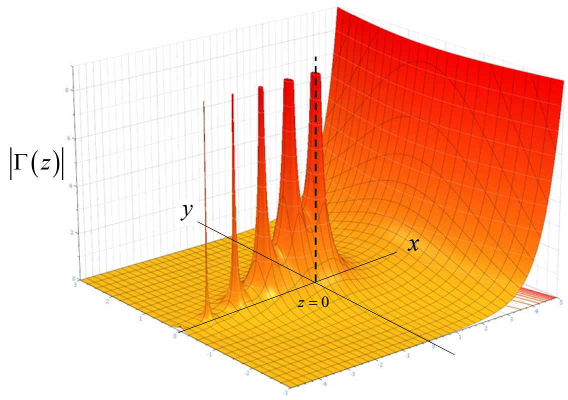
ECE 6382



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Notes 14

The Gamma Function

Notes are from D. R. Wilton, Dept. of ECE

The Gamma Function

- The Gamma function appears in many expressions, including Bessel functions, etc.

- It generalizes the factorial function $n!$ to non-integer values and even complex values.

- It appears in the method of steepest descent (a method for obtaining the asymptotic expansion of a class of integrals).

Definition 1

Definition # 1

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^{z}, \quad z \neq 0, -1, -2,$$

This definition gives the Gamma function a nice property for $z = n$ (a positive integer), as proven on the next slide:

$$\Gamma(n) = (n-1)!$$

(factorial property)

Definition 1 (cont.)

Proof of factorial property:

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z, \quad z \neq 0, -1, -2, \cdots$$

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)\cdots(z+n+1)} n^{z+1} = \Gamma(z) \lim_{n\to\infty} \frac{nz}{(z+n+1)}$$

$$\Rightarrow \boxed{\Gamma(z+1) = z\Gamma(z)}$$

Note that $\Gamma(1) = \lim_{n\to\infty} \frac{\cancel{1\cdot 2\cdot 3\cdots n}}{\cancel{1\cdot 2\cdot 3\cdots n}\,(n+1)} n = 1$, and $\Gamma(2) = 1\cdot\Gamma(1) = 1$,

$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1$, $\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2\cdot 1$, $\Gamma(5) = 4\cdot\Gamma(4) = 4\cdot 3\cdot 2\cdot 1$, etc.

Hence

$$\Gamma(n) = (n-1)! \quad n = 1, 2, 3\ldots$$

or

$$\Gamma(n+1) = n! \quad n = 0, 1, 2, \ldots$$

Definition 2

Definition # 2

$$\Gamma(z) \equiv \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \operatorname{Re} z > 0$$

This is the Euler-integral form of the definition.

Leonard Euler

Note:

$$t^{z-1} = t^{x-1} t^{iy} = t^{x-1} \left(e^{\ln t}\right)^{iy} = t^{x-1} \left(e^{iy \ln t}\right)$$

$\Rightarrow \left|t^{z-1}\right| = t^{x-1} \quad \Rightarrow \; x > 0$ for the integral to converge at $t = 0$

Note:

Definition 1 is the analytic continuation of definition 2 from the right-half plane into the entire complex plane (except at zero and the negative integers).

Equivalent Integral Forms

The following three integral definitions are all equivalent :

$$\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1} dt, \qquad \operatorname{Re} z > 0$$

$$\Gamma(z) = 2\int_0^\infty e^{-s^2} s^{2z-1} ds, \quad \operatorname{Re} z > 0 \qquad (\text{let } t = s^2)$$

$$\Gamma(z) = \int_0^1 \left(\ln \frac{1}{s} \right)^{z-1} ds, \quad \operatorname{Re} z > 0 \qquad (\text{let } t = \ln(1/s))$$

Equivalence of Definitions 1 and 2

Equivalence of definitions #1 and #2

Use $e^{-t} \equiv \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n$,

Define $F(z,n) \equiv \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt;\quad F(z,n) \underbrace{\rightarrow}_{n\to\infty} \int_0^\infty e^{-t}t^{z-1}dt = \Gamma_2(z)$

