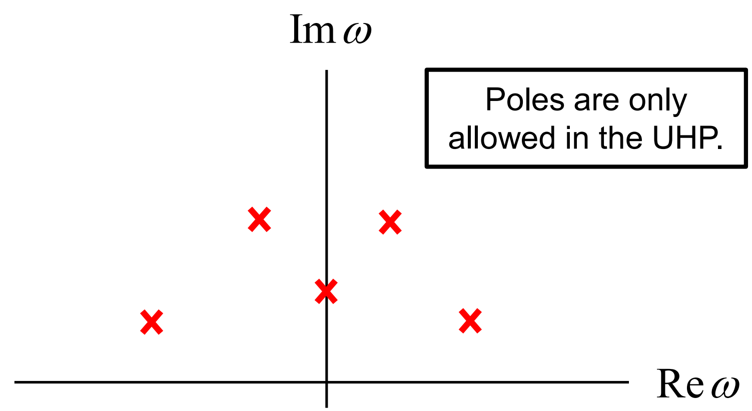
$f(z) \to 0$, z in the LHP

Hilbert Transforms in Circuit Theory



$$H(\omega) \equiv V_{out}(\omega) / V_{in}(\omega)$$

(using j instead of i)



LHP: Lower Half Plane

Assumptions $\left(\exp(j\omega t)\right.$ time convention$\left.\right)$:

$H(\omega)$ is analytic in the LHP$^{(1)}$

$H(\omega) \to 0$, for $\omega \to \infty$ in the LHP$^{(2)}$

Note 1: A pole in the LHP would corresponds to a nonphysical growing response:

$$\omega = \omega_r + j\omega_i \Rightarrow e^{j\omega t} = e^{j\omega_r t}\, e^{-\omega_i t}$$

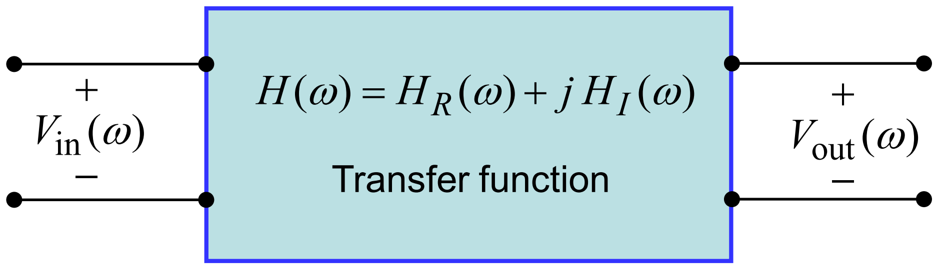
Note 2: The system is assumed to be unable to respond to a signal at very high frequency.

Symmetry property: $H(-\omega) = H^*(\omega) \Rightarrow H_R(-\omega) = H_R(\omega);\; H_I(-\omega) = -H_I(\omega)$

(see Appendix)

Hilbert Transforms in Circuit Theory (cont.)

We thus have:



$$H(\omega) \equiv V_{\text{out}}(\omega) / V_{\text{in}}(\omega)$$

$$H_R(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H_I(\omega')}{\omega' - \omega}\, d\omega'$$

$$H_I(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H_R(\omega')}{\omega' - \omega}\, d\omega'$$

The real and imaginary parts of the transfer function are Hilbert transforms of each other.

Hilbert Transforms in Circuit Theory (cont.)

We can also derive an alternative form:

$$H_R(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H_I(\omega')}{\omega'-\omega} d\omega' = -\frac{1}{\pi}\int_{-\infty}^{0} \frac{H_I(\omega')}{\omega'-\omega} d\omega' - \frac{1}{\pi}\int_{0}^{\infty} \frac{H_I(\omega')}{\omega'-\omega} d\omega'$$

First one :
Use $\omega' \rightarrow -\omega'$

$$= \frac{1}{\pi}\int_{\infty}^{0} \frac{H_I(-\omega')}{-\omega'-\omega} d\omega' - \frac{1}{\pi}\int_{0}^{\infty} \frac{H_I(\omega')}{\omega'-\omega} d\omega'$$

