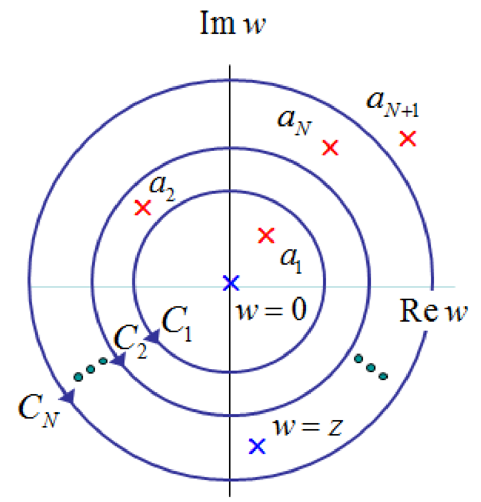


ECE 6382

Fall 2023

David R. Jackson

Notes 12

Series and Product Expansions, and Series Summation

Notes are from D. R. Wilton, Dept. of ECE

Pole Expansion of a Function (Mittag-Leffler Theorem)

Mittag-Leffler Theorem

Assumptions :

- $f(z)$ has simple poles at $z = a_n,\ n = 1, 2, \ldots,$ where $0 < |a_1| < |a_2| < |a_3| < \cdots$
- $f(z)$ has residue b_n at $z = a_n,\ n = 1, 2, \ldots$
- $|f(z)| \le M$, independent of N, on circles C_N of radius R_N that enclose N poles, not passing through any of them and such that $R_N \xrightarrow[N\to\infty]{} \infty$

An extension would be <u>pairs</u> of poles occurring at increasing distance from the origin.

Then

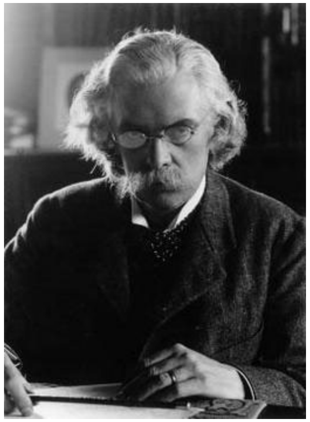
$$f(z) = f(0) + \sum_{n=1}^{\infty} \left(\frac{b_n}{z - a_n} + \frac{b_n}{a_n} \right)$$

Note:
If the sum is finite, this becomes a partial fraction expansion.

This is an expansion in terms of residues!

Note: Each of the two individual series in the sum may not converge.

Note that a pole at the origin is not allowed!
But we can always shift using $z \to z - z_0$.
Or we can add a term to cancel a pole that appears at the origin.



Magnus Gustaf (Gösta) Mittag-Leffler

Proof of Mittag-Leffler Theorem

For $z \neq a_n$, consider the sequence of contour integrals :

$$I_N \equiv \frac{1}{2\pi i} \oint_{C_N} F(w)\, dw \;=\; \sum \text{residues}\, F(w)$$

where

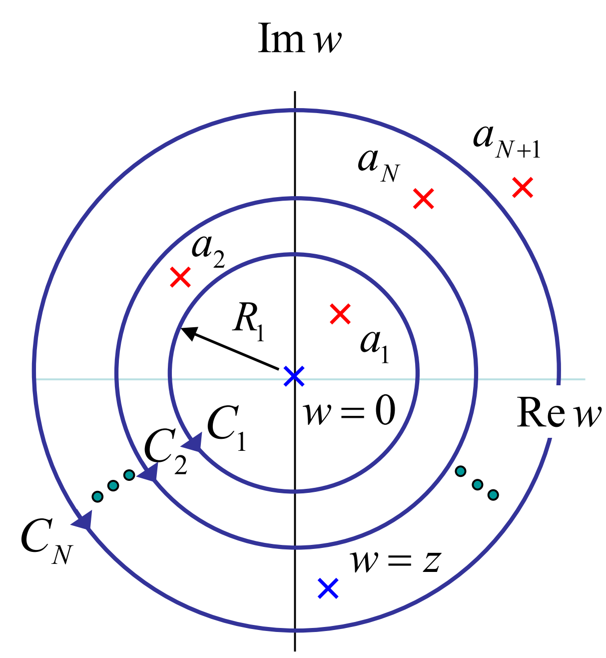
$$F(w) \equiv \frac{f(w)}{w(w-z)}$$

$$\mathrm{Res}\, F(w_p) \;=\; \lim_{w \to w_p} \left(w - w_p\right) F(w)$$

$$\text{Residue}\, F(0) = \frac{f(0)}{-z}$$

$$\text{Residue}\, F(z) = \frac{f(z)}{z}$$

$$\text{Residue}\, F(a_n) = \frac{b_n}{a_n(a_n - z)}$$



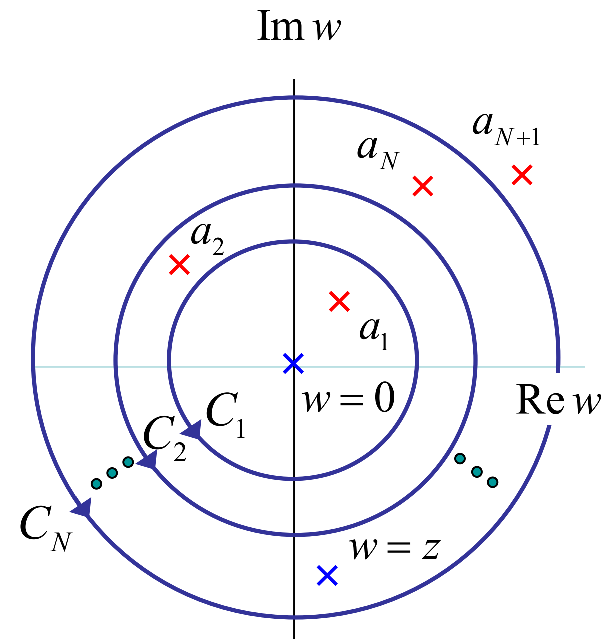
Proof of Mittag-Leffler Theorem

$$I_N \equiv \frac{1}{2\pi i}\oint_{C_N} \frac{f(w)\,dw}{w(w-z)} = \left(\frac{f(0)}{-z}+\frac{f(z)}{z}\right)+\sum_{n=1}^{N}\frac{b_n}{a_n(a_n-z)}$$ (from the residue theorem)

But we also have for $w = R_N e^{i\theta}$ on C_N

$$|I_N| \le \left|\frac{1}{2\pi i}\oint_{C_N}\frac{f(w)\,dw}{w(w-z)}\right| \le \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(R_N e^{i\theta})|\,R_N\,d\theta}{R_N(R_N-|z|)}$$

$$\le \frac{1}{\cancel{2\pi}}\frac{\cancel{2\pi}M\cancel{R_N}}{\cancel{R_N}(R_N-|z|)} \xrightarrow[N\to\infty]{} 0$$

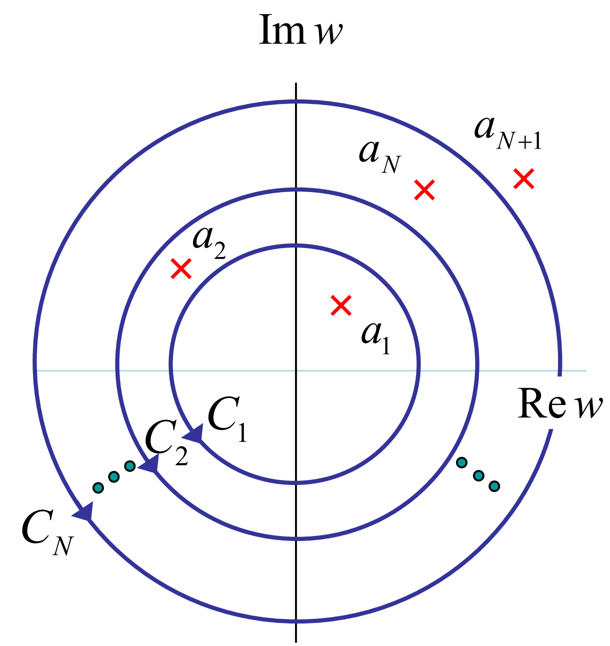


Taking the limit as $N\to\infty$, we have

$$I_N \to 0 \quad \Rightarrow \quad \boxed{\left(\frac{f(0)}{-z}+\frac{f(z)}{z}\right)+\sum_{n=1}^{\infty}\frac{b_n}{a_n(a_n-z)} = 0}$$

