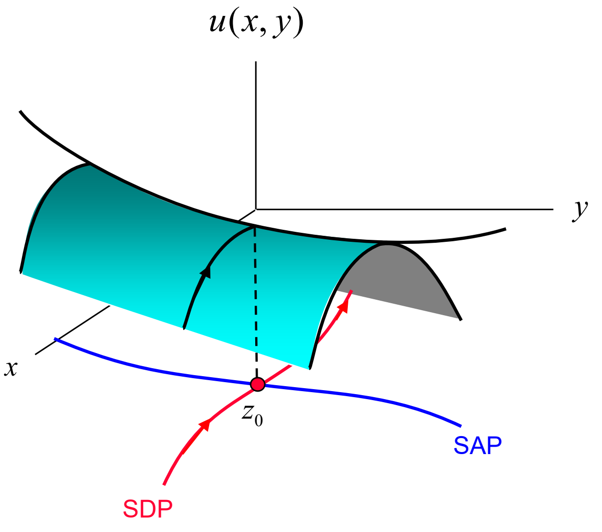


ECE 6382

Fall 2023

David R. Jackson

Notes 15

The Steepest-Descent Method

Notes are adapted from ECE 6341

Steepest-Descent Method

Complex Integral:

$$I(\Omega) = \int_C f(z) e^{\Omega g(z)} dz$$

$$\Omega >> 1$$

The method was published by Peter Debye in 1909. Debye noted in his work that the method was developed in an unpublished note by Bernhard Riemann (1863).

Peter Joseph William Debye (March 24, 1884 – November 2, 1966) was a Dutch physicist and physical chemist, and Nobel laureate in Chemistry.

Georg Friedrich Bernhard Riemann (September 17, 1826 – July 20, 1866) was an influential German mathematician who made lasting contributions to analysis and differential geometry, some of them enabling the later development of general relativity.

http://en.wikipedia.org/wiki/Peter_Debye

http://en.wikipedia.org/wiki/Bernhard_Riemann

Steepest-Descent Method (cont.)

Complex Integral:

$$I(\Omega) = \int_C f(z)\, e^{\Omega g(z)}\, dz$$

We want to obtain an approximate evaluation of the integral when the real parameter Ω is large.

The functions $f(z)$ and $g(z)$ are analytic (except for poles or branch points), so that the path C may be deformed if necessary (possibly adding residue contributions or branch-cut integrals) to go through a saddle point.

Saddle Point (SP): $g'(z_0) = 0$

(This is the point that ends up contributing the most.)

$$\Rightarrow \quad \frac{\partial g(x,y)}{\partial x} = 0, \quad \frac{\partial g(x,y)}{\partial y} = 0$$

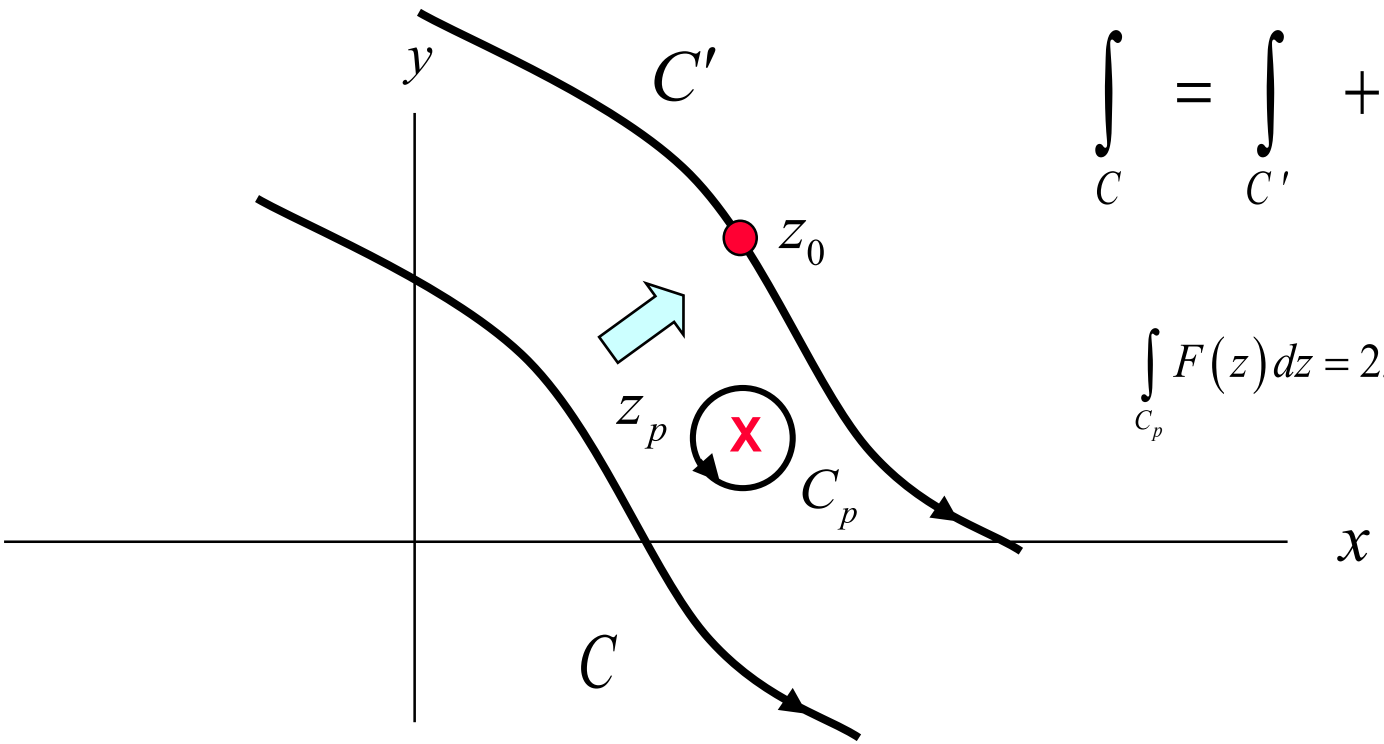
(Let $\Delta z = \Delta x,\ \Delta z = i\Delta y$)

Steepest-Descent Method (cont.)

Path deformation:

If the path does not go through a saddle point, we assume that it can be deformed to do so.

If any singularities are encountered during the path deformation, they must be accounted for (e.g., residue of captured poles).



$$\int_{C} = \int_{C'} + \int_{C_p}$$

$$\int_{C_p} F(z)\,dz = 2\pi i \,\mathrm{Res}\, F(z)\big|_{z=z_p}$$

Steepest-Descent Method (cont.)

