ECE 6382

Fall 2023

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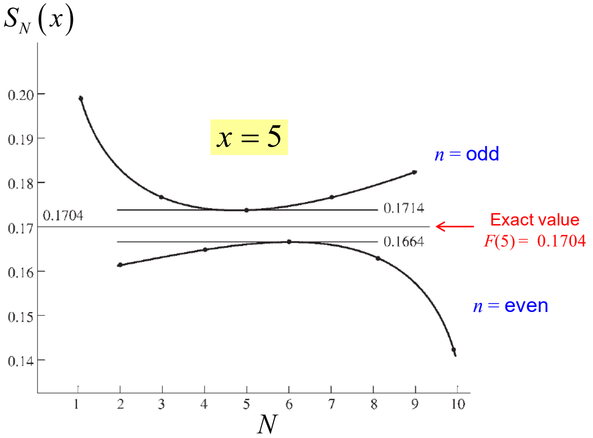


Notes 13

Asymptotic Series

Asymptotic Series

An asymptotic series (as $z \to \infty$) is of the form

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} \qquad \text{as} \quad z \to \infty$$

or

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

Note the “asymptotically equal to” sign.

The asymptotic series shows how the function behaves as z gets large in magnitude.

Important point:
An asymptotic series does not have to be a converging series.

(This is why we do not use an equal sign.)

Asymptotic Series (cont.)

Properties of an asymptotic series:

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

- For a fixed number of terms in the series, the series get more accurate as the magnitude of z increases.
- For a fixed value of z, the series does not necessarily get more accurate as the number of terms increases.
- The series does not necessarily even converge as we increase the number of terms, for a fixed value of z.

Note: We can also talk about $f(w) \sim a_0 + a_1 w + a_2 w^2 + \ldots$ as $w \to 0$

Use: $w = 1/z$

Big O and Small *o* Notation

This notation is helpful for defining and discussing asymptotic series.

Big O notation:

$$f(z) = O(g(z))$$

Qualitatively, this means that f "goes to zero like" g as z gets large (or possibly goes to zero even faster than g).

Definition: There exists a constant k and a radius R such that

$$|f(z)| < k|g(z)|$$

For all $|z| > R$

Big O and Small *o* Notation (cont.)

Examples:

$$\frac{10}{z} = \mathrm{O}\left(\frac{1}{z}\right) \qquad \frac{1}{\sqrt{z}} \neq \mathrm{O}\left(\frac{1}{z}\right)$$

$$\frac{1}{z} + \frac{2}{z^2} = \mathrm{O}\left(\frac{1}{z}\right)$$

$$\sin\left(\frac{1}{z}\right) = \mathrm{O}\left(\frac{1}{z}\right)$$

Note: $\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \ldots$

$$= \frac{1}{z} - \frac{1}{z^3}\left(\frac{1}{3!} - \frac{1}{5!}\frac{1}{z^2} + \ldots\right)$$

Big O and Small *o* Notation (cont.)

Small *o* notation:

$$f(z) = o\left(g(z)\right)$$

Qualitatively, this means that f "gets smaller faster than" g as z gets large.

Definition: For <u>any</u> ε there exists a radius R (which depends on ε) such that

$$\left|f(z)\right| < \varepsilon \left|g(z)\right|$$

For all $\left|z\right| > R$

Big O and Small *o* Notation (cont.)

Examples:

$$\frac{1}{z^2} = o\left(\frac{1}{z}\right) \qquad \frac{1}{z} \neq o\left(\frac{10}{z}\right)$$

$$\frac{1}{z^2} + \frac{3}{z^3} = o\left(\frac{1}{z}\right)$$

$$e^{-z} = o\left(\frac{1}{z^m}\right), \quad m = 1, 2, 3, \ldots \qquad -\frac{\pi}{2} < \arg(z) < \frac{\pi}{2}$$

Note : $e^{-z} = e^{-x} e^{-iy} = e^{-r\cos\theta}\, e^{-ir\sin\theta}$

Big O and Small *o* Notation (cont.)

Note:

$$f(z) = o(g(z)) \;\Rightarrow\; f(z) = O(g(z))$$

$$f(z) = O(g(z)) \;\not\Rightarrow\; f(z) = o(g(z))$$

Examples:

$$\frac{1}{z^2} = o\left(\frac{1}{z}\right) \;\Rightarrow\; \frac{1}{z^2} = O\left(\frac{1}{z}\right)$$

$$\frac{5}{z} = O\left(\frac{1}{z}\right) \;\not\Rightarrow\; \frac{5}{z} = o\left(\frac{1}{z}\right)$$

Definition of Asymptotic Series

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} \qquad \text{as } z \to \infty$$

Definition of asymptotic series:

In order to have an asymptotic series we require the following:

$$f(z) - \sum_{n=0}^{N} \frac{a_n}{z^n} = o\left(\frac{1}{z^N}\right)$$

For <u>any</u> N

As z gets large, the error in stopping at term $n = N$ gets smaller than the last term in the series.

Example:

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \ldots$$

$$\left| f(z) - \left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} \right) \right| = o\left(\frac{1}{z^2}\right)$$

This ensures that the last term kept is meaningful.

Theorem for Asymptotic Series

Theorem

If $f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ as $z \to \infty$

Then $f(z) - \sum_{n=0}^{N} \frac{a_n}{z^n} = \mathrm{O}\left(\frac{1}{z^{N+1}}\right)$ for any N

Note:

$O\left(\frac{1}{z^{N+1}}\right) \to o\left(\frac{1}{z^N}\right)$

$o\left(\frac{1}{z^N}\right) \not\to O\left(\frac{1}{z^{N+1}}\right)$

Example:

$$\left| f(z) - \left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} \right) \right| = O\left(\frac{1}{z^3}\right)$$

Theorem for Asymptotic Series (cont.)