First one :
Use $H_I(-\omega) = -H_I(\omega)$

$$= -\frac{1}{\pi}\int_{0}^{\infty} \frac{H_I(\omega')}{\omega'+\omega} d\omega' - \frac{1}{\pi}\int_{0}^{\infty} \frac{H_I(\omega')}{\omega'-\omega} d\omega'$$

$$= -\frac{1}{\pi}\int_{0}^{\infty} \frac{H_I(\omega')\left((\omega'-\omega)+(\omega'+\omega)\right)}{(\omega'+\omega)(\omega'-\omega)} d\omega'$$

$$= -\frac{1}{\pi}\int_{0}^{\infty} \frac{H_I(\omega')(2\omega')}{\left(\omega'^2-\omega^2\right)} d\omega'$$

This integral starts at zero.

Similarly, we have

$$H_I(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H_R(\omega')}{\omega'-\omega} d\omega' = -\frac{1}{\pi}\int_{0}^{\infty} \frac{H_R(\omega')(2\omega)}{\left(\omega'^2-\omega^2\right)} d\omega'$$

Dispersion Relation: Circuit Theory (cont.)

Summarizing, we have:

$$H_R(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_I(\omega')}{\omega'-\omega}d\omega' = -\frac{2}{\pi}\int_{0}^{\infty}\frac{\omega' H_I(\omega')}{\left(\omega'^2-\omega^2\right)}d\omega'$$

$$H_I(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_R(\omega')}{\omega'-\omega}d\omega' = \frac{2}{\pi}\int_{0}^{\infty}\frac{\omega H_R(\omega')}{\left(\omega'^2-\omega^2\right)}d\omega'$$

Assumptions:

$H(\omega)$ is analytic in the LHP

$H(\omega)\rightarrow 0$, for $\omega\rightarrow\infty$ in the LHP

Kramers-Kronig Relations

Hendrik Anthony “Hans” Kramers

Ralph de Laer Kronig

- The Kramers-Kronig relations are a fundamental set of equations that relate the real and imaginary parts of the complex permittivity.

Kramers-Kronig Relations

Assumption:

The *relative permittivity* $\varepsilon_r(\omega)$ is analytic in the LHP and $\varepsilon_r(\omega) \to 1,\ \omega$ in LHP

Similar to the transfer-function analysis, one obtains the *Kramers-Kronig* dispersion relations:

$$\operatorname{Re}\left(\varepsilon_r(\omega)-1\right) = -\frac{2}{\pi}\int_0^\infty \frac{\omega' \operatorname{Im}\left(\varepsilon_r(\omega')-1\right)}{\omega'^2-\omega^2}\,d\omega'$$

$$\operatorname{Im}\left(\varepsilon_r(\omega)-1\right) = \frac{2}{\pi}\int_0^\infty \frac{\omega \operatorname{Re}\left(\varepsilon_r(\omega')-1\right)}{\omega'^2-\omega^2}\,d\omega'$$

Material parameters :

$\varepsilon_r \equiv$ relative permittivity

$\chi_e \equiv$ electric susceptibility

$\chi_e(\omega) \equiv \varepsilon_r(\omega)-1$

$\underline{P} = \varepsilon_0 \chi_e \underline{E}$ (polarization per unit volume)

$\chi_e(\omega) \to 0$ as $\omega \to \infty$ in LHP

Kramers-Kronig Relations (cont.)

Denote $\varepsilon_r(\omega) = \varepsilon_r'(\omega) - j\varepsilon_r''(\omega)$ (using j instead of i)

The final form of the Kramers-Kronig relations is then:

$$\varepsilon_r'(\omega) = 1 + \frac{2}{\pi} \int_0^{\infty} \frac{\omega' \varepsilon_r''(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$\varepsilon_r''(\omega) = -\frac{2}{\pi} \int_0^{\infty} \frac{\omega \left(\varepsilon_r'(\omega') - 1\right)}{\omega'^2 - \omega^2} d\omega'$$

Note: This shows that if there is no loss ($\varepsilon_r'' = 0$), then the relative permittivity must be 1. Hence, practical materials with $\varepsilon_r' > 1$ will always have some loss.