Denote $$g(z) = u(z) + i\,v(z)$$

Cauchy Reimann eqs.:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)$$

(switch order)

$$= -\frac{\partial^2 u}{\partial y^2}$$

Steepest-Descent Method (cont.)

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ (The function u obeys the 2D Laplace equation.)

$\Rightarrow$ If $u_{xx} < 0,$ then $u_{yy} > 0$

Near the saddle point:

$$u(x,y) \approx u(x_0, y_0) + \frac{1}{2} u_{xx} (x - x_0)^2$$

$$+ \frac{1}{2} u_{yy} (y - y_0)^2 + \underline{u_{xy} (x - x_0)(y - y_0)}$$

(We can rotate coordinates to eliminate this term.)

Steepest-Descent Method (cont.)

$$u(x,y) \approx u(x_0,y_0) + \frac{1}{2}u_{xx}(x-x_0)^2 + \frac{1}{2}u_{yy}(y-y_0)^2 + u_{xy}(x-x_0)(y-y_0)$$

In the rotated coordinate system:

$$u(x',y') \approx u(x'_0,y'_0) + \frac{1}{2}u_{x'x'}(x'-x'_0)^2 + \frac{1}{2}u_{y'y'}(y'-y'_0)^2$$

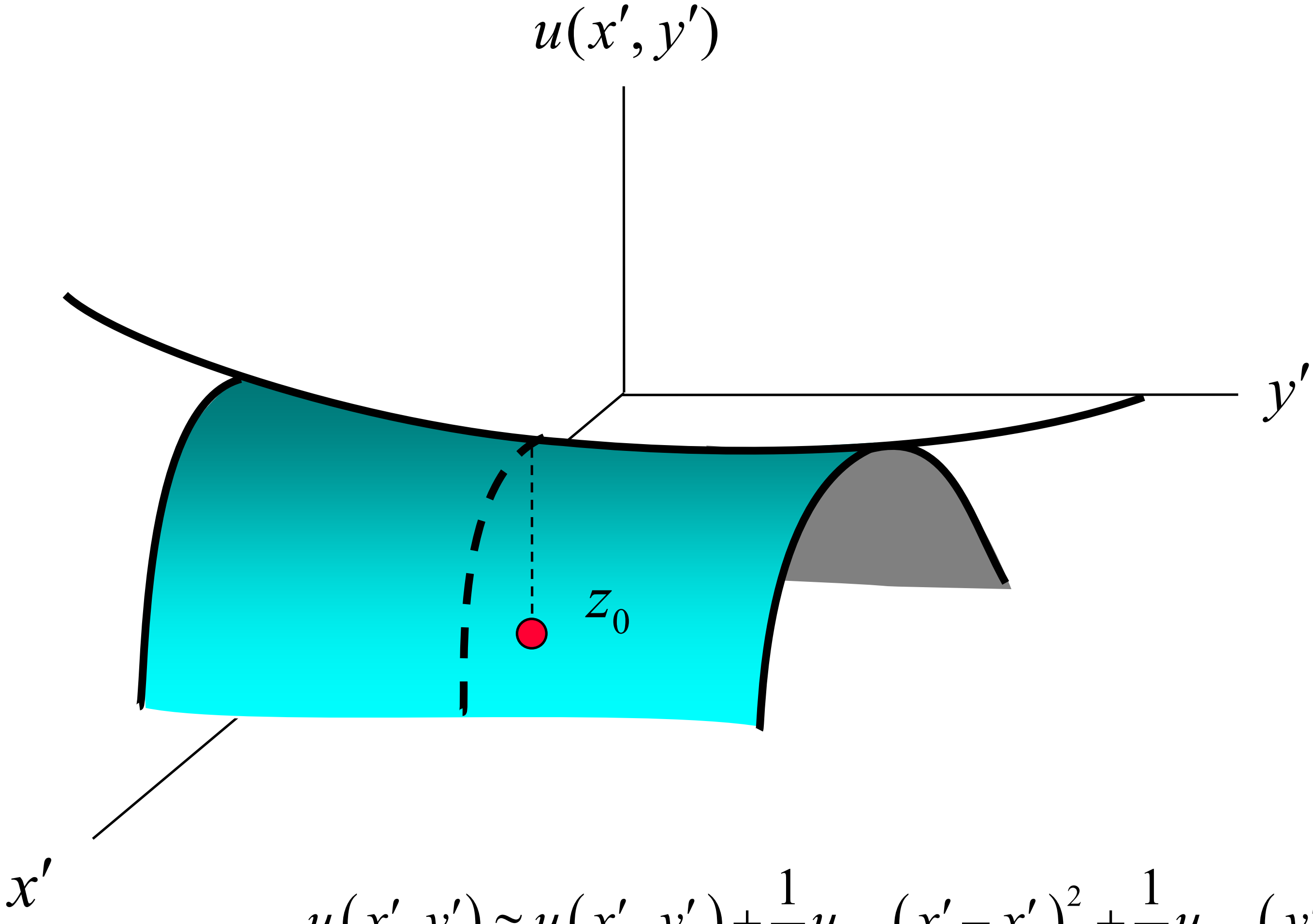
Assume that the coordinate system is rotated so that

$$u_{x'x'} < 0 \qquad u_{y'y'} > 0$$

Note:
The angle of rotation necessary to do this will be clear later
(it is the departure angle θ_{SDP} of the steepest-descent path).