Proof of theorem

Assume $f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ as $z \to \infty$

$$f(z) - \sum_{n=0}^{N+1} \frac{a_n}{z^n} = o\left(\frac{1}{z^{N+1}}\right)$$ (from definition of asymptotic series)

$$\Rightarrow \quad f(z) - \sum_{n=0}^{N} \frac{a_n}{z^n} - \frac{a_{N+1}}{z^{N+1}} = o\left(\frac{1}{z^{N+1}}\right)$$

$$\Rightarrow \quad f(z) - \sum_{n=0}^{N} \frac{a_n}{z^n} = \frac{a_{N+1}}{z^{N+1}} + o\left(\frac{1}{z^{N+1}}\right) = O\left(\frac{1}{z^{N+1}}\right)$$

Summing Asymptotic Series

- One must be careful when summing an asymptotic series, since it may diverge: it is not clear what the optimum number of terms is, for a given value of $z = z_0$.

$$f\left(z_0\right) \approx \sum_{n=0}^{N} \frac{a_n}{z_0^n}$$

What is the optimum N?

General "rule of thumb":

Pick N so that the $N+1$ term in the series is the smallest.

(See the example later.)

Summing Asymptotic Series (cont.)

"Rule of Thumb" Principle:

As x gets large, the error in stopping with term N is approximately given by the first term that is omitted (i.e., the $N+1$ term).

To see this, use:

$$f(z)-\sum_{n=0}^{N+1}\frac{a_n}{z^n}=o\left(\frac{1}{z^{N+1}}\right)$$

(from definition of asymptotic series)

Therefore, we have (separating out the last term from the sum)

$$f(z)=\sum_{n=0}^{N}\frac{a_n}{z^n}+\frac{a_{N+1}}{z^{N+1}}+o\left(\frac{1}{z^{N+1}}\right)$$

Hence:

$$\text{Error}(N)=\left(f(z)-\sum_{n=0}^{N}\frac{a_n}{z^n}\right)\sim\frac{a_{N+1}}{z^{N+1}}$$

This is an asymptotic estimate of the error.

Generation of Asymptotic Series

Various method can be used to generate an asymptotic series expansion of a function:

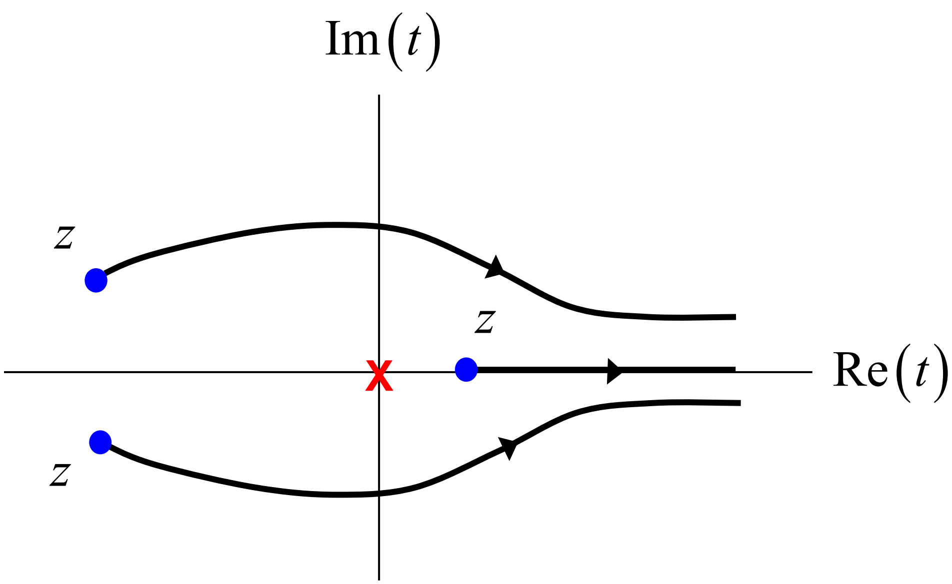
- ❖ Integration by parts*
- ❖ The method of steepest descent
- ❖ Watson's lemma
- ❖ Other specialized techniques

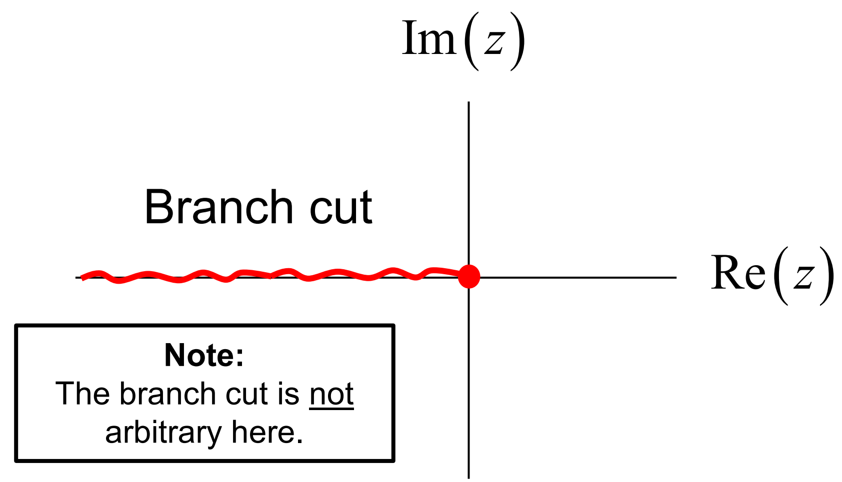
* This method is discussed in this set of notes.