Laplace Transform

The Laplace transform is defined as:

$$F(s) \equiv \int_0^{\infty} f(t)\, e^{-st}\, dt$$

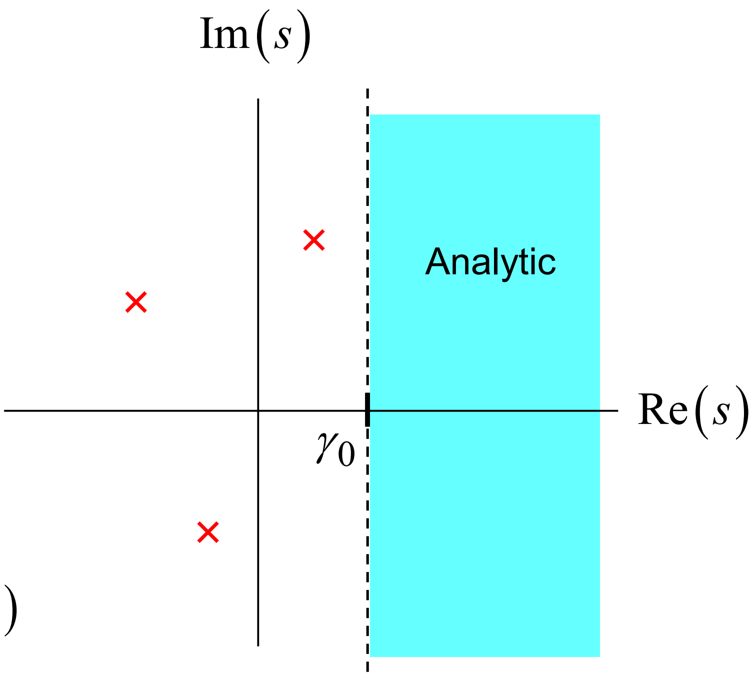
Assume

$$\left| f(t) \right| \le A e^{\gamma_0 t}, \quad t > 0$$

Then $F(s)$ is analytic in the region

$$\mathrm{Re}(s) > \gamma_0$$

Note: $F'(s) = \int_0^{\infty} (-t) f(t)\, e^{-st}\, dt \;\left(\text{valid for } \mathrm{Re}\, s > \gamma_0\right)$



Note: In the Laplace transform we do not care how f is defined for $t < 0$.

Laplace Transform (cont.)

Define a function $g(t)$ (using any $\gamma > \gamma_0$):

$$g(t) \equiv \begin{cases} e^{-\gamma t} f(t), & t > 0 \\ 0, & t < 0 \end{cases}$$

The Fourier transform of $g(t)$ <u>exists</u>.
(The function $g(t)$ stays finite along the entire real axis and tends to zero as $t \to \pm\infty$.)

We then have, from the inverse Fourier transform,

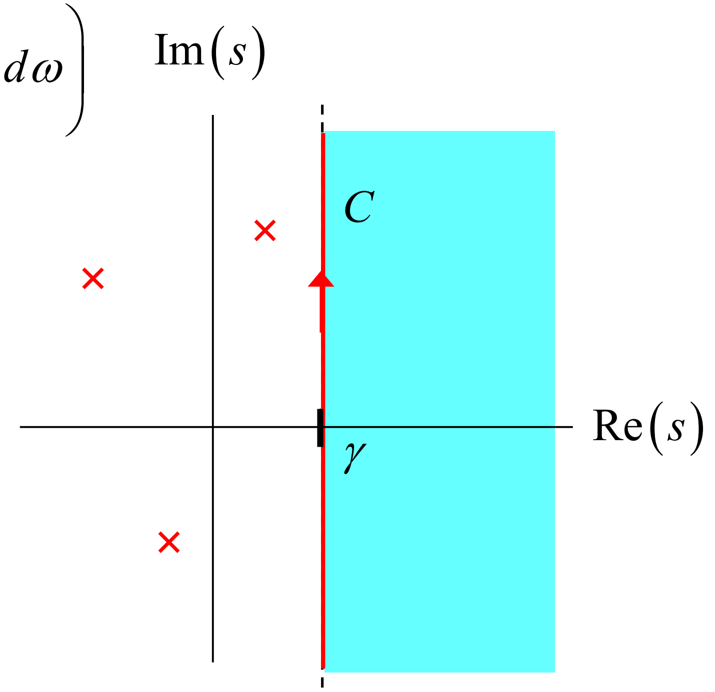
$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega) e^{i\omega t} d\omega \qquad \left(\tilde{g}(\omega) \equiv \int_{-\infty}^{\infty} g(t) e^{-i\omega t} d\omega \right)$$