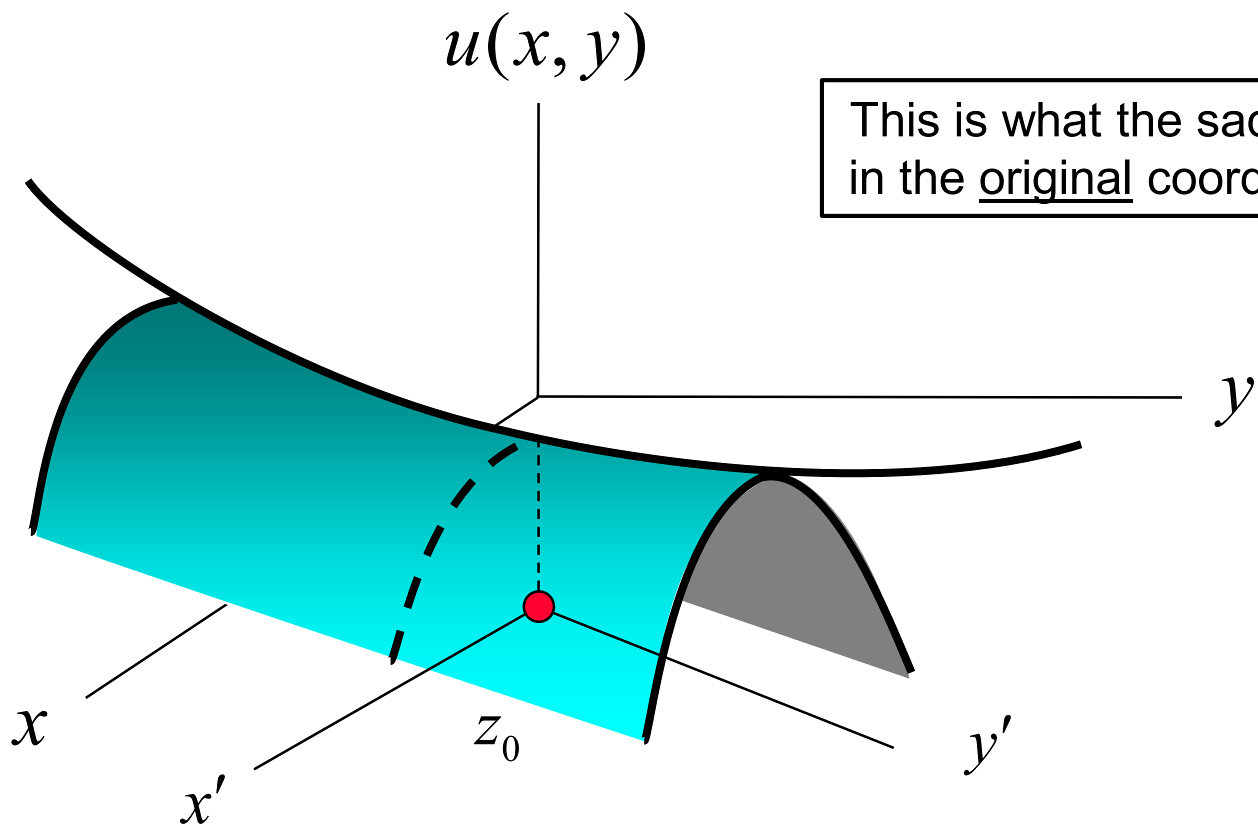
Steepest-Descent Method (cont.)

The $u(x', y')$ function has a "saddle" shape near the SP:



$$u(x', y') \approx u(x'_0, y'_0) + \frac{1}{2} u_{x'x'} (x' - x'_0)^2 + \frac{1}{2} u_{y'y'} (y' - y'_0)^2$$

Steepest-Descent Method (cont.)

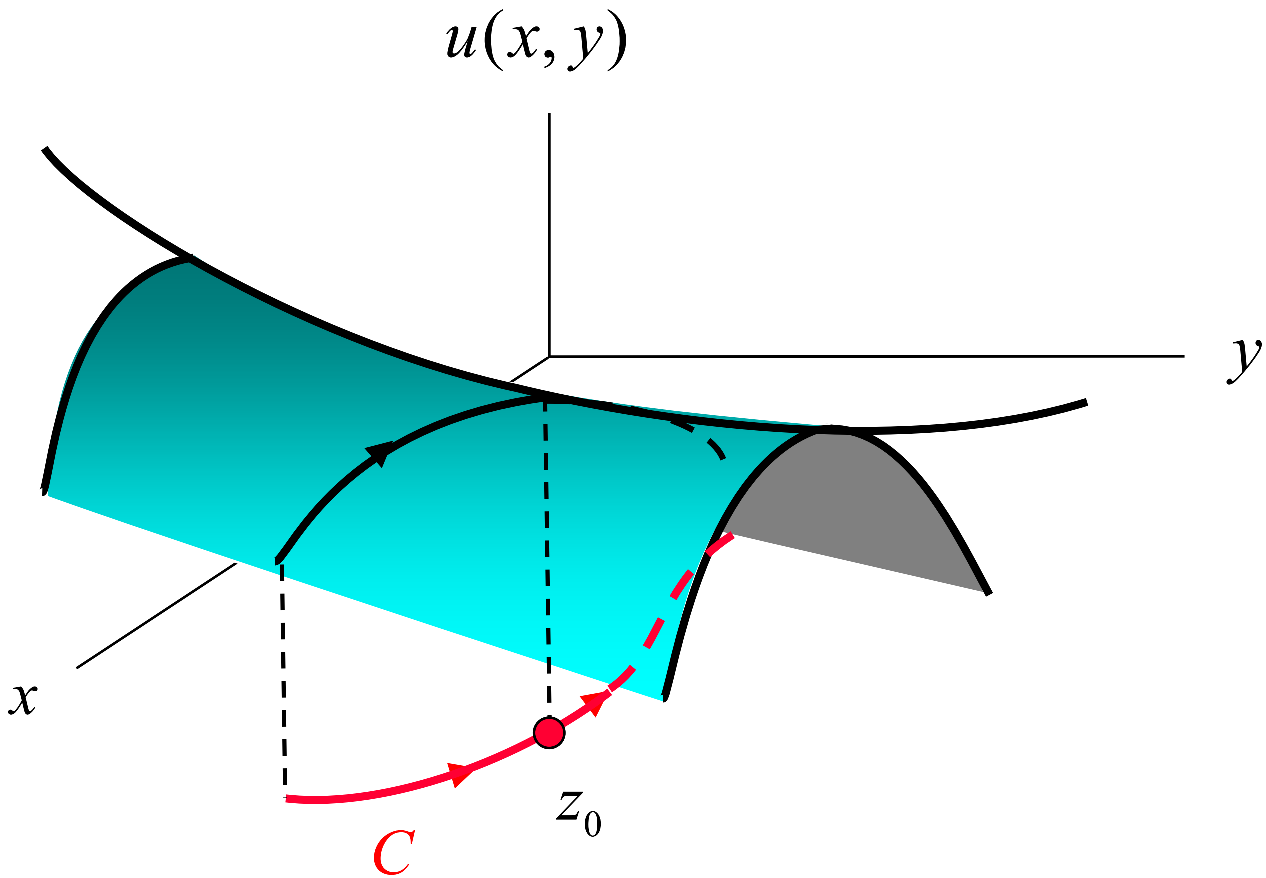


This is what the saddle looks like in the <u>original</u> coordinate system.

Note:
The saddle does not necessarily open along one of the original axes (only when $u_{xy}\,(x_0,\,y_0) = 0$).