Example

The exponential integral function (of order 1):

$$E_1(z) \equiv \int_z^{\infty} \frac{e^{-t}}{t}\, dt, \quad \text{path } C \text{ does not cross the real axis.}$$





Note: $E_1(z)$ is discontinuous (by $2\pi i$) across the negative real axis:

$$E_1(-a-i\varepsilon) - E_1(-a+i\varepsilon) = 2\pi i \,\mathrm{Res}\left(\frac{e^{-t}}{t}\right)_{t=0} = 2\pi i \qquad (a>0,\ \varepsilon \to 0)$$

Example (cont.)

Use integration by parts:

$$\int_a^b u \frac{dv}{dt} dt = uv\Big|_a^b - \int_a^b \frac{du}{dt} v \, dt$$

$$E_1(z) \equiv \int_z^\infty \frac{e^{-t}}{t} dt = \int_z^\infty \underbrace{\frac{1}{t}}_{u} \underbrace{e^{-t}}_{\frac{dv}{dt}} dt$$

$$= \frac{1}{t}\left(-e^{-t}\right)\Big|_z^\infty - \int_z^\infty \left(-\frac{1}{t^2}\right)\left(-e^{-t}\right) dt$$

$$= \frac{e^{-z}}{z} - \int_z^\infty \left(\frac{1}{t^2}\right) e^{-t} \, dt$$

$$= \frac{e^{-z}}{z} - \left[\frac{1}{t^2}\left(-e^{-t}\right)\Big|_z^\infty - \int_z^\infty \left(\frac{-2}{t^3}\right)\left(-e^{-t}\right) dt\right]$$

$$= \frac{e^{-z}}{z} - \frac{e^{-z}}{z^2} + \int_z^\infty \left(\frac{2}{t^3}\right) e^{-t} dt$$

Note:
It is very important which of the two functions is chosen to be u and which one is chosen to be v.

Example (cont.)

Using integration by parts N times:

"Error term"

$$E_1(z) = \frac{e^{-z}}{z} - \frac{e^{-z}}{z^2} + \ldots + \frac{e^{-z}}{z^N}(-1)^{N-1}(N-1)! \; + (-1)^N N! \int_z^\infty \left(\frac{1}{t^{N+1}}\right) e^{-t} dt$$

or

$$E_1(z) = e^{-z}\left(\frac{1}{z} - \frac{1}{z^2} + \ldots + \frac{(-1)^{N-1}(N-1)!}{z^N}\right) + (-1)^N N! \int_z^\infty \left(\frac{1}{t^{N+1}}\right) e^{-t} dt$$

Question: Is this a valid asymptotic series:

$$f(z) \equiv e^z E_1(z) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{z^n} \quad ???$$

Note: $a_0 = 0$ here.

Example (cont.)

Examine the difference term:

$$\Delta_N \equiv e^z E_1(z) - \left(\frac{1}{z} - \frac{1}{z^2} + \ldots + \frac{(-1)^{N-1}(N-1)!}{z^N} \right) = e^z (-1)^N N! \int_z^\infty \left(\frac{1}{t^{N+1}} \right) e^{-t} dt$$

$$|\Delta_N| < |e^z| N! \left(\frac{1}{|z|^{N+1}} \right) \left| \int_z^\infty e^{-t} dt \right| = |e^z| N! \left(\frac{1}{|z|^{N+1}} \right) |e^{-z}|$$

or

$$|\Delta_N| < N! \left(\frac{1}{|z|^{N+1}} \right)$$

Note: $|e^z| \, |e^{-z}| = |e^z e^{-z}| = 1$

so

$$\Delta_N = O\left(\frac{1}{z^{N+1}} \right) = o\left(\frac{1}{z^N} \right)$$

Example (cont.)

Hence, we have:

$$f(z) \equiv e^{z} E_1(z) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{z^n} = \frac{1}{z} - \frac{1}{z^2} + \frac{2}{z^3} - \frac{6}{z^4} + \frac{24}{z^5} - \frac{120}{z^6} + \ldots$$

Question: Is this a converging series? $\sum_{n=1}^{\infty} b_n$ $\left(b_n \equiv \frac{a_n}{z^n} = \frac{(-1)^{n-1}(n-1)!}{z^n} \right)$

Use the d'Alembert ratio test:

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = L$$

$L < 1$: converges

$L > 1$: diverges

$$L = \lim_{n\to\infty} \left| \frac{z^n}{z^{n+1}} \right| \left(\frac{n!}{(n-1)!} \right)$$

$$= \lim_{n\to\infty} \left| \frac{1}{z} \right| (n) = \infty$$

The series diverges!

Example (cont.)