Proof of Mittag-Leffler Theorem

$$\left(\frac{f(0)}{-z}+\frac{f(z)}{z}\right)+\sum_{n=1}^{\infty}\frac{b_n}{a_n(a_n-z)}=0$$



Multiply by z, and solve for $f(z)$:

$$f(z)=f(0)-\sum_{n=1}^{\infty}\frac{zb_n}{a_n(a_n-z)}$$

Then use a partial fraction expansion:

$$f(z)=f(0)-\sum_{n=1}^{\infty}\left(\frac{b_n}{a_n-z}-\frac{b_n}{a_n}\right)=f(0)+\sum_{n=1}^{\infty}\left(\frac{b_n}{z-a_n}+\frac{b_n}{a_n}\right)$$

(proof complete)

Extended Form of the Mittag-Leffler Theorem

Extended Mitag-Leffler Theorem

Asumptions :

- $f(z)$ has simple poles at $z = a_n,\ n = 1, 2, \ldots,$ where $0 < |a_1| < |a_2| < |a_3| < \cdots$
- $f(z)$ has residue b_n at $z = a_n,\ n = 1, 2, \ldots$
- $|f(z)| < M|z|^p$ (for integer p), independent of N, on circles C_N of radius R_N that enclose N poles, not passing through any of them and such that $R_N \xrightarrow[N\to\infty]{} \infty$

Then consideration of the integral

$$I_N = \frac{1}{2\pi i}\oint_{C_N} \frac{f(w)\,dw}{w^{p+1}(w-z)}$$

leads to the this expansion :

$$f(z) = f(0) + zf'(0) + \cdots + \frac{z^p f^{(p)}(0)}{p!} + \sum_{n=1}^{\infty}\left(\frac{b_n z^{p+1} / a_n^{p+1}}{z - a_n}\right)$$

Note: The first p terms are those of a Taylor series.

Example: Pole Expansion of $\cot z$

$\cot z = \dfrac{\cos z}{\sin z}$ has poles $z = n\pi,\ n = 0, \pm 1, \pm 2, \ldots$ with residues

$$\lim_{z \to n\pi} \frac{\cos z}{\frac{d}{dz} \sin z} = \lim_{z \to n\pi} \frac{\cos z}{\cos z} = 1. \quad \text{Hence, } b_n = 1.$$

Then, since a pole at $z = 0$ is not allowed, consider

$$f(z) = \cot z - \frac{1}{z},$$

for which the singularity at the origin has been removed and which has a removeable singularity at the origin with a finite limit :

$$\lim_{z \to 0}\left(\cot z - \frac{1}{z}\right) = \lim_{z \to 0}\left(\frac{\cos z}{\sin z} - \frac{1}{z}\right) = \lim_{z \to 0}\left(\frac{z \cos z - \sin z}{z \sin z}\right)$$

$$= \lim_{z \to 0}\left(\frac{z\left(\cancel{1} - \frac{z^2}{2!} + \right) - \left(\cancel{z} - \frac{z^3}{3!} + \right)}{z\left(z - \frac{z^3}{3!} + \right)}\right) = \lim_{z \to 0} \frac{\left(-\frac{1}{2} + \frac{1}{6}\right) z^3 + \mathcal{O}(z^5)}{z^2 + \mathcal{O}(z^4)} \to 0$$

(The finite limit is actually zero here.)

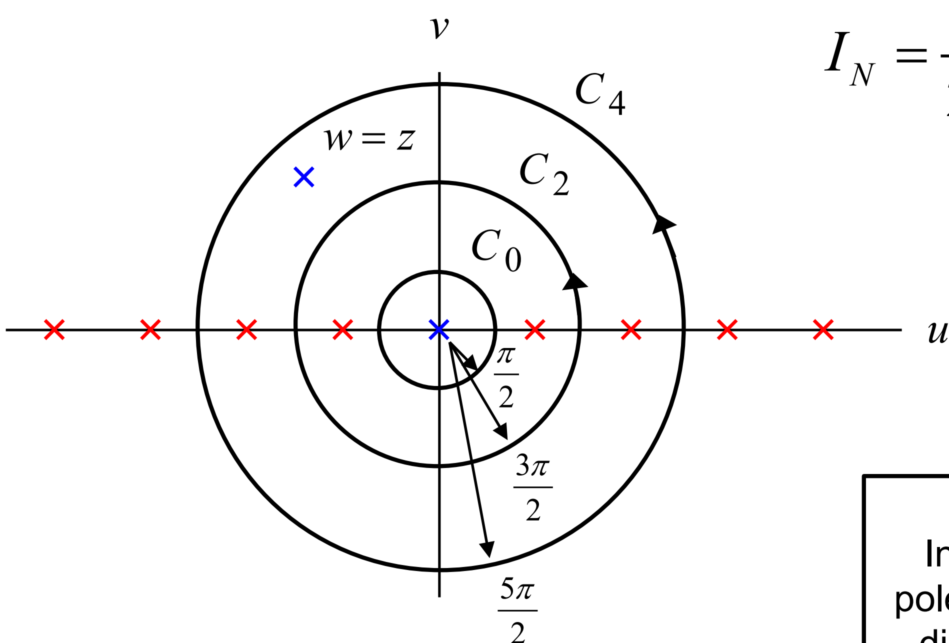
Example: Pole Expansion of $\cot z$ (cont.)

Figure showing the circles

$$f(z) = \cot z - \frac{1}{z}$$

Poles at: $a_n = \pm n\pi,\ n = 1, 2, \ldots$

$$I_N = \frac{1}{2\pi i} \oint_{C_N} \frac{f(w)\,dw}{w(w-z)}$$



Note:
In this case it is <u>pairs</u> of poles that are of increasing distance from the origin.

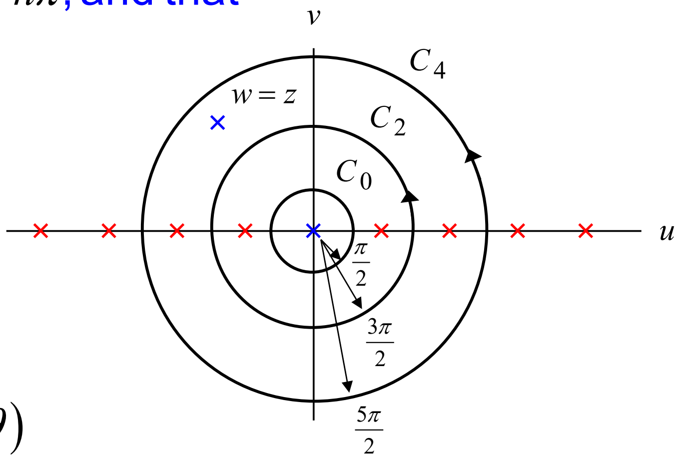
In this case N is always even.

Example: Pole Expansion of $\cot z$ (cont.)