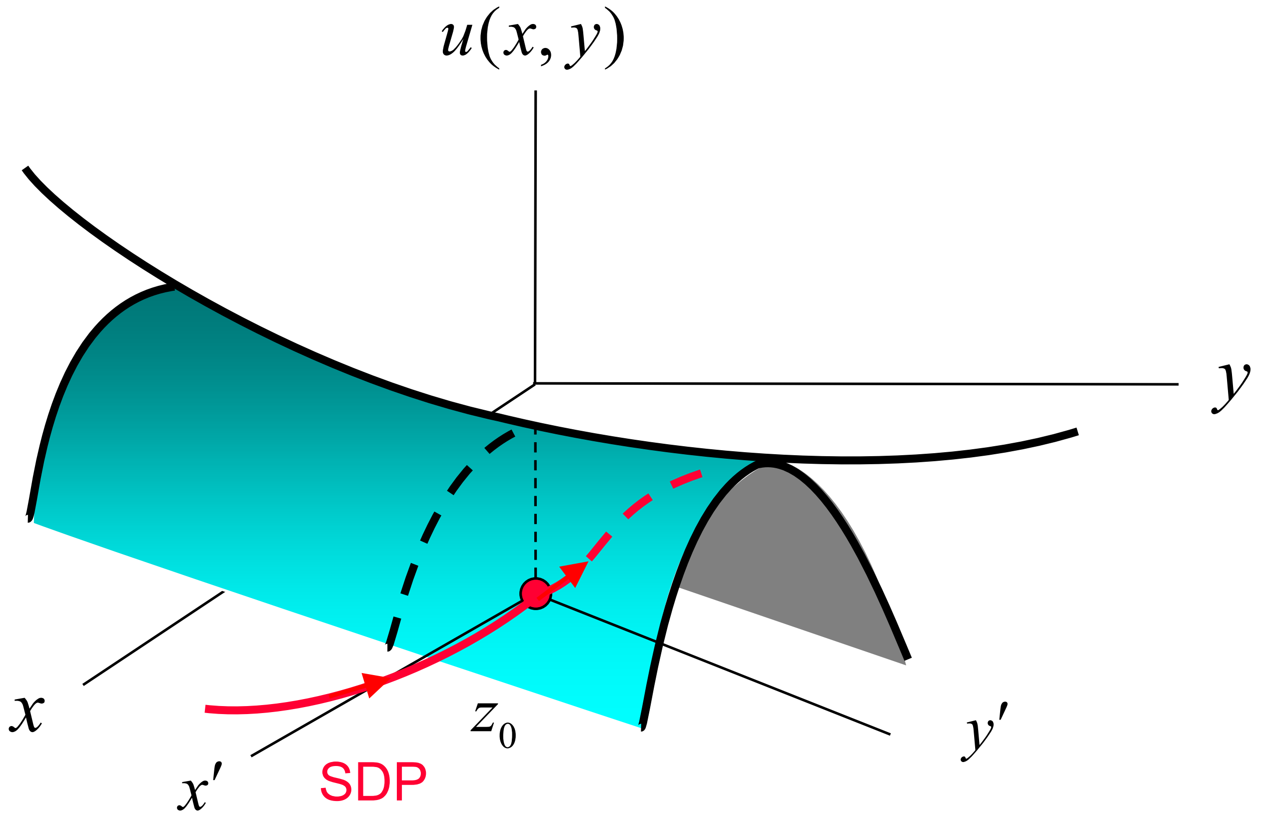
Steepest-Descent Method (cont.)

A <u>descending</u> path is one on which the u function <u>decreases</u> away from the saddle point.



A 3D view of a descending path C.

Steepest-Descent Method (cont.)



The steepest-descent path (SDP) is the one for which we descend the fastest on the saddle.

Steepest-Descent Method (cont.)

Along any descending path C we will have convergence:

$$I(\Omega) = \int_C f(z) e^{\Omega g(z)} dz$$

$$= e^{\Omega g(z_0)} \int_C f(z) e^{\Omega[g(z) - g(z_0)]} dz$$

$$= e^{\Omega g(z_0)} \int_C f(z) e^{i\Omega[v(z) - v(z_0)]} e^{\Omega[u(z) - u(z_0)]} dz$$

Exponentially decreasing function

Steepest-Descent Method (cont.)

Behavior on a Descending Path

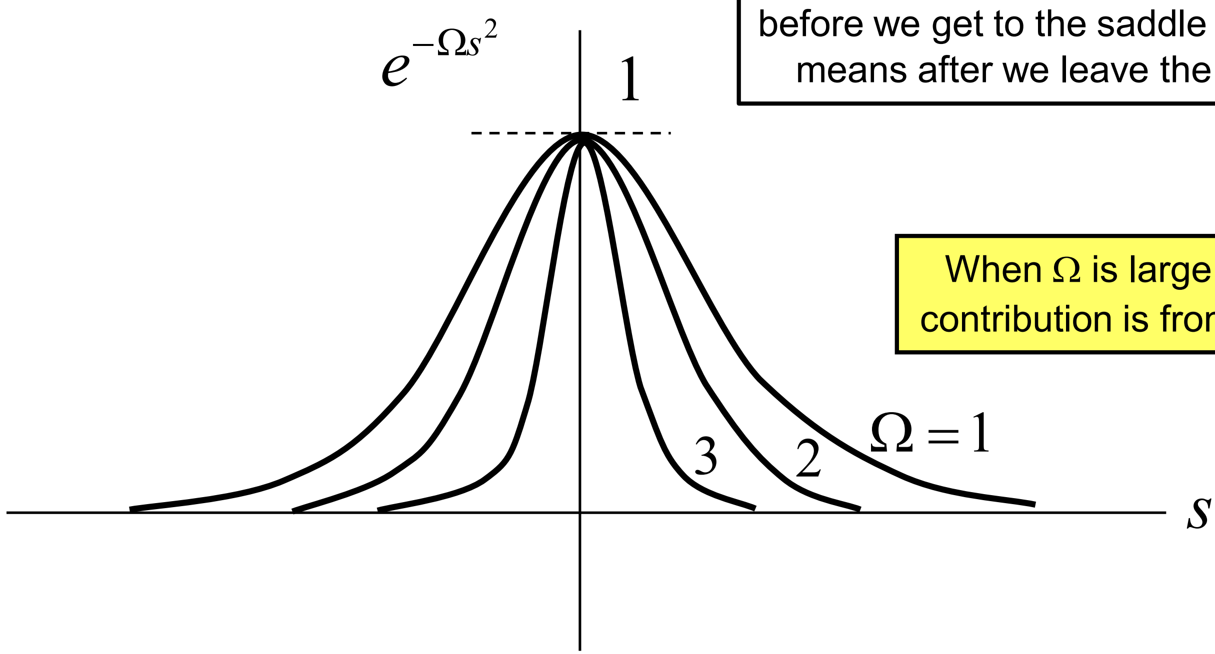
$$I(\Omega) = e^{\Omega g(z_0)} \int_C f(z)\, e^{i\Omega[v(z)-v(z_0)]}\, e^{\Omega[u(z)-u(z_0)]} dz$$

Let $s^2 \equiv u(z_0) - u(z)$

$$e^{-\Omega(u(z_0)-u(z))} = e^{-\Omega s^2}$$

Note:
The parameter s is related to the distance along the path from the saddle point.