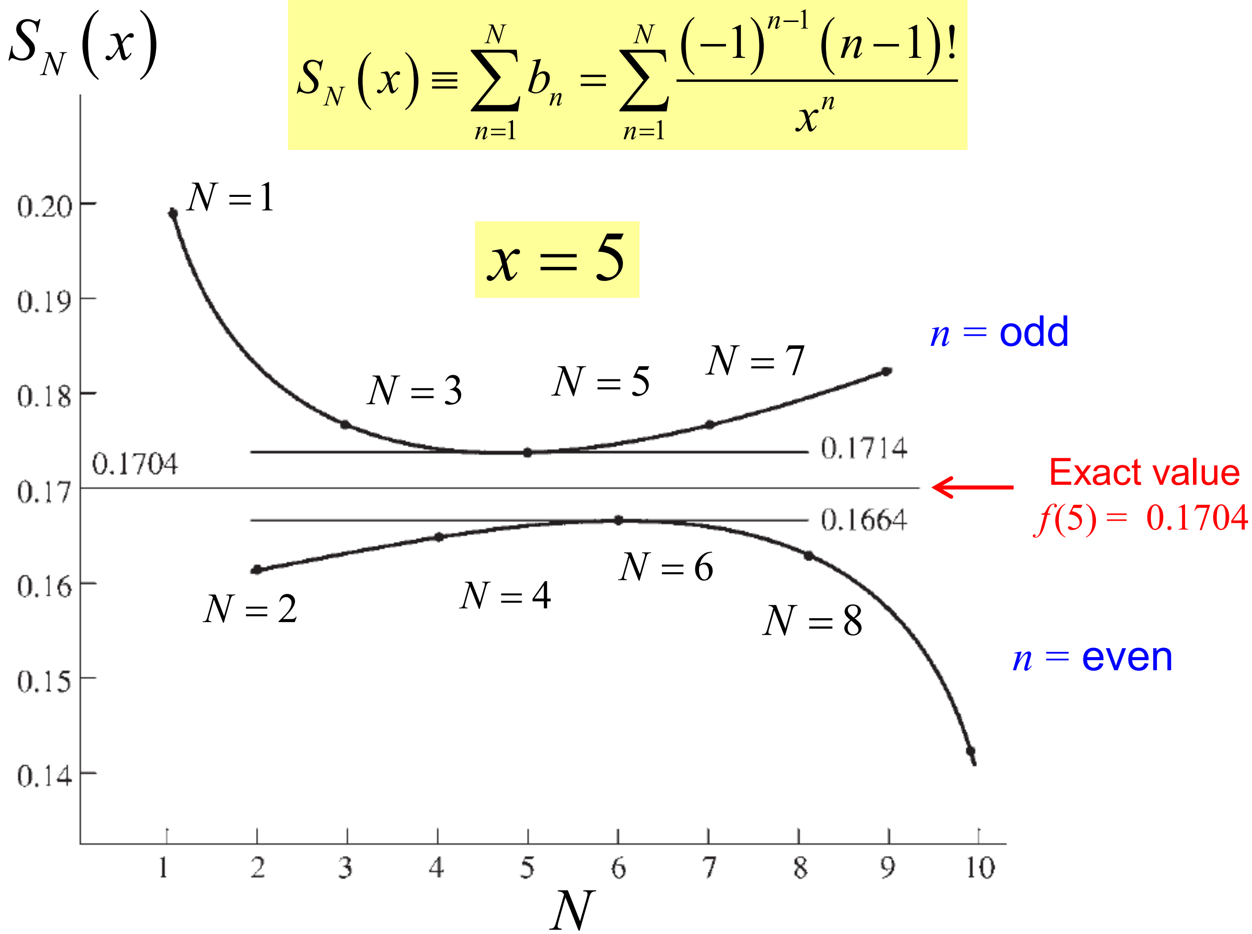
$x = 5$

$$f(x) \equiv e^{x} E_1(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$b_n \equiv \frac{a_n}{x^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

n	b_n
1	0.2
2	-0.04
3	0.016
4	-0.00916
5	**0.00768**
6	**-0.00768**
7	0.00922
8	-0.013
9	0.021
10	-0.037

$$S_N(x) \equiv \sum_{n=1}^{N} b_n = \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^n}$$

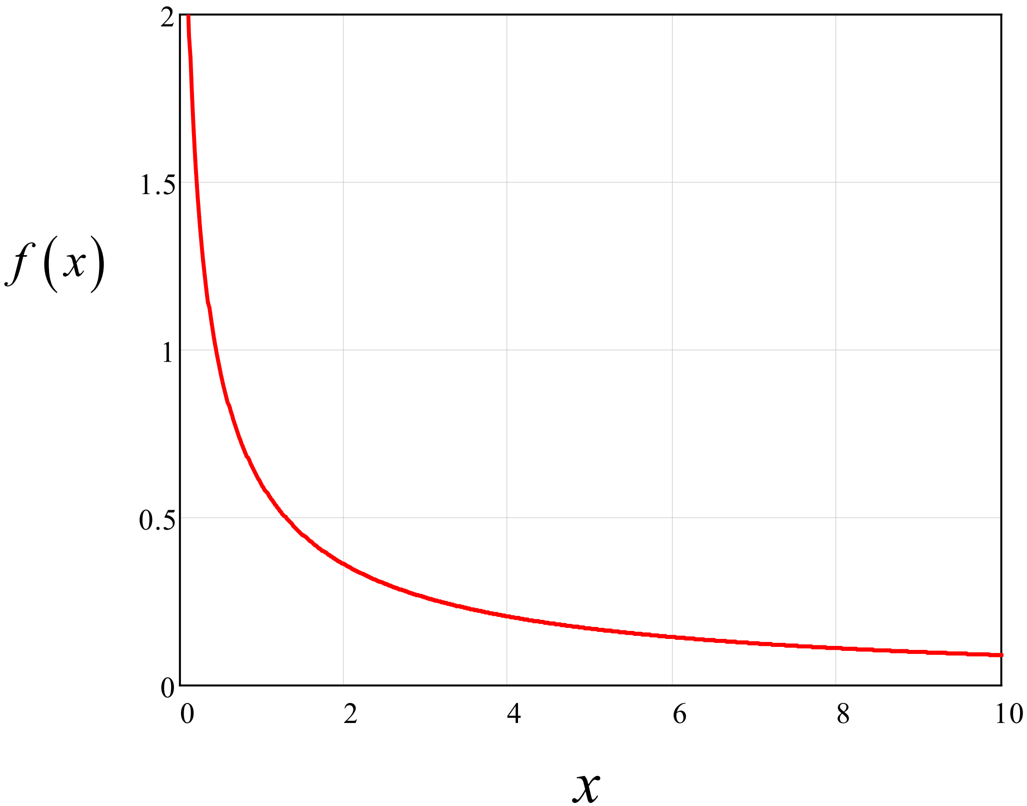


Using $n = 5$ or 6 is optimum for $x = 5$. This is also where $|b_n|$ is the smallest.