Then, from the relation between f and g:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega) e^{\gamma t} e^{i\omega t} d\omega, \quad t > 0$$

Let $s \equiv \gamma + i\omega$

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{g}\left(-i(s-\gamma)\right) e^{st} ds, \quad t > 0$$

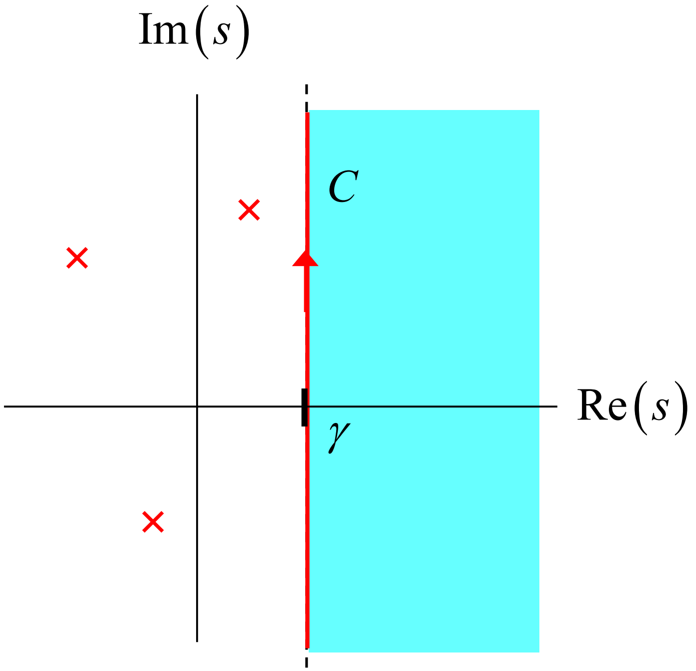


Laplace Transform (cont.)

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{g}\left(-i\left(s-\gamma\right)\right) e^{st} ds, \quad t > 0$$

We have for the integrand term:

$$\begin{aligned}
\tilde{g}\left(-i\left(s-\gamma\right)\right) &= \int_{-\infty}^{\infty} g(t) e^{-i\left(-i\left(s-\gamma\right)\right)t} dt \\
&= \int_{0}^{\infty} g(t) e^{-i\left(-i\left(s-\gamma\right)\right)t} dt \\
&= \int_{0}^{\infty} g(t) e^{-\left(s-\gamma\right)t} dt \\
&= \int_{0}^{\infty} f(t) e^{-st} dt \\
&= F(s)
\end{aligned}$$



Laplace Transform (cont.)

Hence, we have:

$$f(t) = f_B(t) \equiv \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds, \quad t > 0$$

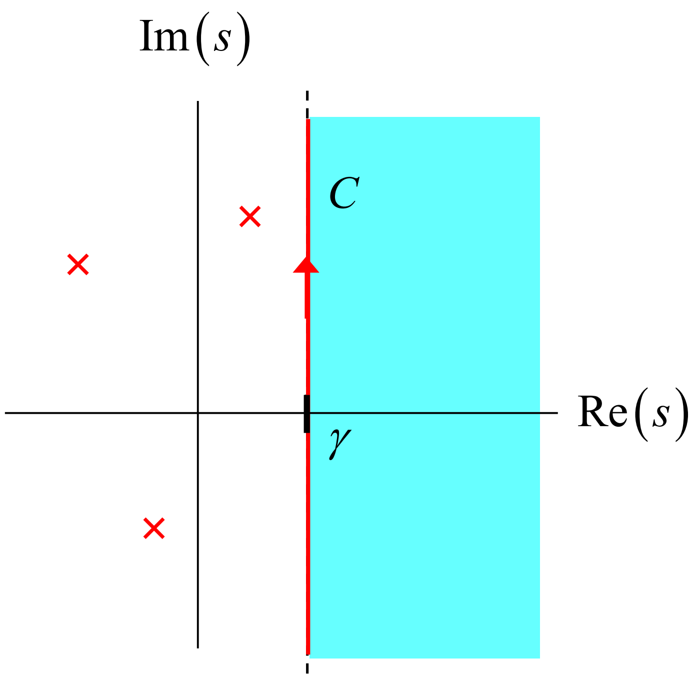
"Inverse Laplace transform"