The convention is that negative s means before we get to the saddle point, positive s means after we leave the saddle point.

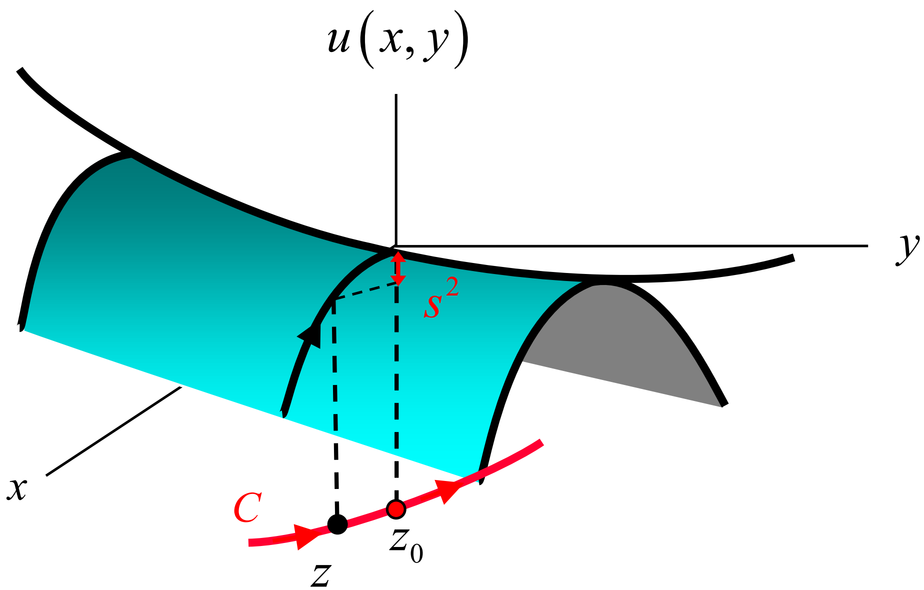


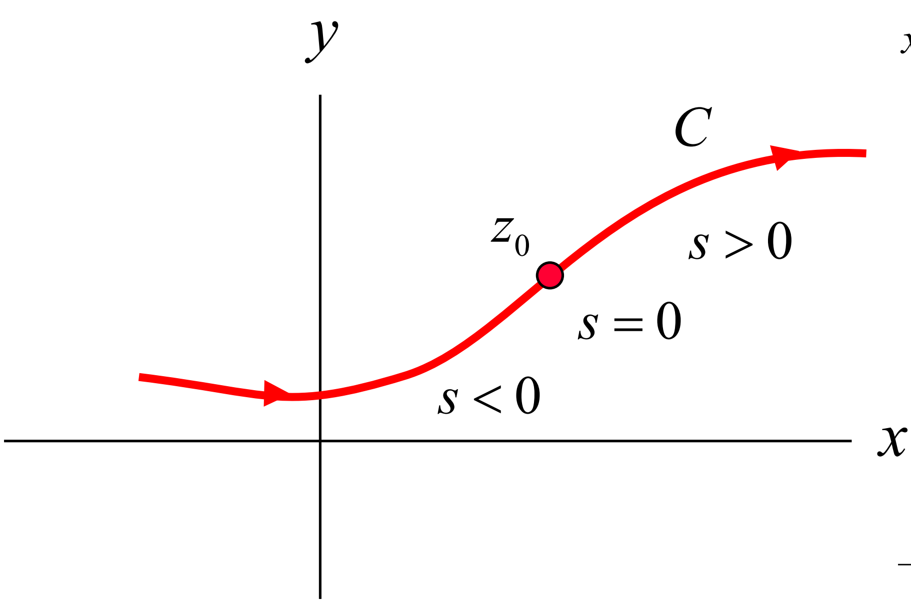
When Ω is large, most of contribution is from near z_0.

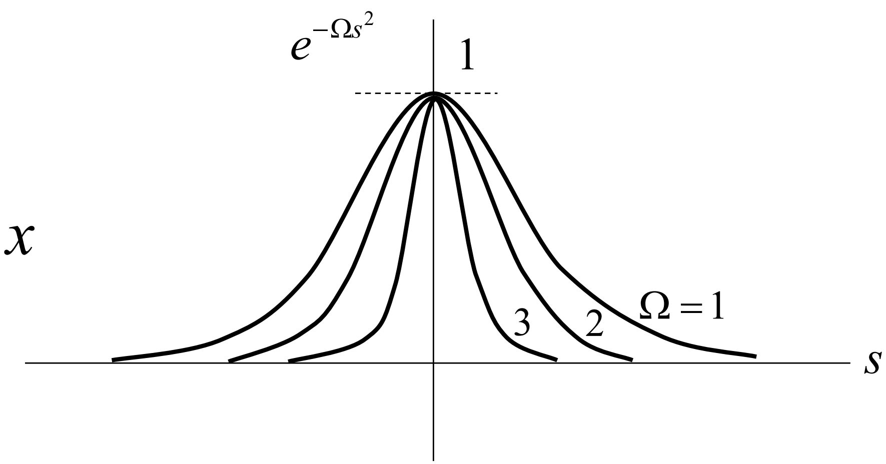
Steepest-Descent Method (cont.)

Sketches of a descending path:

$$s^2 \equiv u(z_0) - u(z)$$







Steepest-Descent Method (cont.)

Along any <u>descending path</u>:

$$I(\Omega) = e^{\Omega g(z_0)} \int_C f(z) e^{\Omega[g(z)-g(z_0)]} dz$$

$$\sim f(z_0) e^{\Omega g(z_0)} \int_C e^{\Omega[g(z)-g(z_0)]} dz$$

$$= f(z_0) e^{\Omega g(z_0)} \int_C e^{i\Omega[v(z)-v(z_0)]} \, e^{\Omega[u(z)-u(z_0)]} dz$$