Example (cont.)

$$f(x) \equiv e^{x} E_1(x) = e^{x} \int_{x}^{\infty} \frac{e^{-t}}{t} \, dt$$

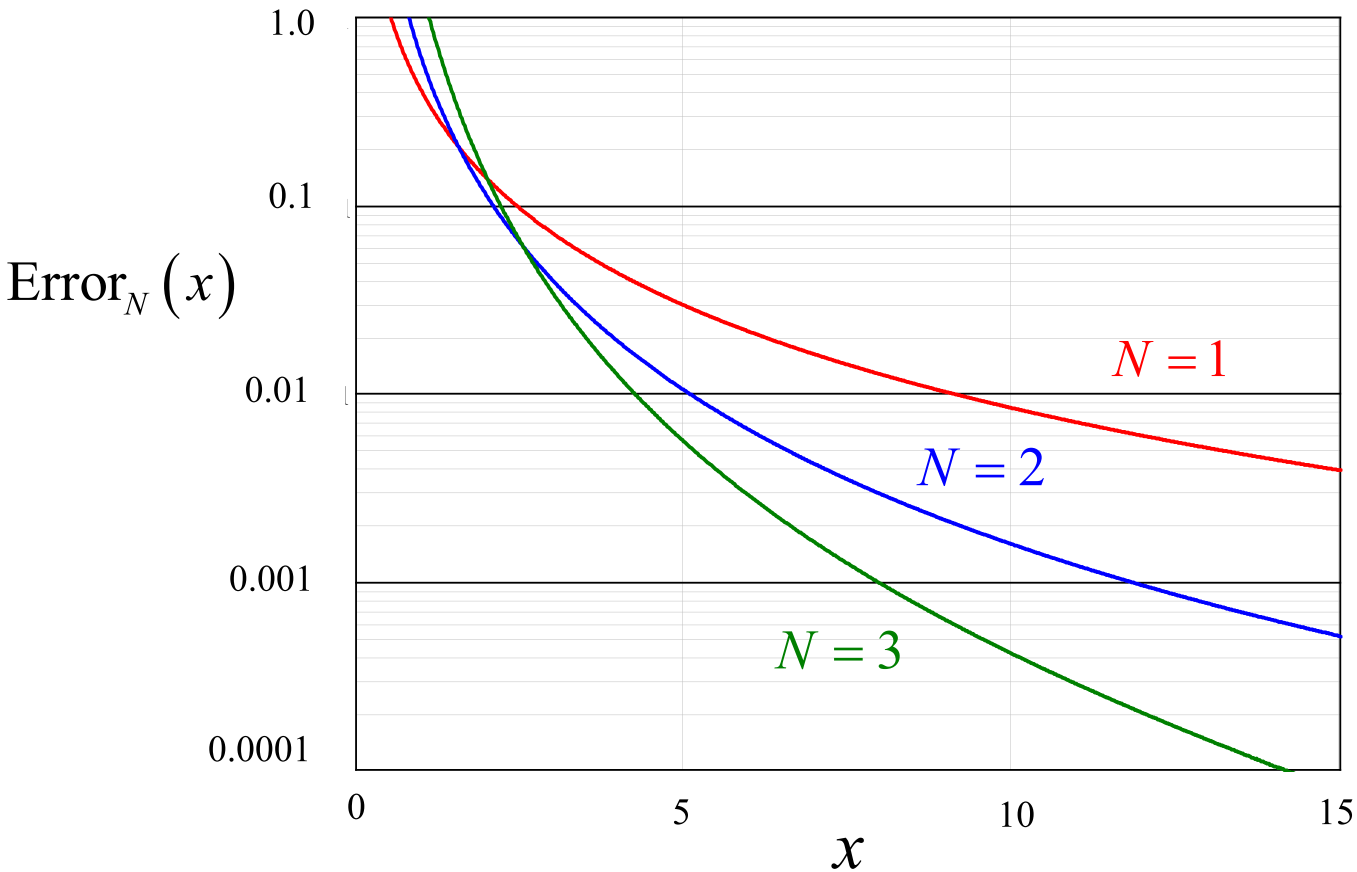
Exact function



Example (cont.)

$$\text{Error}_N(x) \equiv \left| f(x) - \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^n} \right|$$