Letting $w=\frac{t}{n}$ and integrating by parts n times,

$$F(z,n) = n^z\int_0^1 (1-w)^n w^{z-1}dw = n^z \overbrace{\frac{1\cdot 2\cdot 3\cdots(n-1)n}{z(z+1)(z+2)\cdots(z+n-1)}\underbrace{\int_0^1 w^{z+n-1}dw}_{\frac{1}{z+n}}}^{\text{Factor appearing in Definition \#1}}$$

Hence $\lim_{n\to\infty} F(z,n) = \Gamma_1(z)$

(Please see next slide.)

$$\Gamma_1(z) \equiv \lim_{n\to\infty}\frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)}n^z,\quad z\neq 0,-1,-2,$$

Equivalence of Definitions 1 and 2 (cont.)

Integration by parts development:

$$I \equiv \int_0^1 \underbrace{(1-w)^n}_{u} \underbrace{w^{z-1}}_{\frac{dv}{dw}} dw$$

Integrate by parts once:

$$I = (1-w)^n \frac{w^z}{z}\Bigg|_0^1 - \int_0^1 -n(1-w)^{n-1} \frac{w^z}{z} dw$$

$$= 0 + \int_0^1 n(1-w)^{n-1} \frac{w^z}{z} dw$$

$$= \frac{n}{z} \int_0^1 (1-w)^{n-1} w^z dw$$

Equivalence of Definitions 1 and 2 (cont.)

$$I \equiv \int_0^1 \underbrace{(1-w)^n}_{u} \underbrace{w^{z-1}}_{\frac{dv}{dw}} dw$$

Integrate by parts twice:

$$I = \frac{n}{z}\int_0^1 (1-w)^{n-1} w^z dw$$

$$= \frac{n}{z}(1-w)^{n-1}\frac{w^{z+1}}{z+1}\Bigg|_0^1 - \frac{n}{z}\int_0^1 (n-1)(1-w)^{n-2}(-1)\frac{w^{z+1}}{z+1}dw$$

$$= 0 + \frac{n(n-1)}{z(z+1)}\int_0^1 (1-w)^{n-2} w^{z+1} dw$$

$$= \frac{n(n-1)}{z(z+1)}\int_0^1 (1-w)^{n-2} w^{z+1} dw$$

$$\int_0^1 w^{z+n-1} dw$$

$$\uparrow$$

After n times:

$$I = \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{z(z+1)(z+2)\cdots(z+n-1)}\int_0^1 (1-w)^{n-n} w^{z+n-1} dw$$

Definition 3

Definition # 3

The Weierstrass product form can be shown to be equivalent to definitions #1 and #2.

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

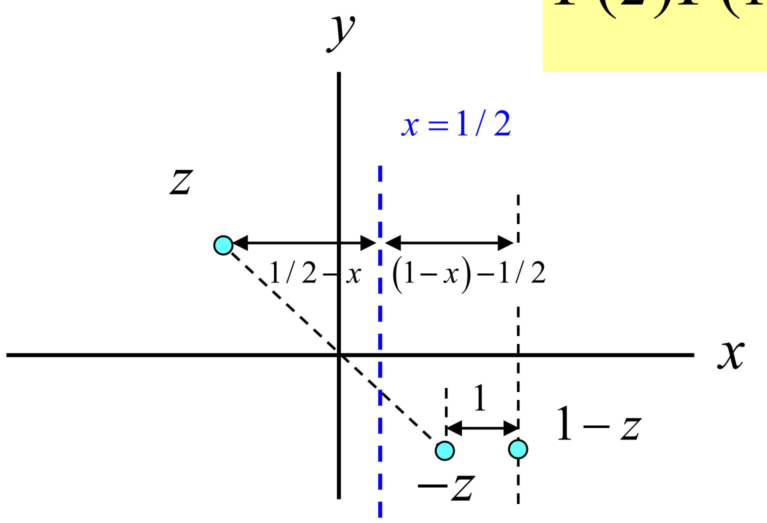
where $\gamma = 0.5772156619\ldots$ is the Euler - Mascheroni constant.

Euler Reflection Formula

Euler Reflection Formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

(Proof omitted.)