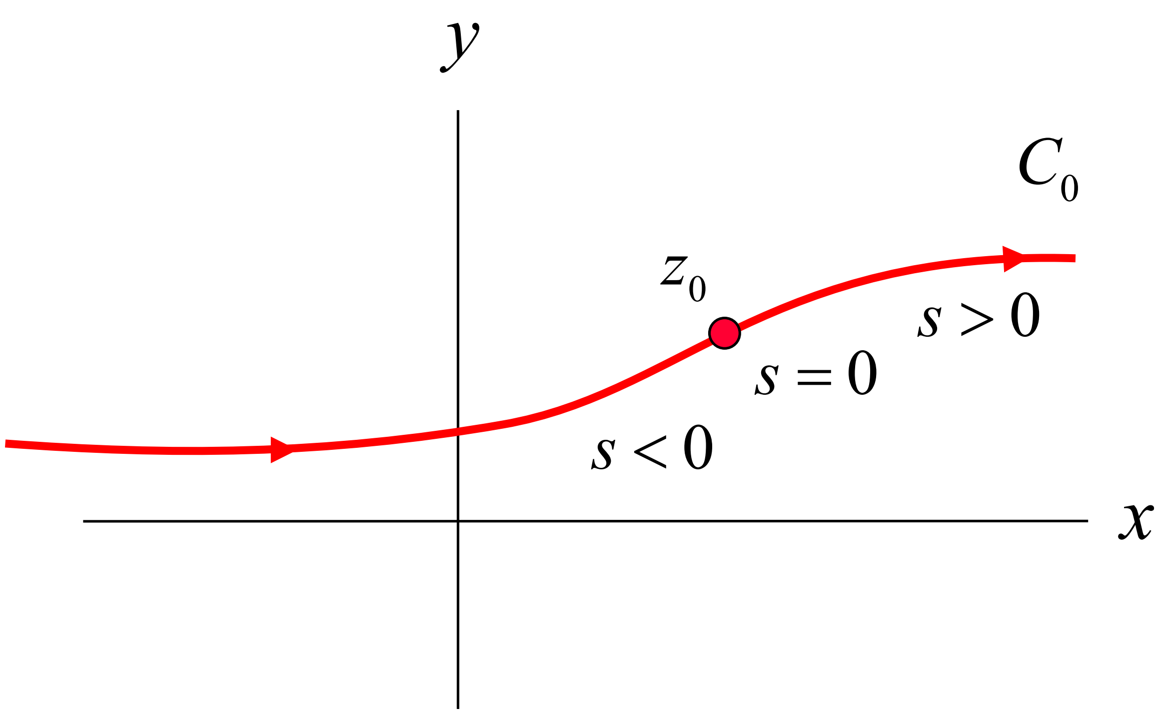
Both the phase <u>and</u> amplitude change along an arbitrary descending path C.

Important Point:
If we can find a path along which the phase does not change (v is constant), the integrand will have a purely exponentially decaying behavior (no phase term), making the integral easier to evaluate.

Steepest-Descent Method (cont.)

Choose a path C_0 of constant phase:

$$C_0 : \; v(z) = v(z_0) = \text{constant}$$



Steepest-Descent Method (cont.)

Gradient Property (proof follows next):

$$\nabla u\left(x,y\right)\ \text{is parallel to}\ C_0$$

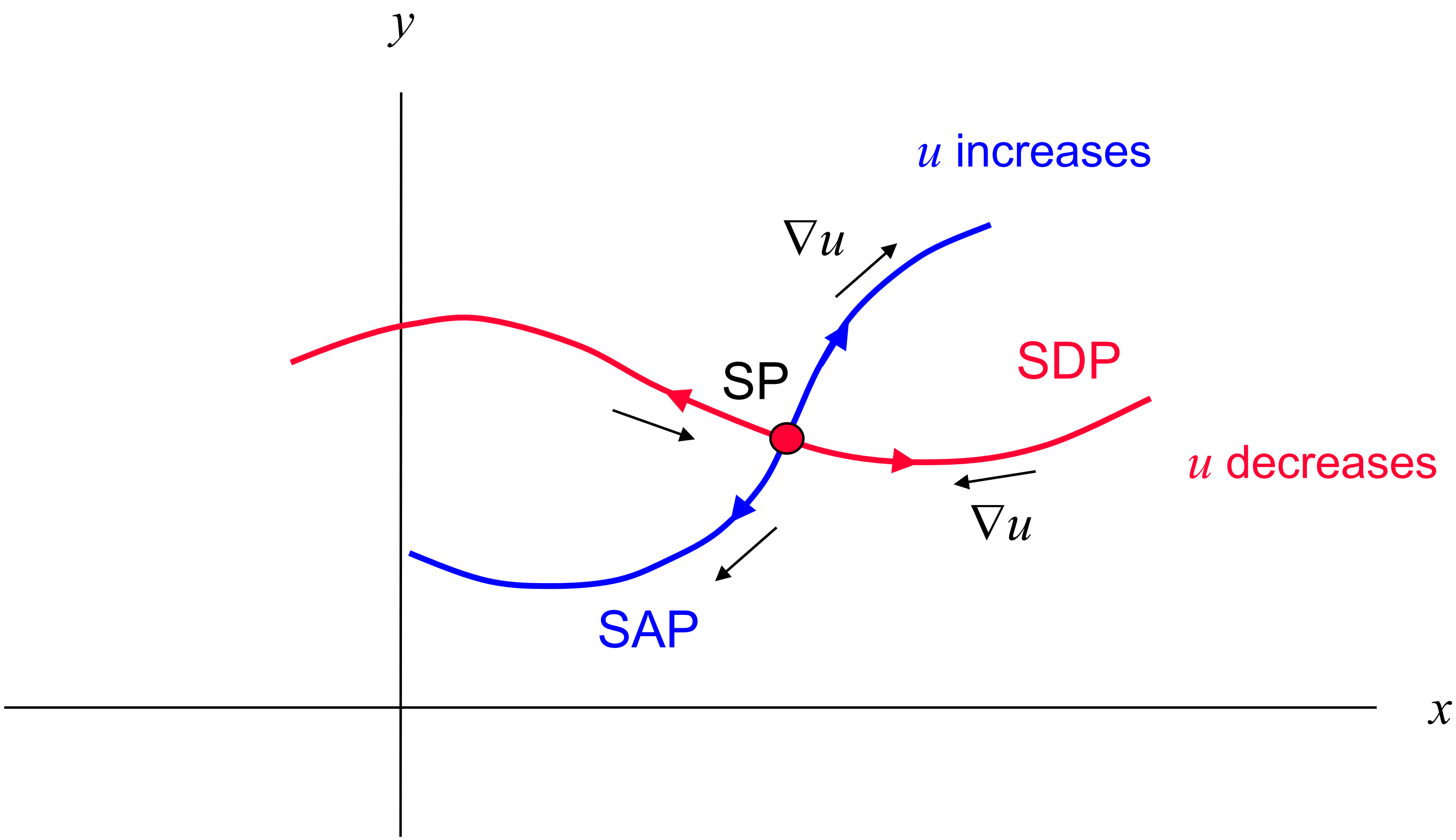
Hence C_0 (path of constant $v(z)$) is either a "path of steepest descent" (SDP) or a "path of steepest ascent" (SAP).