Assumption : $|f(z)| < M$, independent of N, on circles C_N

To check that $\cot z = \dfrac{\cos z}{\sin z}$ is bounded on C_N, use $z = R_N e^{i\theta}$ where

$R_N = \dfrac{(2N-1)\pi}{2}$ so that C_N threads between the poles at $z = n\pi$, and that



$$|\cot z|^2 = \left|\frac{\cos\left(R_N e^{i\theta}\right)}{\sin\left(R_N e^{i\theta}\right)}\right|^2 = \left|\frac{\cos\left[R_N\left(\cos\theta + i\sin\theta\right)\right]}{\sin\left[R_N\left(\cos\theta + i\sin\theta\right)\right]}\right|^2$$

$$= \left|\frac{\cos\left(R_N\cos\theta\right)\cosh\left(R_N\sin\theta\right) - i\sin\left(R_N\cos\theta\right)\sinh\left(R_N\sin\theta\right)}{\sin\left(R_N\cos\theta\right)\cosh\left(R_N\sin\theta\right) + i\cos\left(R_N\cos\theta\right)\sinh\left(R_N\sin\theta\right)}\right|^2$$

$$= \frac{\cos^2\left(R_N\cos\theta\right)\cosh^2\left(R_N\sin\theta\right) + \sin^2\left(R_N\cos\theta\right)\sinh^2\left(R_N\sin\theta\right)}{\sin^2\left(R_N\cos\theta\right)\cosh^2\left(R_N\sin\theta\right) + \cos^2\left(R_N\cos\theta\right)\sinh^2\left(R_N\sin\theta\right)}$$

$$= \frac{\cosh^2\left(R_N\sin\theta\right) - \sin^2\left(R_N\cos\theta\right)}{\cosh^2\left(R_N\sin\theta\right) - \cos^2\left(R_N\cos\theta\right)}$$

where we've used $\sinh^2 x = \cosh^2 x - 1$ and $\sin^2 x + \cos^2 x = 1$ in both numerator and denominator.

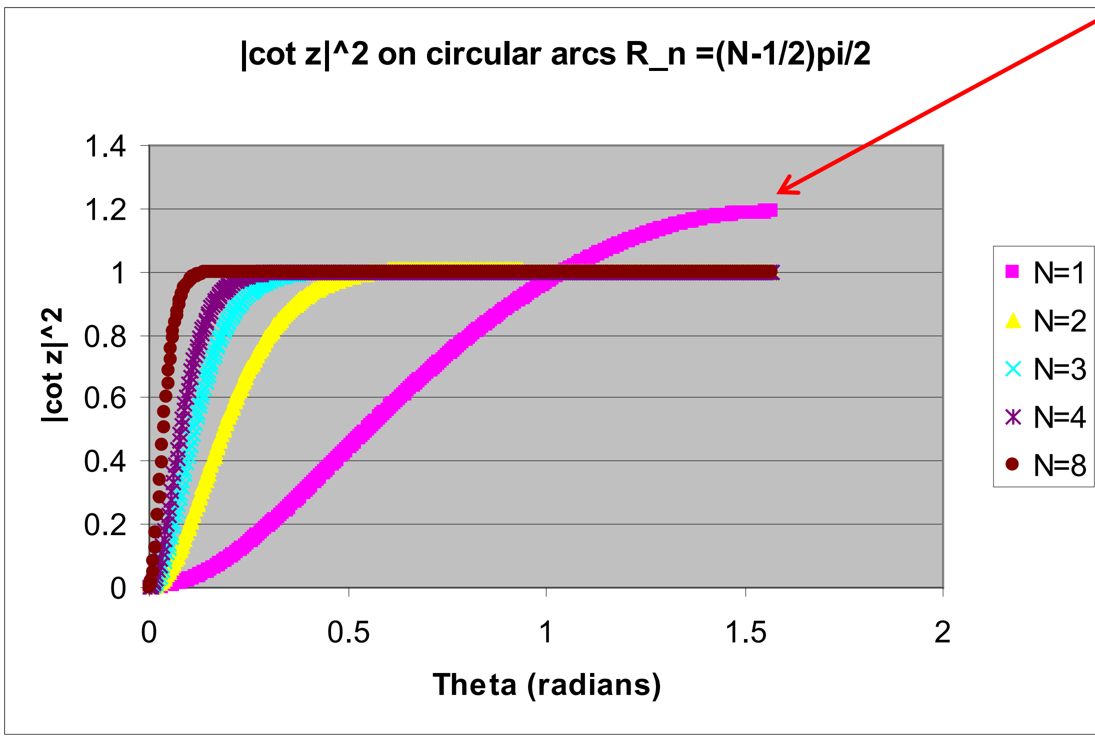
Example: Pole Expansion of $\cot z$ (cont.)

Now if we plot the expression

$$\left|\cot z\right|^2 = \frac{\cosh^2\left(R_N \sin\theta\right) - \sin^2\left(R_N \cos\theta\right)}{\cosh^2\left(R_N \sin\theta\right) - \cos^2\left(R_N \cos\theta\right)} \qquad \left(R_N = \frac{(2N-1)\pi}{2}\right)$$

for $0 \le \theta \le \pi/2$, for various values of N, it appears that $\left|\cot z\right|^2 < 1.18823$

so that $\left|\cot z\right| < M = \sqrt{1.18823}$, independent of N.



It isn't necessary that the paths C_N be circular; see the next slide.

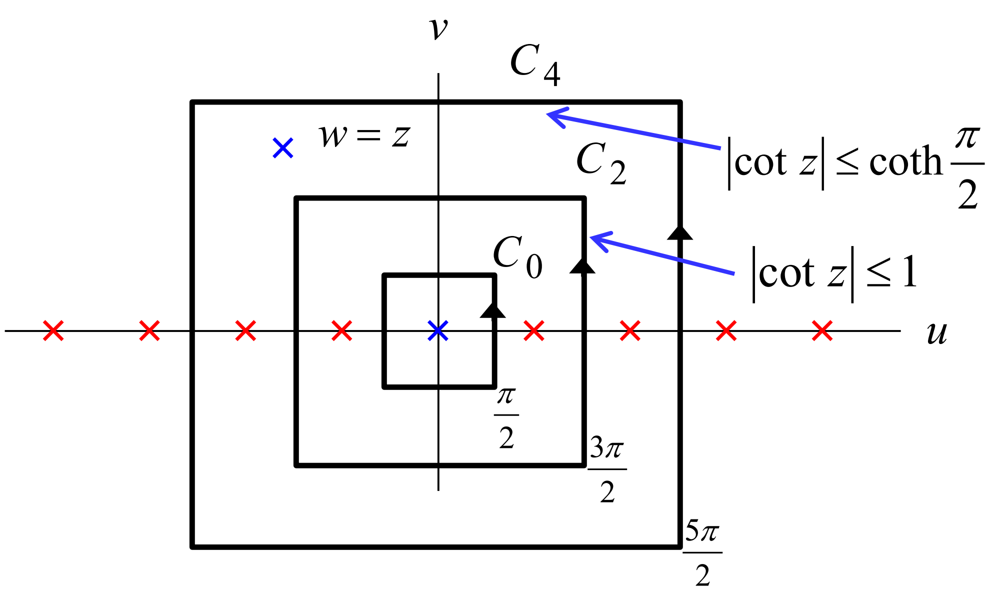
Example: Pole Expansion of $\cot z$ (cont.)