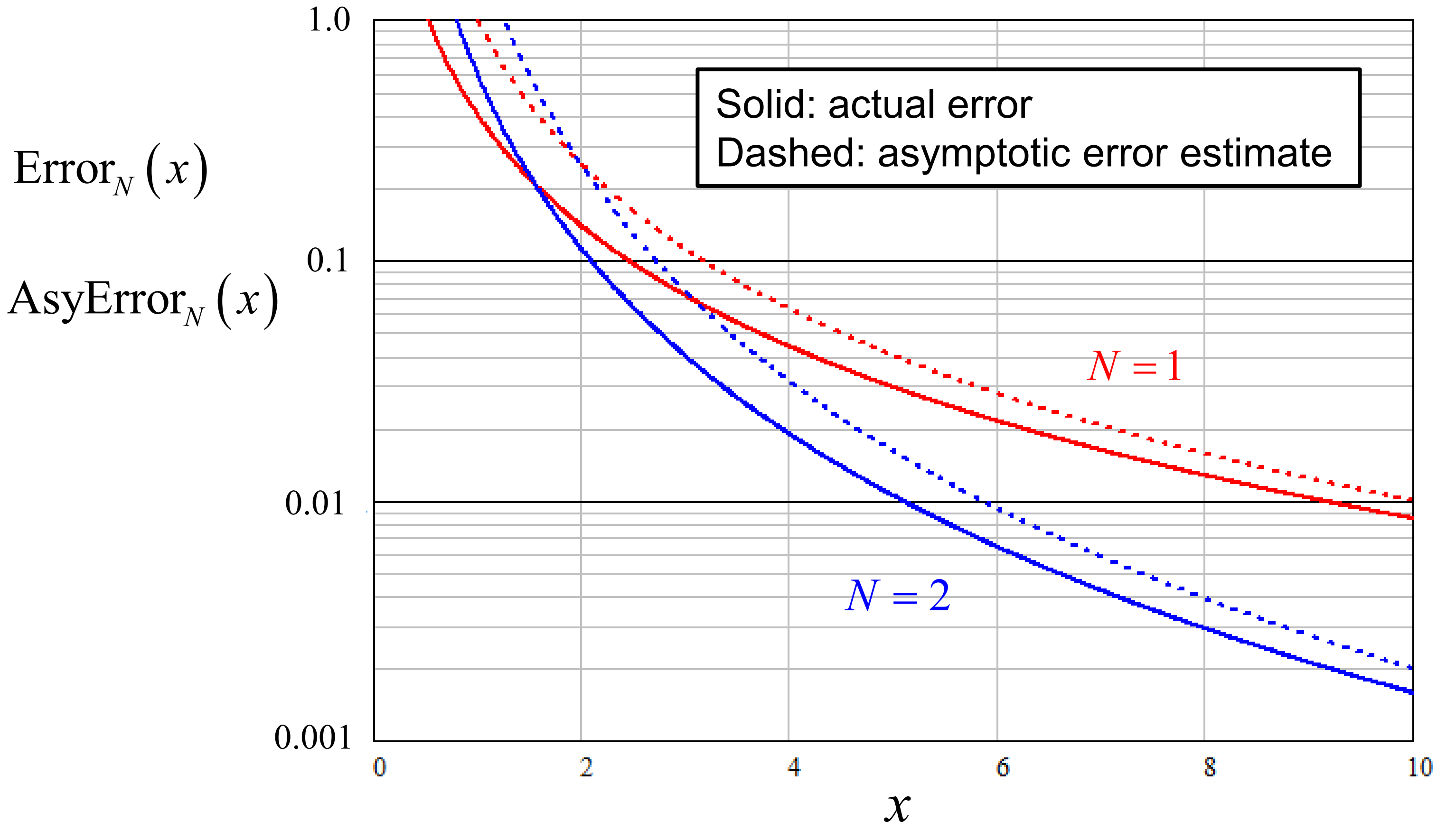
Exact error



Example (cont.)

$$\text{Error}_N(x) \equiv \left| f(x) - \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^n} \right|$$

$$\text{AsyError}_N \equiv \left| \frac{(-1)^N N!}{x^{N+1}} \right|$$



Note on Converging Series

Assume that a series converges for all $|z| > R$ (for some R), so that

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

Then it must also be a valid asymptotic series:

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

Proof:

$$f(z) = \sum_{n=0}^{N} \frac{a_n}{z^n} + \underbrace{\sum_{n=N+1}^{\infty} \frac{a_n}{z^n}}_{\text{Is this } o\left(1/z^N\right)?} = \sum_{n=0}^{N} \frac{a_n}{z^n} + \frac{1}{z^{N+1}}\left(a_{N+1} + \frac{a_{N+2}}{z} + \ldots\right)$$

Note on Converging Series (cont.)

$$f(z) = \sum_{n=0}^{N} \frac{a_n}{z^n} + \frac{1}{z^{N+1}} \left(a_{N+1} + \frac{a_{N+2}}{z} + \ldots \right)$$

Converging series

We note that

$$a_{N+1} + \frac{a_{N+2}}{z} + \ldots \rightarrow a_{N+1} \quad \text{as} \quad z \rightarrow \infty$$

$$\Rightarrow \quad \frac{1}{z^{N+1}} \left(a_{N+1} + \frac{a_{N+2}}{z} + \ldots \right) = O\left(\frac{1}{z^{N+1}} \right) = o\left(\frac{1}{z^{N}} \right)$$

Note on Converging Series (cont.)

Example:

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \ldots$$

The point $z = 0$ is an isolated essential singularity, and there are no other singularities out to infinity.

➡ This Laurent series converges for all $z \neq 0$ (hence for $|z| > R$).

➡ This is a valid asymptotic series.

$$e^{1/z} \sim 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \ldots$$

Stationary-Phase Method

The stationary phase method is sometimes useful for getting the leading term of the asymptotic series for integrals of this form:

$$I(\Omega) = \int_a^b f(x)\, e^{i\Omega g(x)}\, dx$$

$$\Omega \to \infty$$

Note: The integral must be along the real axis.

Assumption: There is a "stationary phase point" inside the integration interval.

$$g'(x_0) = 0 \qquad x_0 \in (a,b)$$

Stationary-Phase Method (cont.)