SDP: $u(x,y)$ decreases as fast as possible along the path away from the saddle point.

SAP: $u(x,y)$ increases as fast as possible along the path away from the saddle point.

Note: The integral will not converge along the SAP (we don't want this one!).

Steepest-Descent Method (cont.)



Note: $v(x,y)$ is constant along both paths.

Steepest-Descent Method (cont.)

Proof $\nabla u(x,y)$ is parallel to C_0

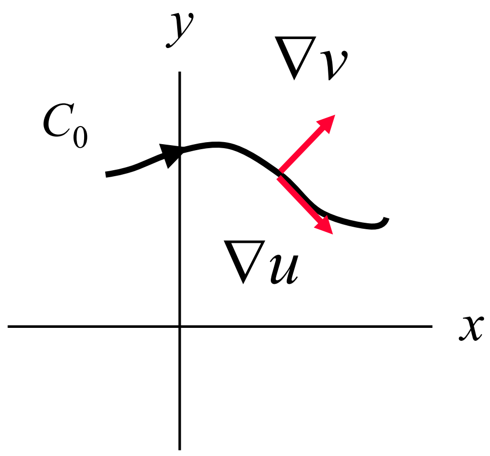
$$\nabla u = \underline{\hat{x}}\frac{\partial u}{\partial x} + \underline{\hat{y}}\frac{\partial u}{\partial y}$$

$$\nabla v = \underline{\hat{x}}\frac{\partial v}{\partial x} + \underline{\hat{y}}\frac{\partial v}{\partial y}$$

$$\begin{aligned}\nabla u \cdot \nabla v &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \\ &= \frac{\partial u}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\ &= 0\end{aligned}$$

C.R. Equations

Hence, $\nabla u \perp \nabla v$



Also, $\nabla v \perp C_0$ (v is constant on C_0)

Hence $\nabla u \,||\, C_0$

Steepest-Descent Method (cont.)

$$I(\Omega) \sim f(z_0)e^{\Omega g(z_0)} \int_C e^{i\Omega[v(z)-v(z_0)]} e^{\Omega[u(z)-u(z_0)]} dz$$

Because the v function is <u>constant</u> along the SDP, we have:

$$I(\Omega) \sim f(z_0)e^{\Omega g(z_0)} \int_{\mathrm{SDP}} e^{\Omega[u(z)-u(z_0)]} dz$$

or

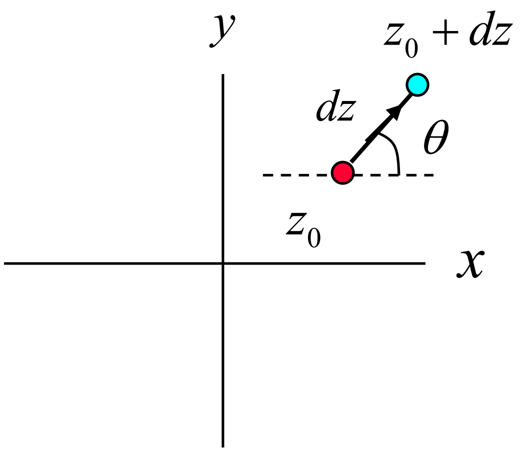
$$I(\Omega) \sim f(z_0)e^{\Omega g(z_0)} \int_{\mathrm{SDP}} e^{\Omega \underbrace{[g(z)-g(z_0)]}_{\text{Real}}} dz$$

Steepest-Descent Method (cont.)

Local behavior near SP

$$g(z) \approx g(z_0) + \cancel{g'(z_0)(z - z_0)} + \frac{1}{2} g''(z_0)(z - z_0)^2$$

so $$g(z) - g(z_0) \approx \frac{1}{2} g''(z_0)(z - z_0)^2$$



Denote $$\begin{cases} g''(z_0) = R\, e^{i\alpha} \quad (\text{known constant}) \\ z - z_0 = dz = re^{i\theta} \end{cases}$$

$$g(z) - g(z_0) \approx \frac{1}{2} R\, r^2\, e^{i(\alpha + 2\theta)}$$

Steepest-Descent Method (cont.)

$$g(z)-g(z_0) \approx \frac{1}{2}\left(Rr^2\right)e^{i(\alpha+2\theta)} \quad \Rightarrow \quad \begin{cases} u(z)-u(z_0) \approx \frac{1}{2}\left(Rr^2\right)\cos(\alpha+2\theta) \\ v(z)-v(z_0) \approx \frac{1}{2}\left(Rr^2\right)\sin(\alpha+2\theta) \end{cases}$$

SAP: $\alpha + 2\theta = 0 + 2\pi n$

$n = 0,1: \quad \theta = -\frac{\alpha}{2}, -\frac{\alpha}{2} + \pi$

$$\begin{aligned} u(z)-u(z_0) &= \frac{1}{2}Rr^2 \\ v(z)-v(z_0) &= 0 \end{aligned}$$