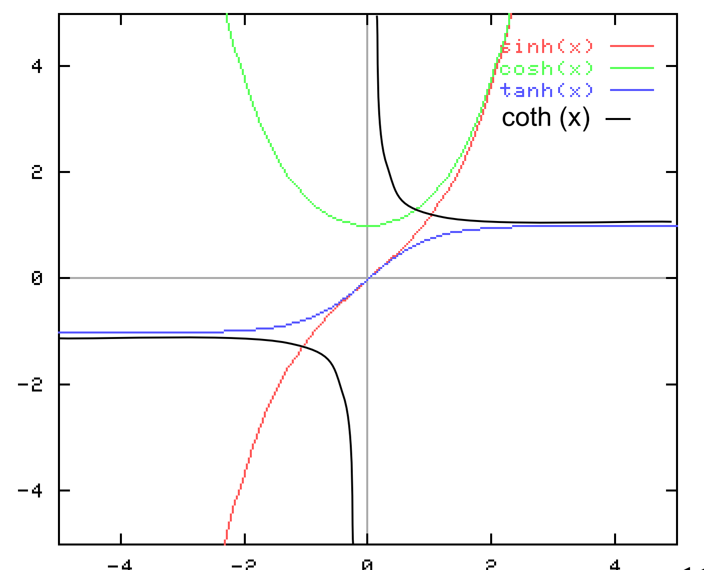
Alternatively, we could consider the expression on the contours shown below :

$$|\cot z|^2 = \left|\frac{\cos(x+iy)}{\sin(x+iy)}\right|^2 = \left|\frac{\cos x\cosh y - i\sin x\sinh y}{\sin x\cosh y + i\cos x\sinh y}\right|^2 = \frac{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y}{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}$$

$$\le \begin{cases} \tanh^2 y \le 1, \quad x = \pm(2N-1)\pi/2 \\ \dfrac{\cos^2 x\cosh^2 y + \sin^2 x\cosh^2 y}{\sin^2 x\sinh^2 y + \cos^2 x\sinh^2 y} = \coth^2 y \le \coth^2 \pi/2, \quad y = \pm(2N-1)\pi/2 \end{cases}$$

It is easy to then show that $I_N = \dfrac{1}{2\pi i}\oint_{C_N} \dfrac{f(w)\,dw}{w(w-z)} \xrightarrow[N\to\infty]{} 0$ on these C_N also.





Example: Pole Expansion of $\cot z$ (cont.)

Hence, we have

$$f(z)=\cot z-\frac{1}{z}=f(0)+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{b_n}{(z-a_n)}+\frac{b_n}{a_n}\right)\qquad (a_n=\pm n\pi,\ n=1,2,\ldots)$$

$$=f(0)+\sum_{n=1}^{\infty}\left(\frac{b_n}{(z-a_n)}+\frac{b_n}{a_n}\right)+\sum_{n=-1}^{-\infty}\left(\frac{b_n}{(z-a_n)}+\frac{b_n}{a_n}\right)$$

$$=\quad 0+\sum_{n=1}^{\infty}\left(\frac{1}{(z-n\pi)}+\frac{1}{n\pi}\right)+\sum_{n=-1}^{-\infty}\left(\frac{1}{(z-n\pi)}+\frac{1}{n\pi}\right)$$

$$=\quad 0+\sum_{n=1}^{\infty}\left(\frac{1}{(z-n\pi)}+\cancel{\frac{1}{n\pi}}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{(z+n\pi)}-\cancel{\frac{1}{n\pi}}\right)$$

Combining the two series and putting over a common demoninator,

$$\boxed{\cot z=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{2z}{z^2-n^2\pi^2}\right)}$$

This result could be helpful for summing this series in closed form.

Other Pole Expansions

- $$\frac{1}{\sin z} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$
- $$\frac{1}{\sinh z} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2\pi^2}$$
- $$\frac{1}{\cos z} = \sum_{n=0}^{\infty} \frac{(2n+1)\pi}{\left(\frac{(2n+1)\pi}{2}\right)^2 - z^2}$$
- $$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{(2n+1)\pi}{\left(\frac{(2n+1)\pi}{2}\right)^2 + z^2}$$
- $$\tan z = \sum_{n=0}^{\infty} \frac{2z}{\left(\frac{(2n+1)\pi}{2}\right)^2 - z^2}$$
- $$\tanh z = \sum_{n=0}^{\infty} \frac{2z}{\left(\frac{(2n+1)\pi}{2}\right)^2 + z^2}$$
- $$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$
- $$\coth z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2\pi^2}$$

The Mittag-Leffler theorem generalizes the partial fraction representation of a rational function to meromorphic functions that have an infinite number of poles.

Infinite Product Expansion of Entire Functions

Weierstrass's Factorization Theorem

Karl Theodor Wilhelm Weierstrass

Assumptions:

- $f(z)$ is an entire function
- $f(z)$ has simple zeros at $z = a_n,\ n = 1, \dots,$

  where $0 < |a_1| < |a_2| < |a_3| < \cdots$
- $\left|\dfrac{f'(z)}{f(z)}\right| < M$, independent of N, on circles C_N of radius R_N that do not pass through zeros of $f(z)$ and such that $R_N \xrightarrow[N\to\infty]{} \infty$

Then

$$f(z) = f(0)\, e^{\frac{z f'(0)}{f(0)}} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}$$

There exists a generalization to multiple-order zeros (Schaum's *Complex Variables*, p. 267).

Product Expansion Formula

Proof :

Near any simple zero a_n, $f(z)$ must have the form $f(z)=(z-a_n)g(z)$,
where $g(z)$ is analytic and non-vanishing at $z=z_0$. Hence the *logarithmic derivative,*

$$\frac{f'(z)}{f(z)}=\frac{d}{dz}\ln f(z)=\frac{d}{dz}\ln(z-a_n)+\frac{d}{dz}\ln g(z)=\frac{1}{(z-a_n)}+\frac{g'(z)}{g(z)},$$

has a simple pole at each point $z=a_n$ with residue 1. By the Mittag-Leffler theorem, $\frac{f'(z)}{f(z)}=\frac{f'(0)}{f(0)}+\sum_{n=1}^{\infty}\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right)$, which, on integrating both sides, yields

$$\int_0^z\frac{f'(z)}{f(z)}\,dz=\ln f(z)-\ln f(0)=\ln\frac{f(z)}{f(0)}=z\frac{f'(0)}{f(0)}+\sum_{n=1}^{\infty}\left(\ln\left(\frac{(z-a_n)}{-a_n}\right)+\frac{z}{a_n}\right).$$

Upon exponentiating, we obtain the desired result,

$$f(z)=f(0)e^{\frac{zf'(0)}{f(0)}}e^{\sum_{n=1}^{\infty}\left[\ln\left(1-\frac{z}{a_n}\right)+\frac{z}{a_n}\right]}=f(0)e^{\frac{zf'(0)}{f(0)}}\prod_{n=1}^{\infty}e^{\ln\left(1-\frac{z}{a_n}\right)}e^{\frac{z}{a_n}}=f(0)e^{\frac{zf'(0)}{f(0)}}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}$$

Useful Product Expansions

- $$\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$$
- $$\sinh z = z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2\pi^2}\right)$$
- $$\cos z = \prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)$$
- $$\cosh z = \prod_{n=1}^{\infty}\left(1+\frac{4z^2}{(2n-1)^2\pi^2}\right)$$

$$\Rightarrow \tan z = z\prod_{n=1}^{\infty}\frac{\left(1-\frac{z^2}{n^2\pi^2}\right)}{\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)}$$

$$\Rightarrow \tanh z = z\prod_{n=1}^{\infty}\frac{\left(1+\frac{z^2}{n^2\pi^2}\right)}{\left(1+\frac{4z^2}{(2n-1)^2\pi^2}\right)}$$

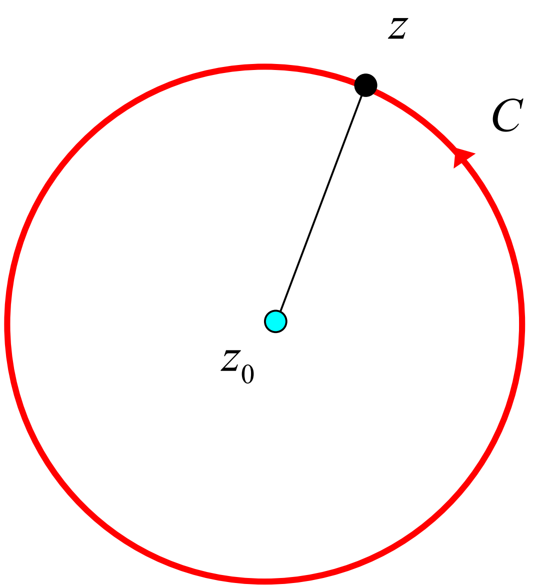
Product expansions generalize the factorization of the numerator and denominator polynomials of a rational function into products of their roots.