$$I(\Omega) = \int_a^b f(x)\, e^{i\Omega g(x)}\, dx$$

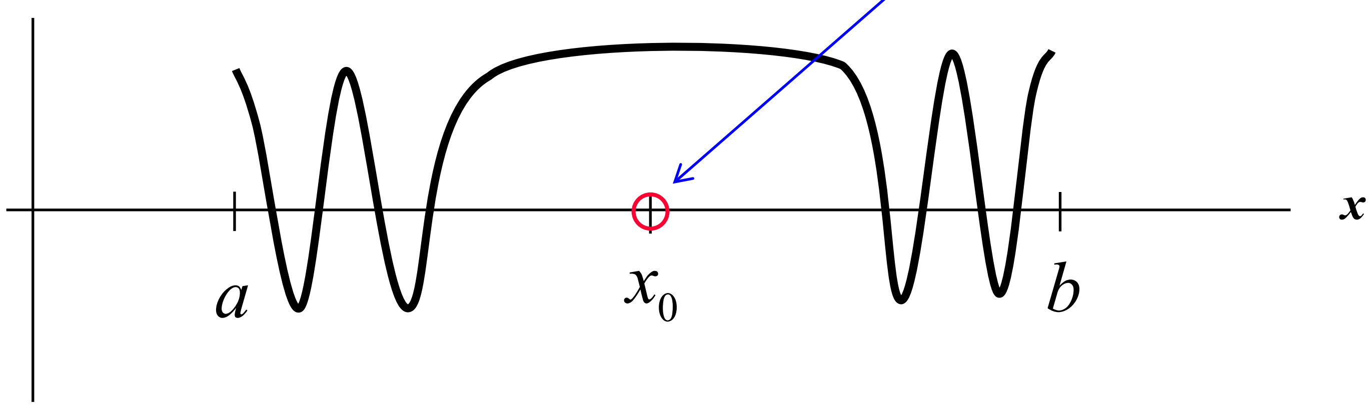
$$g'(x_0) = 0 \qquad x_0 \in (a,b)$$ (stationary phase point)

Assume $f(x_0) \neq 0, \;\; g''(x_0) \neq 0$

$$g(x) \approx g(x_0) + g'(x_0)(x - x_0) + \frac{1}{2} g''(x_0)(x - x_0)^2$$

$$\mathrm{Re}\left[f(x)\, e^{j\Omega g(x)} \right] = f(x) \cos(\Omega g(x))$$

Stationary-phase point



Stationary-Phase Method (cont.)

Assumption:

$$I \sim \int_{x_0-\Delta}^{x_0+\Delta} f(x)\, e^{i\Omega g(x)}\, dx$$

If $\Delta \to 0$ (slow enough)

$$\left(\Delta\sqrt{\Omega} \gg 1\right)$$

(This is justified a little later.)

Stationary-Phase Method (cont.)

$$I \sim \int_{x_0-\Delta}^{x_0+\Delta} f(x)\, e^{i\Omega g(x)} dx$$

Since $\Delta \to 0$

$$f(x) \approx f(x_0)$$

Hence

$$I \sim f(x_0) \int_{x_0-\Delta}^{x_0+\Delta} e^{i\Omega g(x)} dx$$

Stationary-Phase Method (cont.)

$$I \sim f(x_0) \int_{x_0-\Delta}^{x_0+\Delta} e^{i\Omega g(x)} dx$$

$$g(x) \approx g(x_0) + \cancel{g'(x_0)(x-x_0)} + \frac{1}{2} g''(x_0)(x-x_0)^2$$

so

$$I(\Omega) \approx f(x_0) e^{i\Omega g(x_0)} \int_{x_0-\Delta}^{x_0+\Delta} e^{i\frac{1}{2} g''(x_0)\Omega(x-x_0)^2} dx$$

or

$$I(\Omega) \approx f(x_0) e^{i\Omega g(x_0)} \int_{x_0-\Delta}^{x_0+\Delta} e^{\pm i\frac{1}{2}|g''(x_0)|\Omega(x-x_0)^2} dx$$

where

$$g''(x_0) = \pm |g''(x_0)|$$

Stationary-Phase Method (cont.)

Let

$$s = (x - x_0)\sqrt{\frac{\Omega\left|g''(x_0)\right|}{2}} \qquad ds = dx\sqrt{\frac{\Omega\left|g''(x_0)\right|}{2}}$$

Then

$$I(\Omega) \approx f(x_0)\,e^{i\Omega g(x_0)}\sqrt{\frac{2}{\Omega\left|g''(x_0)\right|}}\int_{-S_L}^{+S_L} e^{\pm is^2}\,ds$$

where

$$S_L = \Delta\sqrt{\frac{\Omega\left|g''(x_0)\right|}{2}}$$

As $\Omega \to \infty, \quad S_L \to \infty$ if $\Delta\sqrt{\Omega} \to \infty$

Stationary-Phase Method (cont.)