"Bromwich integral"

The **Bromwich** contour C is chosen to the <u>right</u> of all singularities of the function $F(s)$.

$$|f(t)| \leq A e^{\gamma_0 t}, \quad t > 0$$

$$\gamma > \gamma_0$$



Laplace Transform (cont.)

Consider the case:

$t < 0$

$$f_B(t) \equiv \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds$$

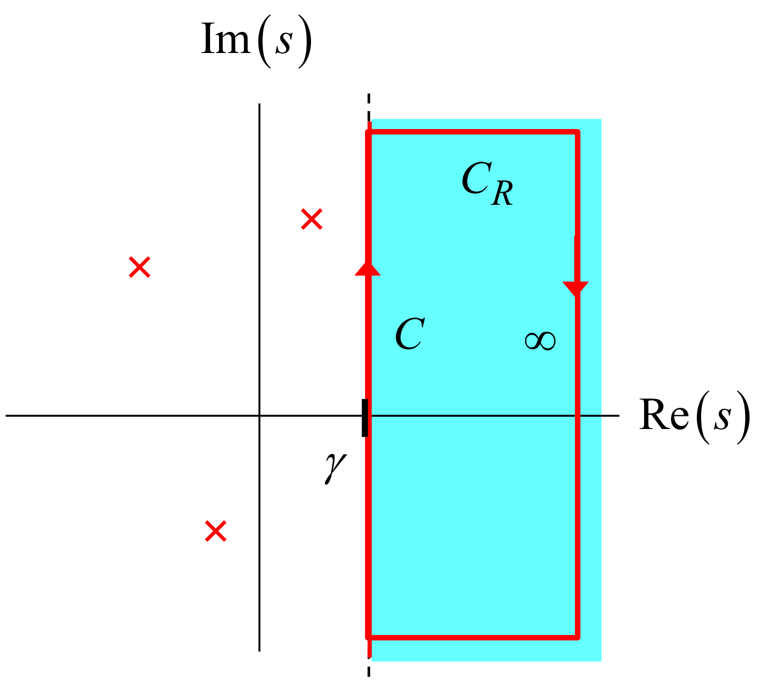
Close the contour to the right.

The integrand is analytic inside C_R.

Only the vertical path C contributes as $C_R \to \infty$:

Right: $e^{st} \to 0$

Top and bottom: $F(s) \to 0$ (assumption)