The Argument Principle

First consider the following integral:

$$I \equiv \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_C \frac{d}{dz}\ln f(z)\,dz$$

"log derivative"



where $f(z) = (z - z_0)^M$

(The integer M can be <u>either</u> positive or negative.)

zero pole

$$I = \frac{1}{2\pi i}\oint_C \frac{d}{dz}\ln\left[(z-z_0)^M\right]dz = \frac{1}{2\pi i}\oint_C M\frac{d}{dz}\ln(z-z_0)\,dz = \frac{M}{2\pi i}\oint_C \frac{1}{z-z_0}\,dz = M$$

Simple pole with residue 1

So we have

$$I = M$$

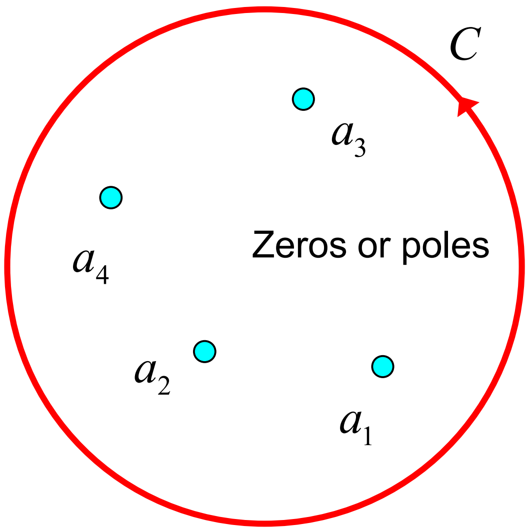
Note: The path C does not have to be a circle.

The Argument Principle (cont.)

- Next, we consider extending this to an <u>arbitrary</u> function that is analytic inside a region except for poles of finite order.
- The function may also have zeros.

$$I \equiv \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \oint_C \frac{d}{dz} \ln f(z)\, dz$$

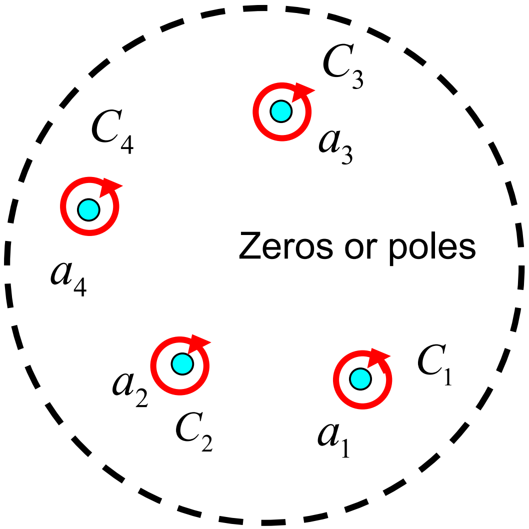
The integrand in analytic everywhere except where f has a zero or a pole.



We can shrink the path C down to small paths that go around each zero and pole.

The Argument Principle (cont.)

$$I = \frac{1}{2\pi i} \sum_n \oint_{C_n} \frac{d}{dz} \ln f(z)\, dz$$



Assume that $f(z)$ has a pole or a zero of order (multiplicity) M_n at $z = a_n$.

$M_n > 0$: zero of order M_n

$M_n < 0$: pole of order $-M_n$

Taylor series for $g(z)$ (valid till we hit the closest singularity to a_n)

Near a_n we can write

$$f(z) = (z - a_n)^{M_n} \left(b_0 + b_1 (z - a_n) + b_2 (z - a_n)^2 + \cdots \right) = (z - a_n)^{M_n} g(z)$$

where $g(z)$ is analytic and non-vanishing at $z = a_n$.

The Argument Principle (cont.)

Near a_n: $f(z) = (z - a_n)^{M_n}\, g(z), \quad g(a_n) \neq 0$

Consider the logarithmic derivative near a_n :

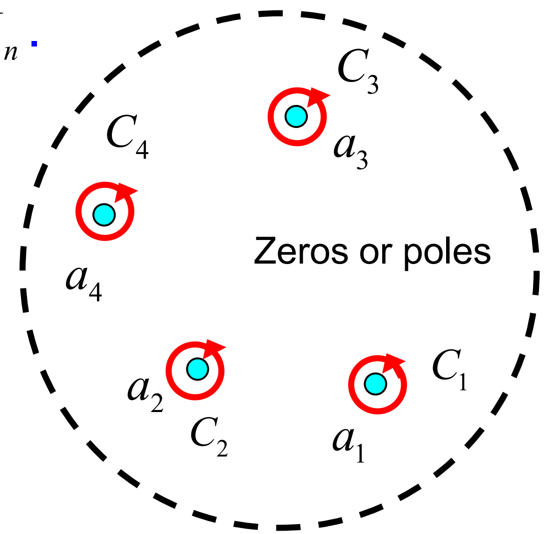
$$\frac{d}{dz}\ln f(z) = \frac{d}{dz}\ln\left((z-a_n)^{M_n} g(z)\right) = M_n \frac{d}{dz}\ln(z-a_n) + \frac{d}{dz}\ln g(z)$$

$$= \frac{M_n}{(z-a_n)} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{Analytic at } a_n}$$

This function has a simple pole at $z = a_n$ with residue M_n .

Therefore

$$\boxed{\oint_C \frac{d}{dz}\ln f(z)\, dz = 2\pi i \sum_n M_n}$$



The Argument Principle (cont.)

Summary

$$\frac{1}{2\pi i}\oint_C \frac{d}{dz}\ln f(z)\,dz = Z - P$$

where $\begin{cases} Z = \text{number of zeros inside } C \\ P = \text{number of poles inside } C \end{cases}$

Note: In counting the zeros and poles, we include multiplicities. (For example, at a double zero, we add 2; at a double pole we add -2).

Note: We assume that the function only has zeros and poles of finite order, and no other singularities.

The Argument Principle (cont.)

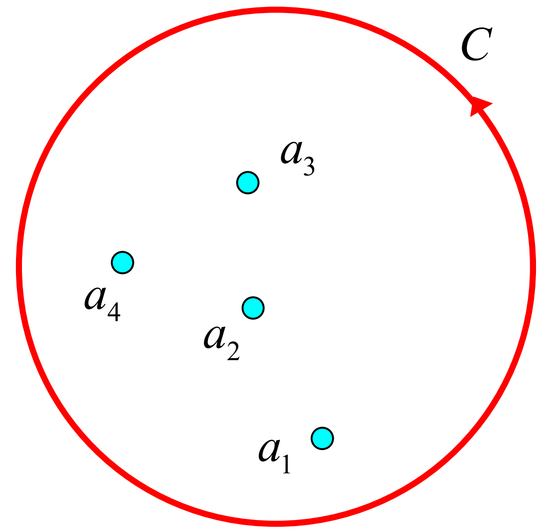
Since the integrand is an exact differential, we also have:

$$\frac{1}{2\pi i}\oint_C \frac{d}{dz}\ln f(z)\,dz = \frac{1}{2\pi i}\ln f(z)\Big|_{\text{beginning point of } C}^{\text{endpoint of } C}$$

Use $f(z) = |f(z)|\,e^{i\arg(f(z))}$

$$= \underbrace{\frac{1}{2\pi i}\ln|f(z)|\Big|_{\text{beginning point of } C}^{\text{endpoint of } C}}_{=0} + \frac{\arg f(z)}{2\pi}\Bigg|_{\text{beginning point of } C}^{\text{endpoint of } C}$$

$$= \frac{1}{2\pi}\times \text{ change in } \arg f(z) \text{ as } z \text{ goes around } C.$$



Note: The argument must change continuously!