Geometric interpretation of reflection formula:

In the horizontal direction, the two points are reflections about the $x = 1/2$ line.

Note: We can use this formula along with definition #2 to find $\Gamma(z)$ for $\mathrm{Re}(z) < 0$.

$$\Gamma(z) = \frac{\pi}{\sin \pi z} \frac{1}{\Gamma(1-z)}, \quad \mathrm{Re}\, z < 0, \quad z \neq 0, -1, -2, \ldots$$

Euler Reflection Formula (cont.)

A special result that occurs frequently is $\Gamma(1/2)$.

To calculate this, use the reflection formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

Set $z = 1/2$:

$$\Gamma(1/2) = \sqrt{\pi}$$

Summary of Factorial Properties

Summary of Factorial Generalization

$$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

Integers

$n = 1, 2, 3\ldots$

$$x! = \Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt$$

Real numbers

$x > -1$

$$z! = \Gamma(z+1) = \int_0^\infty e^{-t} t^z \, dt$$

Complex numbers

$\mathrm{Re}(z) > -1$

Summary of Factorial Properties (cont.)

Summary of Factorial Generalization (cont.)

$$z! = \Gamma(z+1) = \int_0^\infty e^{-t} t^z \, dt \quad (\operatorname{Re} z > -1)$$

+

$$\Gamma(z) = \frac{\pi}{\sin \pi z} \frac{1}{\Gamma(1-z)}$$

Complex numbers

$z \neq -1, -2 \ldots$

Pole Behavior

Simple poles of $\Gamma(z)$ are at $n = 0, -1, -2, -3, \ldots$

Recall: $\Gamma(z+1) = z\Gamma(z) \quad \& \quad \Gamma(1) = 1$

Use

$$\Gamma(z+1) = z\Gamma(z) \;\Rightarrow\; \Gamma(z) = \frac{\Gamma(z+1)}{z}$$

$\Gamma(z)$ has simple pole at $z = 0$
Residue $= 1$

$$\Gamma(z+2) = (z+1)\Gamma(z+1) \;\Rightarrow\; \Gamma(z+1) = \frac{\Gamma(z+2)}{z+1}$$

$$\Rightarrow z\Gamma(z) = \frac{\Gamma(z+2)}{z+1}$$

$$\Rightarrow \Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)}$$

$\Gamma(z)$ has simple pole at $z = -1$
Residue $= -1$

Pole Behavior (cont.)

$$\Gamma(z+3) = (z+2)\Gamma(z+2) \;\Rightarrow\; \Gamma(z+2) = \frac{\Gamma(z+3)}{z+2}$$

$$\Rightarrow (z+1)(z)\Gamma(z) = \frac{\Gamma(z+3)}{z+2}$$

$$\Rightarrow \Gamma(z) = \frac{\Gamma(z+3)}{z(z+1)(z+2)}$$

$\Gamma(z)$ has simple pole at $z = -2$
Residue $= +1/2$

$$\Gamma(z+4) = (z+3)\Gamma(z+3) \;\Rightarrow\; \Gamma(z+3) = \frac{\Gamma(z+4)}{z+3}$$

$$\Rightarrow (z+2)(z+1)(z)\Gamma(z) = \frac{\Gamma(z+4)}{z+3}$$

$$\Rightarrow \Gamma(z) = \frac{\Gamma(z+4)}{z(z+1)(z+2)(z+3)}$$

$\Gamma(z)$ has simple pole at $z = -3$
Residue $= -1/6$

Pole Behavior (cont.)

Residues at Poles

In general (after $n+1$ steps), we will have:

$\Gamma(z)$ has simple pole at $z = -n$

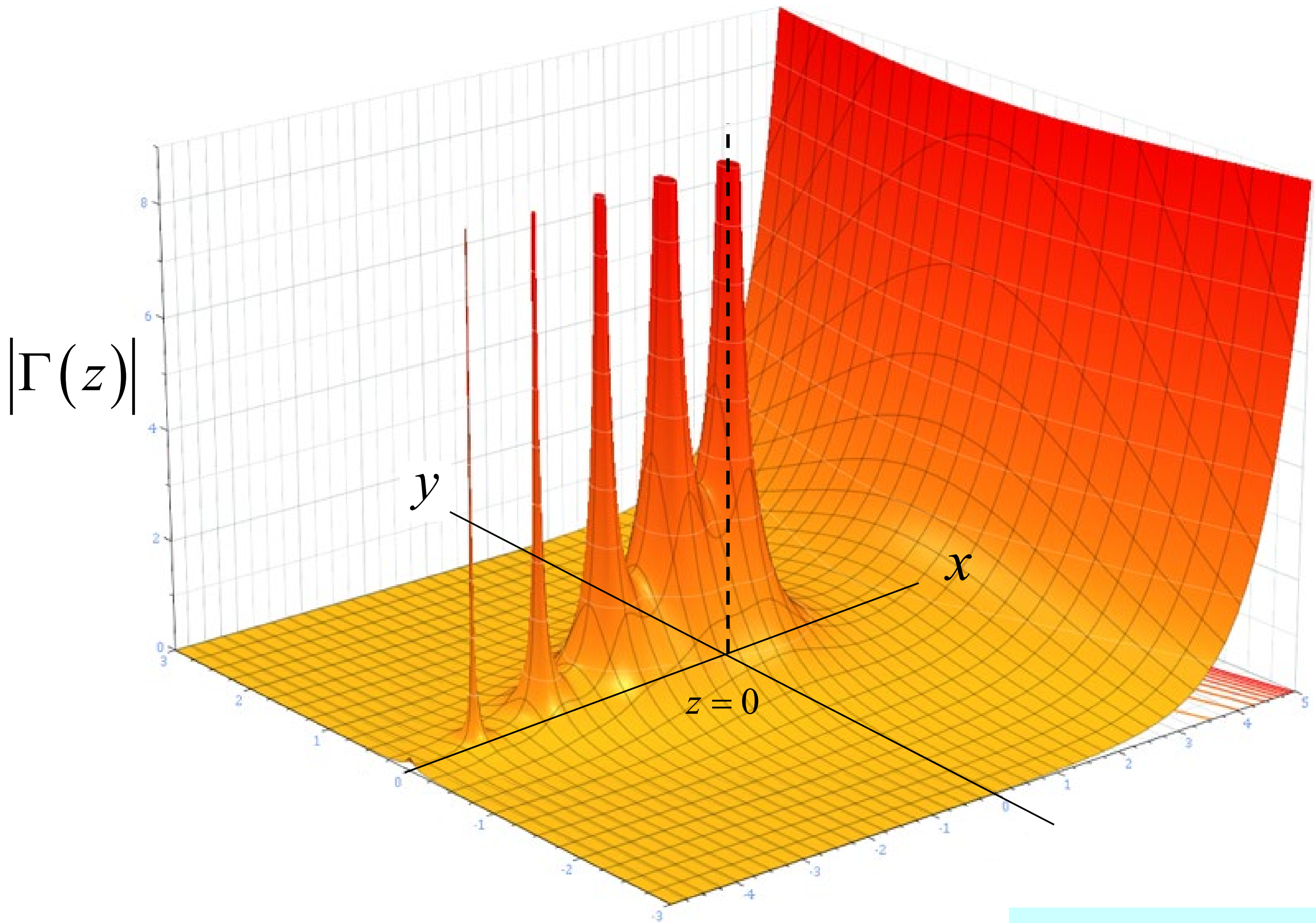
$$\Gamma(z)=\frac{\Gamma(z+n+1)}{z(z+1)(z+2)(z+3)\ldots(z+n)}$$

$$\begin{aligned}\operatorname{Res}\Gamma(z)_{z=-n} &= \lim_{z\to -n}(z+n)\left(\frac{\Gamma(-n+(n+1))}{z(z+1)(z+2)(z+3)\ldots(z+n)}\right)\\ &= \left.\frac{1}{z(z+1)(z+2)(z+3)\ldots(z+n-1)}\right|_{z=-n}\\ &= \frac{1}{(-n)(-n+1)\ldots(-3)(-2)(-1)}\\ &= \frac{(-1)^n}{(n)(n-1)\ldots(3)(2)(1)}\end{aligned}$$