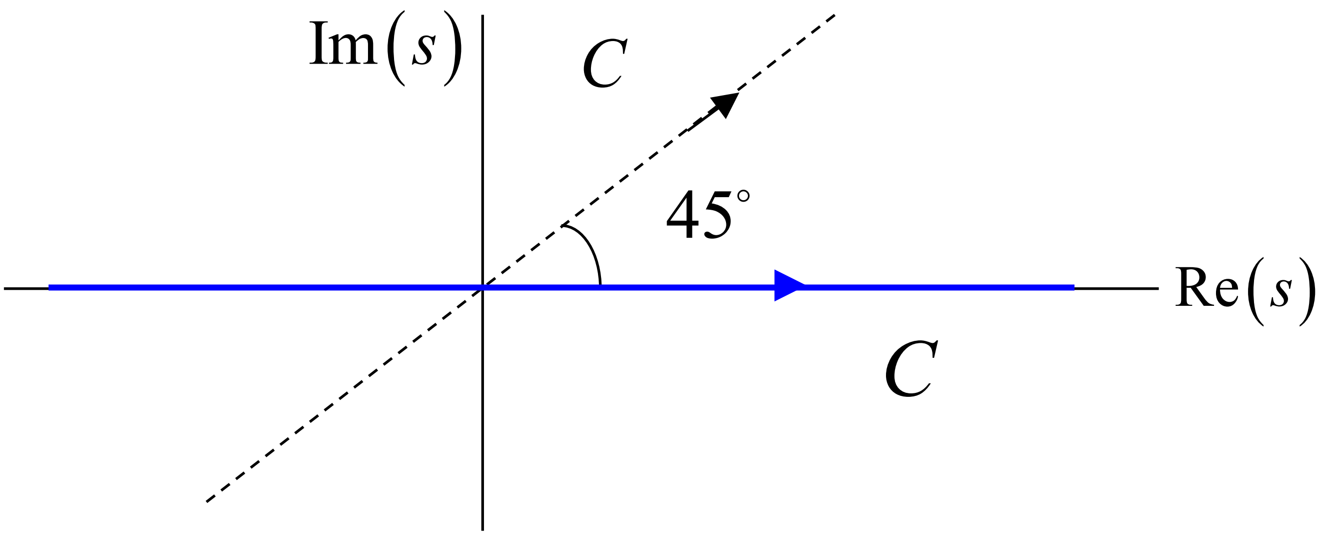
Therefore

$$I_L \equiv \int_{-S_L}^{+S_L} e^{\pm i s^2}\, ds \to \int_{-\infty}^{+\infty} e^{\pm i s^2}\, ds \qquad \left(\Delta\sqrt{\Omega} \to \infty\right)$$

Let

$$I_L^+ \equiv \int_{-\infty}^{+\infty} e^{+i s^2}\, ds = \int_C e^{+i s^2}\, ds$$

(We can change the path by Cauchy's theorem.)



Stationary-Phase Method (cont.)

$$I_L^+ \equiv \int_{-\infty}^{+\infty} e^{+is^2}\,ds = \int_C e^{+is^2}\,ds$$

Let

$$s = t e^{i\frac{\pi}{4}} \quad \Rightarrow \quad \begin{cases} s^2 = t^2 e^{i\frac{\pi}{2}} = it^2 \\ ds = dt\, e^{i\frac{\pi}{4}} \end{cases}$$

Hence

$$I_L^+ = e^{j\frac{\pi}{4}} \int_{-\infty}^{+\infty} e^{i\left(it^2\right)}\,dt$$

$$= e^{i\frac{\pi}{4}} \int_{-\infty}^{+\infty} e^{-t^2}\,dt$$

$$= e^{i\frac{\pi}{4}} \sqrt{\pi}$$

Similarly,

$$I_L^- \equiv \int_{-\infty}^{+\infty} e^{-is^2}\,ds$$

$$= e^{-i\frac{\pi}{4}} \sqrt{\pi}$$

Stationary-Phase Method (cont.)

$$I(\Omega)=\int_a^b f(x)e^{i\Omega g(x)}dx$$

Final result:

$$I\sim f(x_0)e^{i\Omega g(x_0)}\sqrt{\frac{2\pi}{\Omega\left|g''(x_0)\right|}}e^{\pm i\frac{\pi}{4}}\qquad \Omega\to\infty$$

$$+,\quad g''(x_0)>0$$
$$-,\quad g''(x_0)<0$$

Note: $I=O\left(\frac{1}{\sqrt{\Omega}}\right)$

Example

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\,d\theta$$

This is an integral form of the Bessel function of the first kind of order zero.

❖ We want to evaluate $J_0(x)$ for large x.

(We let x be called Ω.)

$$J_0(\Omega) = \frac{1}{\pi}\int_0^{\pi} \cos(\Omega\sin\theta)\,d\theta$$

Example (cont.)

$$J_0(\Omega) = \frac{1}{\pi}\int_0^{\pi} \cos(\Omega \sin\theta)\, d\theta$$

$$= \frac{1}{\pi}\mathrm{Re}\int_0^{\pi} e^{i\Omega\sin\theta}\, d\theta$$

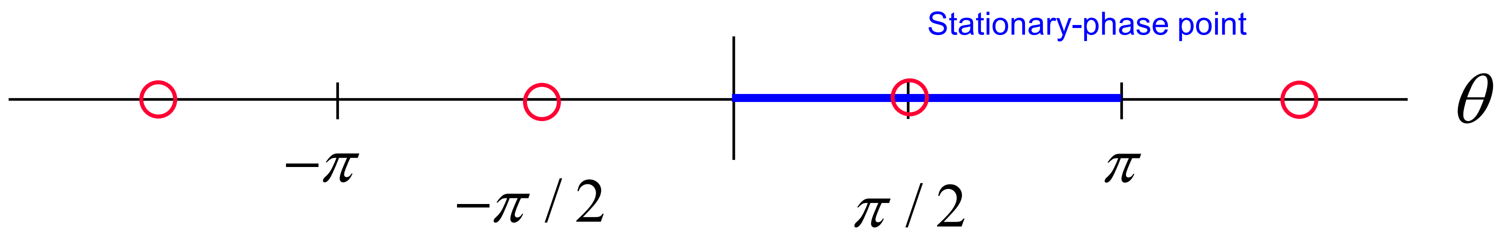
$$f(\theta) = (1)$$

$$g(\theta) = \sin\theta$$

$$g'(\theta_0) = \cos\theta_0 = 0$$

$$\Rightarrow \quad \theta_0 = \frac{\pi}{2} + n\pi$$

Example (cont.)



$$g(\theta) = \sin\theta$$
$$g'(\theta) = \cos\theta$$
$$g''(\theta) = -\sin\theta$$

$$g(\theta_0) = \sin\theta_0 = 1$$

$$g''(\theta_0) = -\sin\theta_0 = -1 < 0$$

Hence, we have:

$$J_0(\Omega) \sim \frac{1}{\pi}\operatorname{Re}\left\{ e^{i\Omega(1)} \sqrt{\frac{2\pi}{\Omega|-1|}} e^{-i\frac{\pi}{4}} \right\}$$

Note: In the final result we can relabel $\Omega \to x$.

Example (cont.)

Hence, the final result is:

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right)$$

as $x \to \infty$