Hence

$$Z - P = \frac{1}{2\pi}\times \text{ change in } \arg f(z) \text{ (radians) as } z \text{ goes around } C.$$

This is the result from which the “argument principle” gets its name.

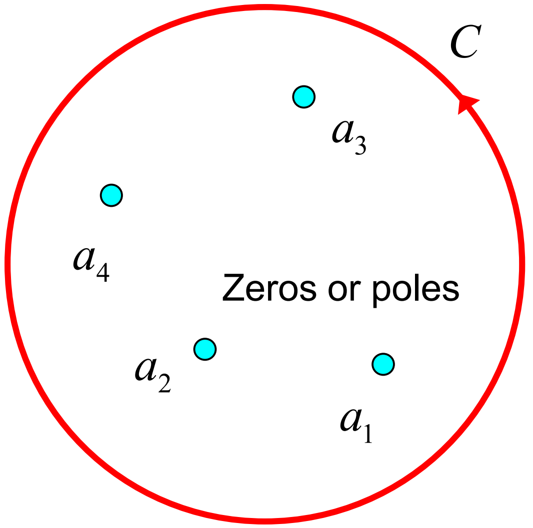
The Argument Principle (cont.)

Summary of Argument Principle (three different forms):

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = Z - P$$

$$\frac{1}{2\pi i}\oint_C \frac{d}{dz}\ln f(z)\,dz = Z - P$$

$$\frac{1}{2\pi}\times \text{ change in } \arg f(z) \text{ as } z \text{ goes around } C \;=\; Z - P$$



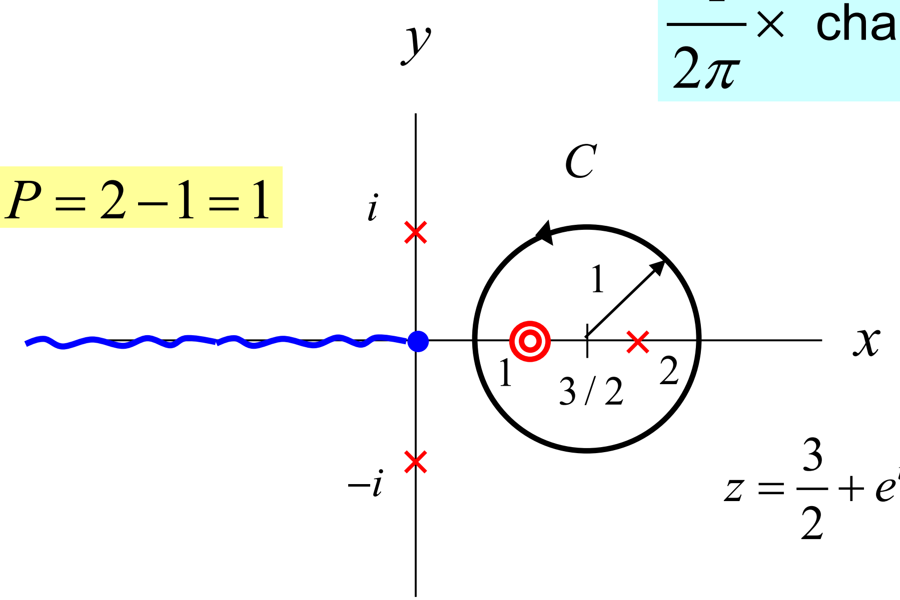
The Argument Principle (cont.)

Example

$$f(z) = \sqrt{z}\left(\frac{(z-1)^2}{(z^2+1)(z-2)}\right)$$

$$\frac{1}{2\pi} \times \text{ change in } \arg f(z) = Z - P = 1$$

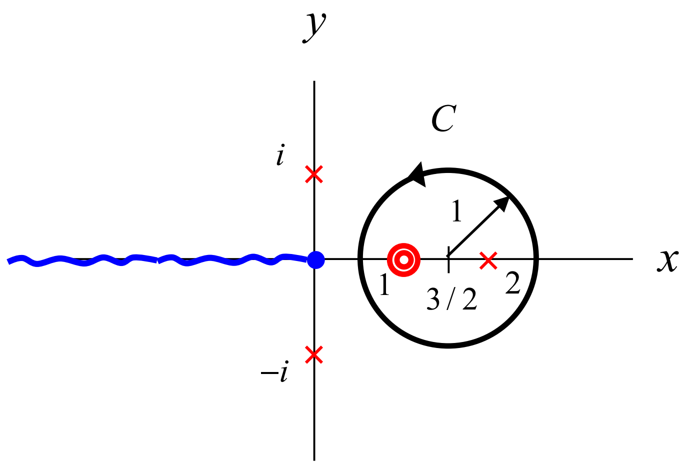
$Z - P = 2 - 1 = 1$



$$z = \frac{3}{2} + e^{i\phi}$$

The path C is chosen as circle centered at $(3/2, 0)$ of radius 1.

The Argument Principle (cont.)