$$\Rightarrow f_B(t) = \frac{1}{2\pi i} \oint_{C_R} F(s) e^{st} ds$$

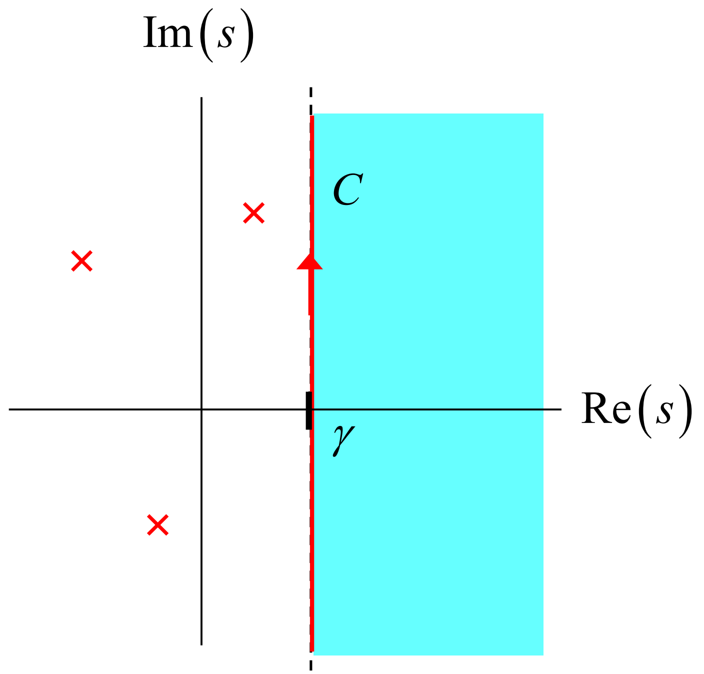
$$\Rightarrow f_B(t) = 0$$

(by Cauchy's theorem)

Laplace Transform (cont.)

Summary

$$f_B(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds = \begin{cases} f(t), & t > 0 \\ 0, & t < 0 \end{cases}$$



Note:
This inverse Laplace transform (Bromwich) integral gives us zero for $t < 0$, no matter how the original function $f(t)$ was defined for $t < 0$.

Laplace Transform (cont.)

Evaluation of the Bromwich integral for the case of poles only:

$t > 0$

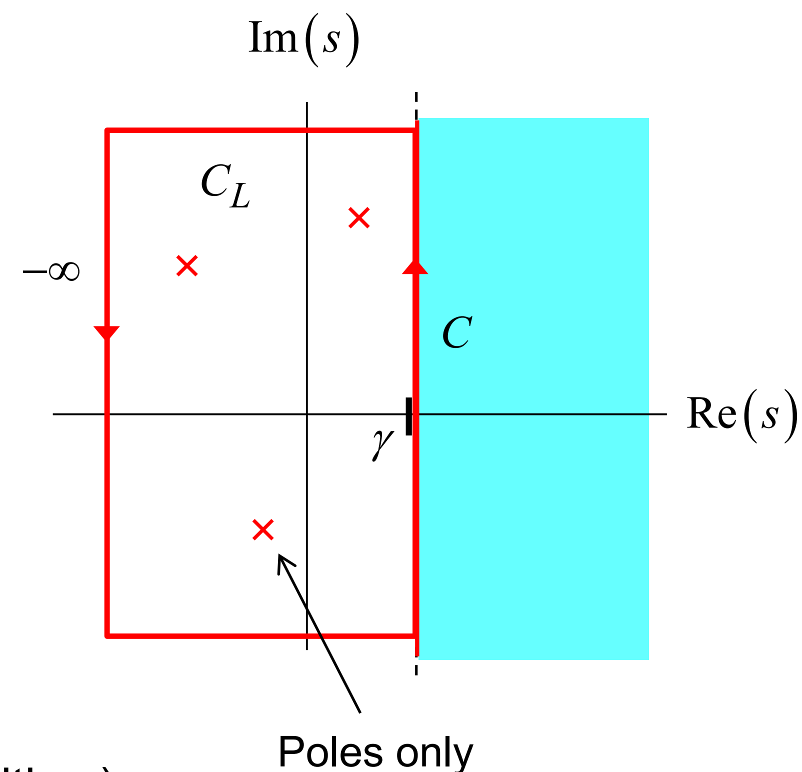
$$f(t) = f_B(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds$$

Close the contour to the left.

Only the vertical path C contributes as $C_L \to \infty$:

Left: $e^{st} \to 0$

Top and bottom: $F(s) \to 0$ (assumption)



$$\Rightarrow f(t) = \frac{1}{2\pi i} \oint_{C_L} F(s) e^{st} ds$$

$\Rightarrow f(t) = \sum$ residues at poles to the left of C

(This assumes that there are only pole singularities.)

Laplace Transform (cont.)