Hence

$$\operatorname{Res}\Gamma(z)_{z=-n}=\frac{(-1)^n}{n!}$$

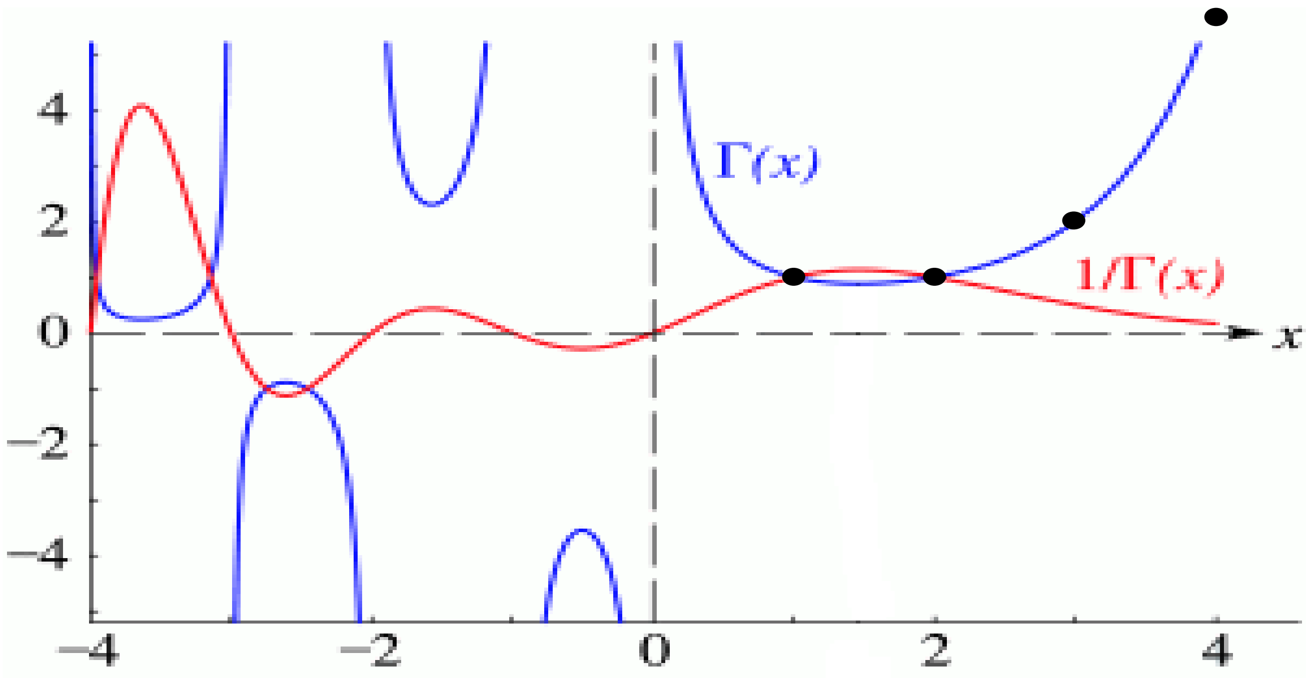
Plot of Gamma Function



Note: There are simple poles at $z = 0, -1, -2, \ldots$

$$\operatorname{Res} \Gamma(z)_{z=-n} = \frac{(-1)^n}{n!}$$

Plot of Gamma Function (cont.)



$\Gamma(x)$ and $1 / \Gamma(x)$

Note: $\Gamma(x)$ never goes to zero.

In fact, $1 / \Gamma(z)$ is analytic everywhere.

Asymptotic Form of Gamma Function

Sterling's formula (asymptotic series for large argument):

$$\Gamma(z) \sim z^z e^{-z} \sqrt{\frac{2\pi}{z}} \left(1 + \underbrace{\frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \ldots}_{w} \right)$$

as $z \to \infty$

Taking the ln of both sides, we also have

$$\ln \Gamma(z) \sim z \ln z - z - \frac{1}{2} \ln\left(\frac{z}{2\pi}\right) + \left(\frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} + \ldots \right)$$

Note : $\ln(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \ldots$