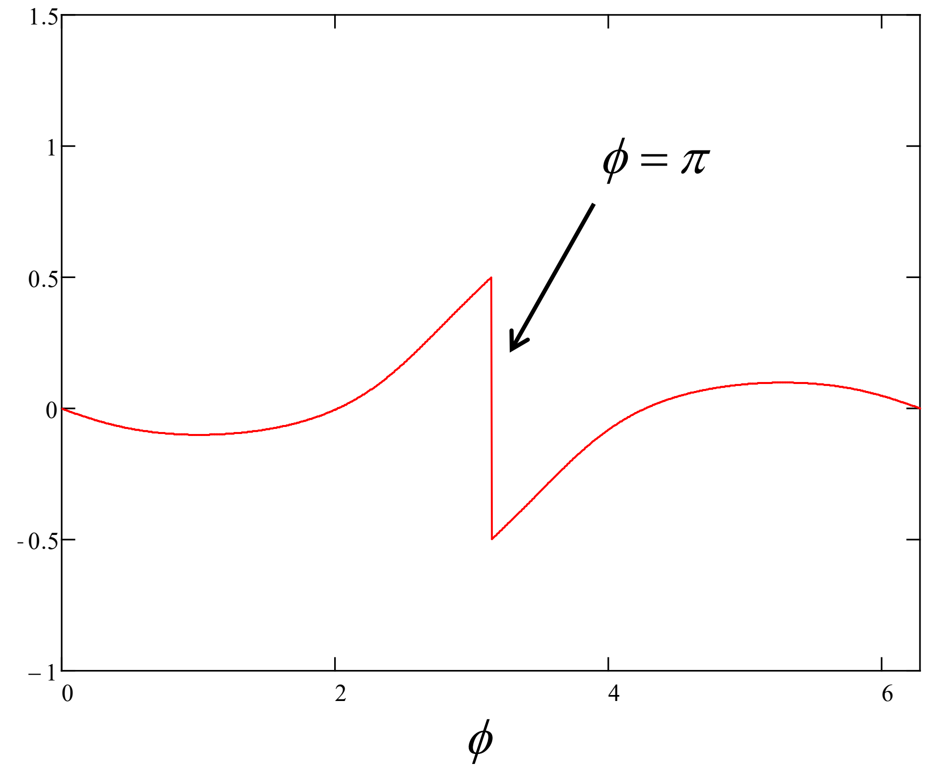


$$f(z)=\sqrt{z}\left(\frac{(z-1)^2}{(z^2+1)(z-2)}\right)$$

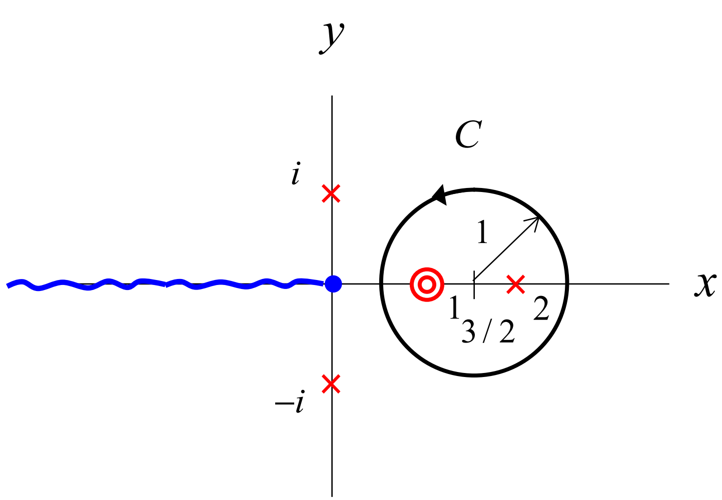
$$z=\frac{3}{2}+(1)e^{i\phi}$$

The original plot from Mathcad

$$\frac{\arg(f(\phi))}{2\pi}$$



The Argument Principle (cont.)

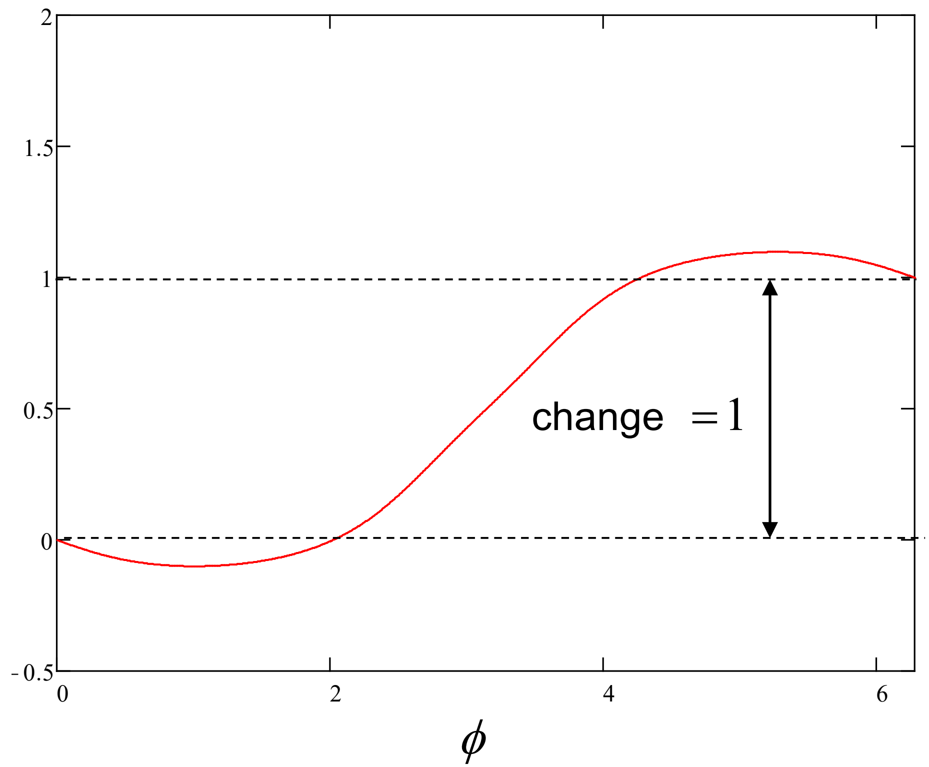


$$f(z)=\sqrt{z}\left(\frac{(z-1)^2}{(z^2+1)(z-2)}\right)$$

$$z=\frac{3}{2}+(1)e^{i\phi}$$

$$\frac{\arg(f(\phi))}{2\pi}$$

We add 2π after $\phi=\pi$.



$$\frac{1}{2\pi}\times \text{ change in } \arg f(z) \quad =1$$

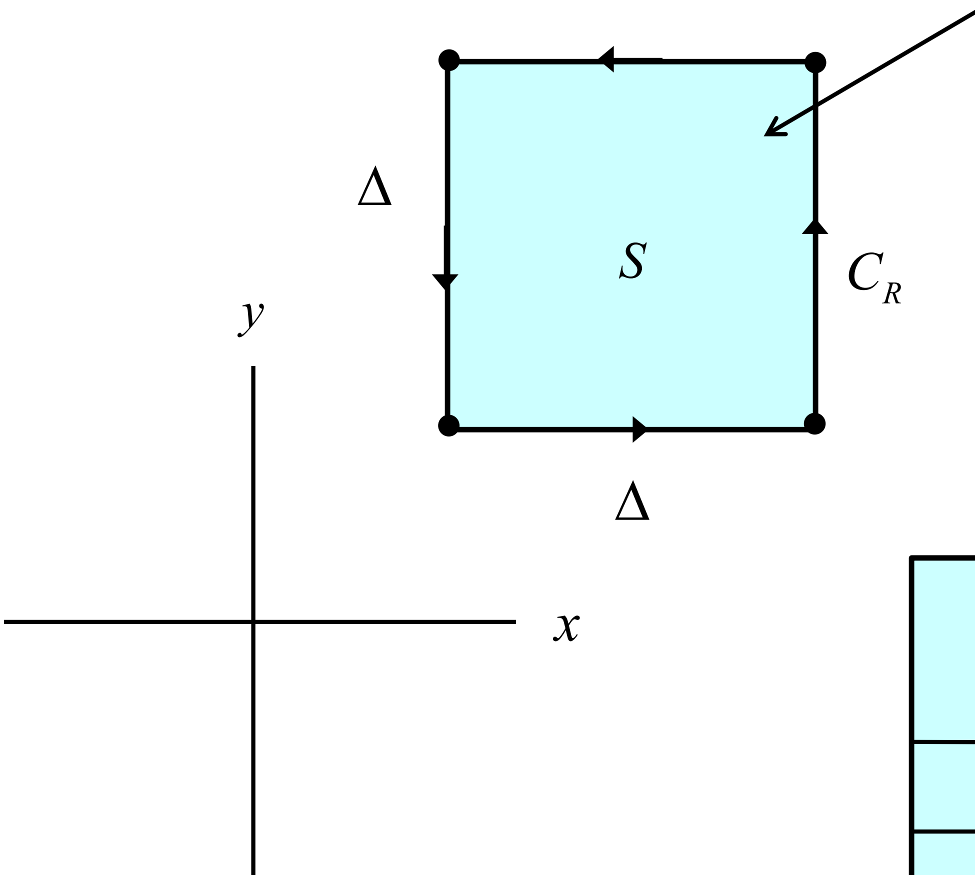
$\Rightarrow$ $Z-P=1$

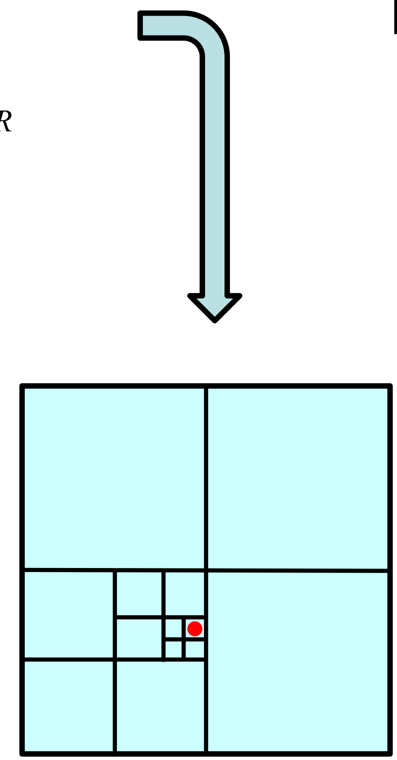
Root finding with the Argument Principle

We can keep subdividing a region to help us locate a zero.