Example $f(t) = e^{at}$ Note : $\gamma_0 = a \;\Rightarrow\; \gamma > a$

$$F(s) = \int_0^{\infty} e^{at}\, e^{-st}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt = \frac{-1}{s-a} e^{-(s-a)t} \Big|_0^{\infty}$$

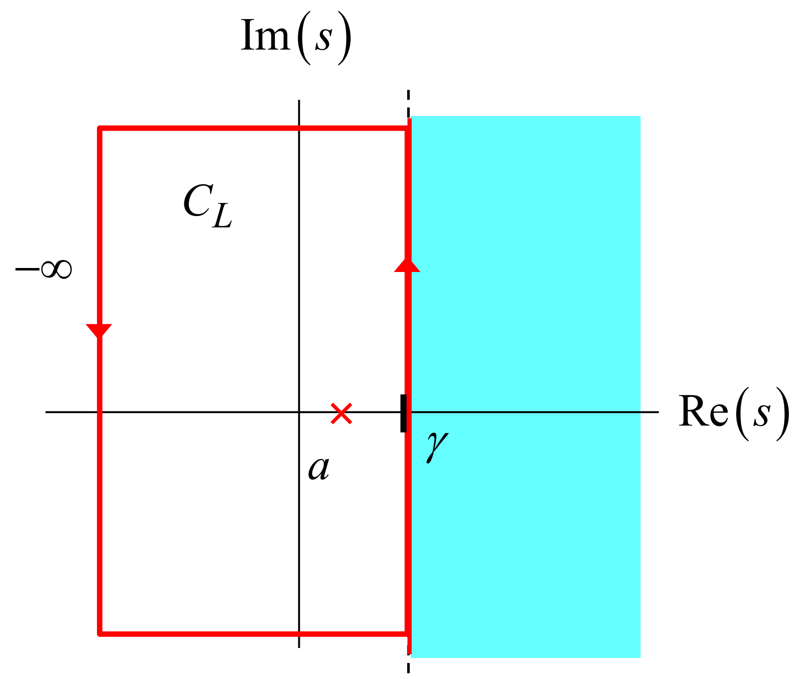
$$F(s) = \frac{1}{s-a}, \quad \operatorname{Re}(s) > a$$

For $t > 0$:

$$f(t) = f_B(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{s-a} e^{st}\, ds, \quad \gamma > a$$

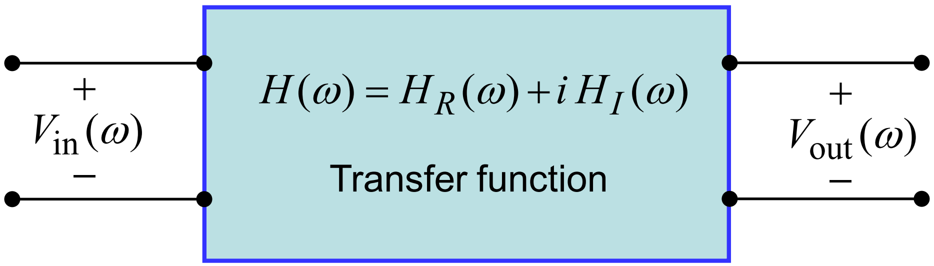
$$f_B(t) = \frac{1}{2\pi i}\left(2\pi i \operatorname{Res}\left(\frac{1}{s-a} e^{st}\right)_{s=a}\right)$$

$$\Rightarrow \quad f_B(t) = e^{at}, \; t > 0$$



Appendix

Proof of symmetry property: $H(-\omega) = H^*(\omega)$



$$H(\omega) \equiv V_{\text{out}}(\omega) / V_{\text{in}}(\omega)$$

Impulse response:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{i\omega t}\, d\omega = \text{real} \qquad H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, dt$$

so

Use $\omega' = -\omega$

$$h^*(t) = h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^*(\omega) e^{-i\omega t}\, d\omega = -\frac{1}{2\pi}\int_{\infty}^{-\infty} H^*(-\omega') e^{i\omega' t}\, d\omega' = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^*(-\omega') e^{i\omega' t}\, d\omega' = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^*(-\omega) e^{i\omega t}\, d\omega$$

Relabel $\omega' \to \omega$

Hence:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^*(-\omega) e^{i\omega t}\, d\omega \Rightarrow F^{-1}\left[H(\omega)\right] = F^{-1}\left[H^*(-\omega)\right] \Rightarrow H^*(-\omega) = H(\omega)$$