(we assume that there are no poles in the region.)

Region S where we want to locate a zero





Error after N divisions:

$$\left(\delta x,\ \delta y\right) \le \frac{\Delta}{2^N}$$

Summation of Series

The residue theorem is frequently used to sum series. Some important results are obtained from integrals I of various functions over the contour C shown below.

- $\sum_{n=-\infty}^{\infty} f(n) = -$(sum of residues of $\pi f(z)\cot \pi z$ at the poles z_p of $f(z)$)
- $\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -$(sum of residues of $\pi f(z)\csc \pi z$ at the poles z_p of $f(z)$)
- $\sum_{n=-\infty}^{\infty} f\left(\frac{2n+1}{2}\right) = -$(sum of residues of $\pi f(z)\tan \pi z$ at the poles z_p of $f(z)$)
- $\sum_{n=-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right) = -$(sum of residues of $\pi f(z)\sec \pi z$ at the poles z_p of $f(z)$)

We next illustrate the method for the first formula above.

Summation of Series (cont.)

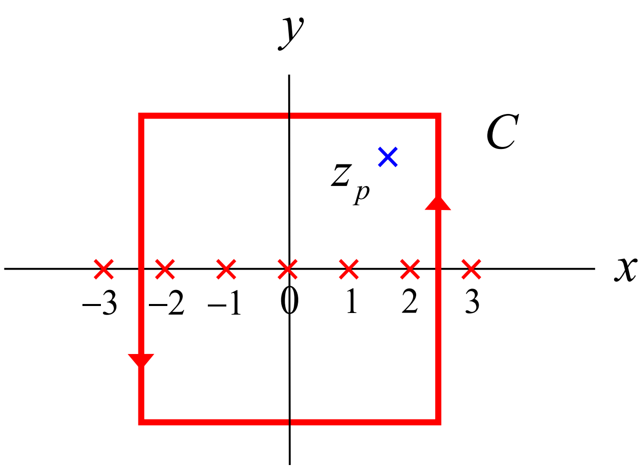
Example

Derive the following result:

$$\sum_{n=-\infty}^{\infty} f(n) = -(\text{sum of residues of } \pi f(z)\cot \pi z \text{ at the poles } z_p \text{ of } f(z))$$

Method: Assume the following integral:

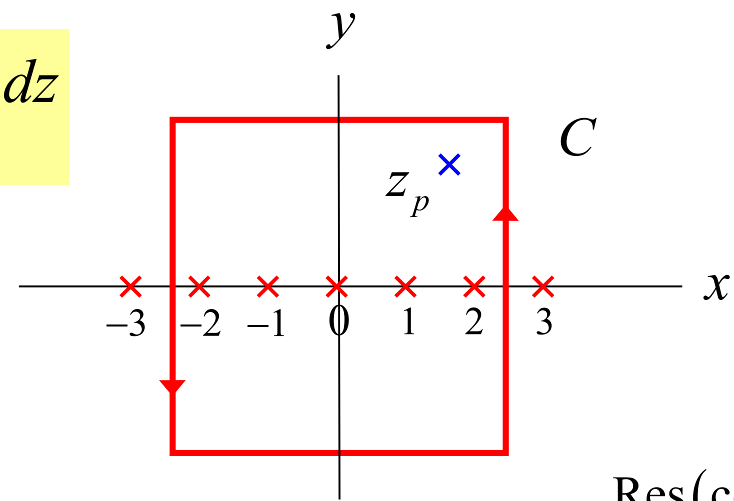
$$I \equiv \oint_C \pi f(z) \cot \pi z \, dz$$



Note: We assume that the integral I over the contour C vanishes as it tends to infinity.

Summation of Series (cont.)

$$I \equiv \oint_C \pi f(z) \cot \pi z \, dz$$



The poles z_p are the poles of the function f.

$$\operatorname{Res}\left(\cot \pi z\right)_{z=n} = 1/\pi$$

$$I \equiv \oint_C \pi f(z) \cot \pi z \, dz = 2\pi i \sum_{n=-\infty}^{\infty} \pi f(n) \operatorname{Res}\left(\cot \pi z\right)_{z=n} + 2\pi i \sum_p \pi \operatorname{Res} f\left(z_p\right) \cot\left(\pi z_p\right)$$

If $I \to 0$ as the path increases, we have:

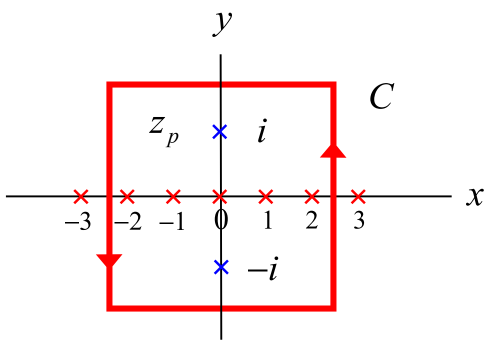
$$0 = 2\pi i \sum_{n=-\infty}^{\infty} \pi f(n)\left(1/\pi\right) + 2\pi i \sum_p \pi \operatorname{Res} f\left(z_p\right) \cot\left(\pi z_p\right)$$

$$\boxed{\sum_{n=-\infty}^{\infty} f(n) = -\sum_p \pi \operatorname{Res} f\left(z_p\right) \cot\left(\pi z_p\right)}$$

Summation of Series (cont.)

Example

Evaluate this sum: $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}$



Use

$$\sum_{n=-\infty}^{\infty} f(n) = -(\text{sum of residues of } \pi f(z)\cot \pi z \text{ at the poles } z_p \text{ of } f(z)\ )$$

We have: $f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z+i)(z-i)} \Rightarrow f(z)$ has poles at $z = +i, -i$

$$\operatorname{Res} \pi f(z)\cot(\pi z)\Big|_{z=\pm i} = \lim_{z\to\pm i}\left(z\mp i\right)\frac{\pi\cot(\pi z)}{(z+i)\left(z-i\right)} = \lim_{z\to\pm i}\left(z\mp i\right)\frac{\pi\cos(\pi z)}{(z+i)\left(z-i\right)\sin(\pi z)}$$

$$= \frac{\pi\cos(\pm i\pi)}{\pm 2i\sin(\pm i\pi)} = -\frac{\pi\cosh(\pi)}{2\sinh(\pi)} = -\frac{\pi\coth(\pi)}{2} \quad \text{for } \textit{both} \text{ residues}$$

(gives us factor of 2)

Hence

$$\boxed{\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} = \pi\coth(\pi)}$$

Kummer Acceleration

Summation formulas are very useful for the acceleration of series by adding and subtracting terms (Kummer acceleration).

Example:

$$S \equiv \sum_{n=1}^{\infty} \frac{1}{n^2 + \tanh(n)}$$

(slowly converging series)

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{n^2 + \tanh(n)} - \frac{1}{n^2+1} \right) + \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

(Kummer acceleration)

Note: $\displaystyle \underbrace{\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}}_{\pi \coth(\pi)} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$\Rightarrow \quad S = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{n^2 + \tanh(n)} - \frac{1}{n^2+1} \right)}_{\text{(rapidly converging series)}} + \frac{1}{2}\left(\pi \coth(\pi) - 1 \right)$$

Kummer Acceleration (cont.)