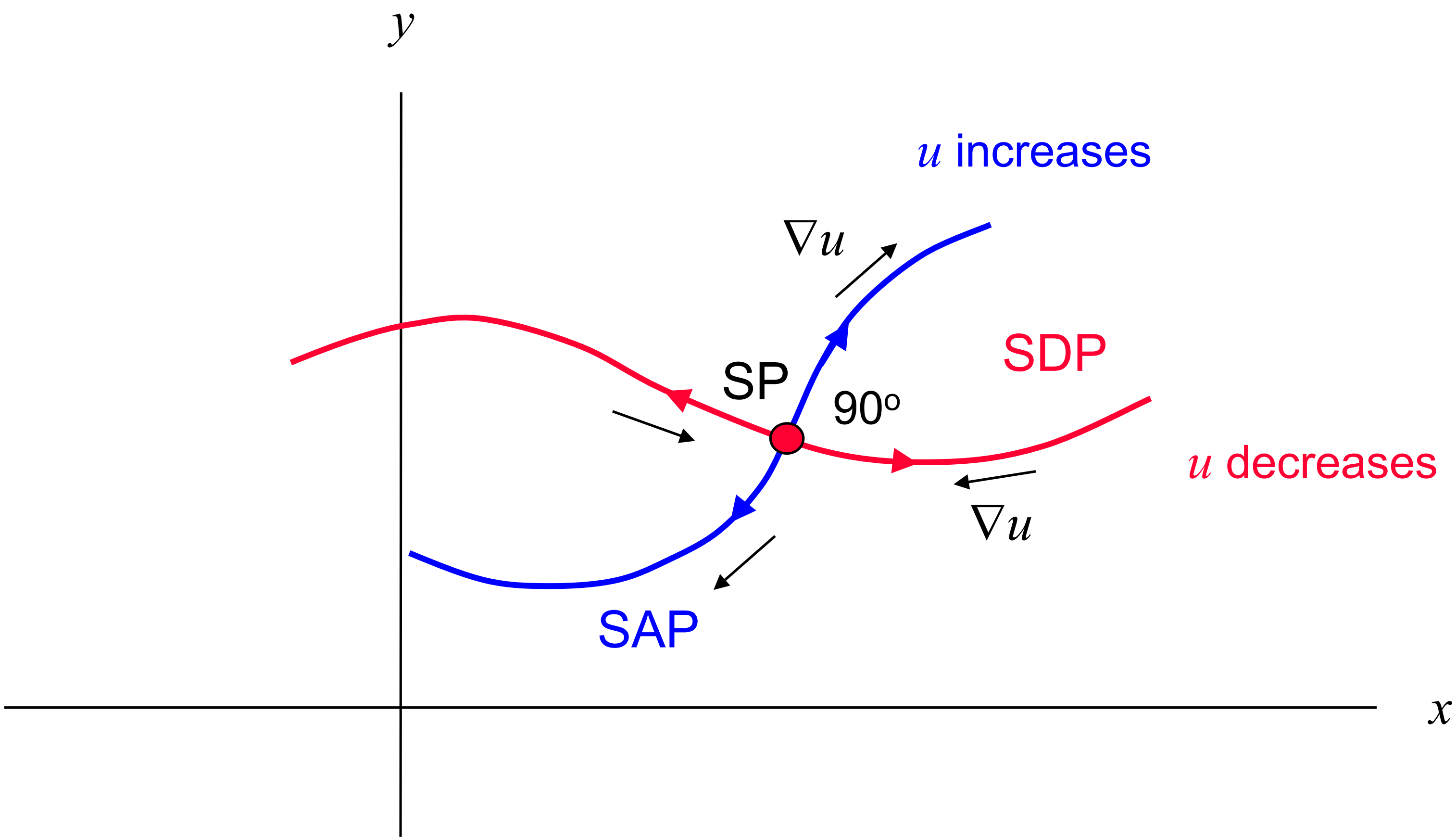
SDP: $\alpha + 2\theta = \pi + 2\pi n$

$n = 0,-1: \quad \theta = -\frac{\alpha}{2} + \frac{\pi}{2}, -\frac{\alpha}{2} - \frac{\pi}{2}$

$$\begin{aligned} u(z)-u(z_0) &= -\frac{1}{2}Rr^2 \\ v(z)-v(z_0) &= 0 \end{aligned}$$

Conclusion: The two paths meet 90° apart at the saddle point.

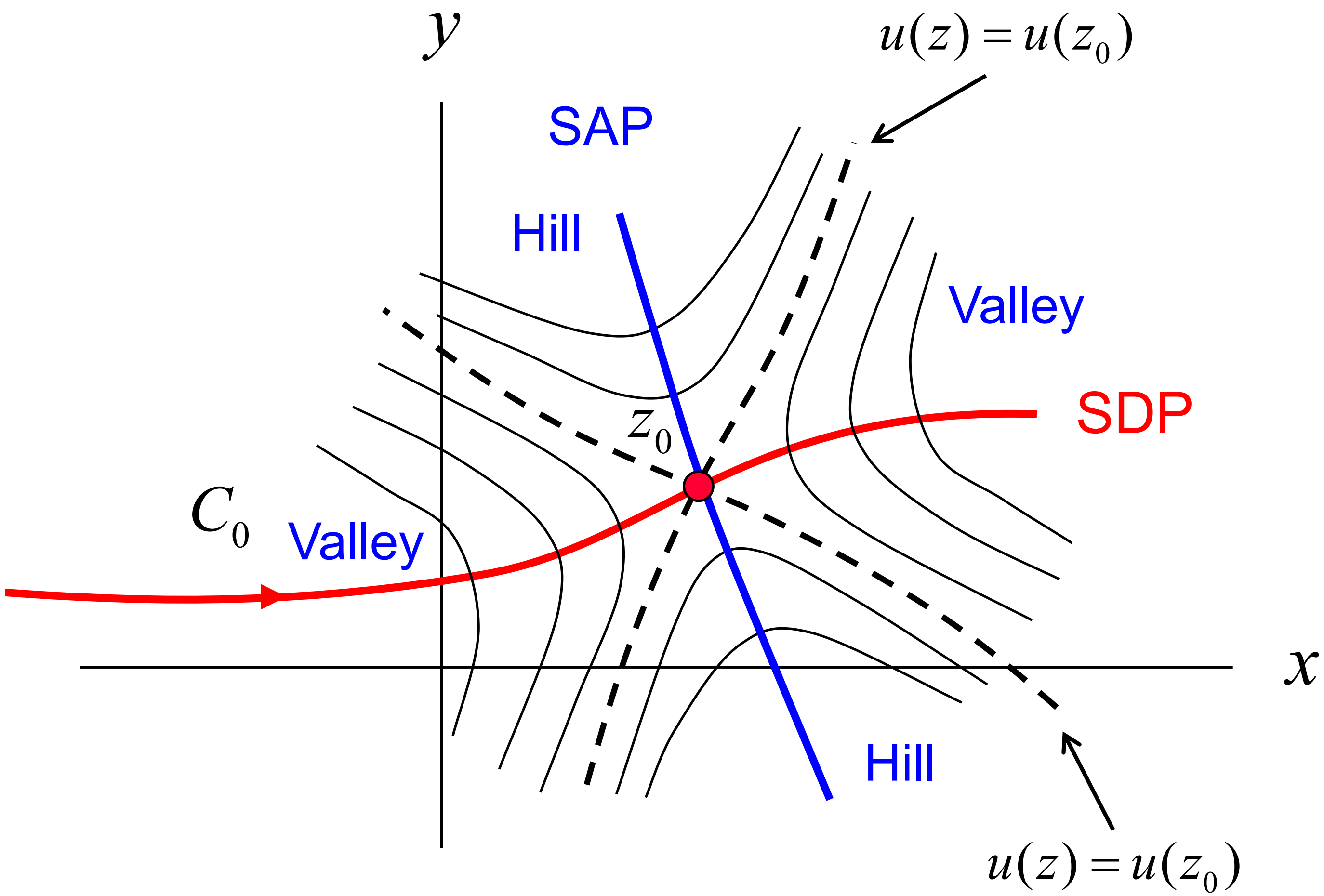
Steepest-Descent Method (cont.)



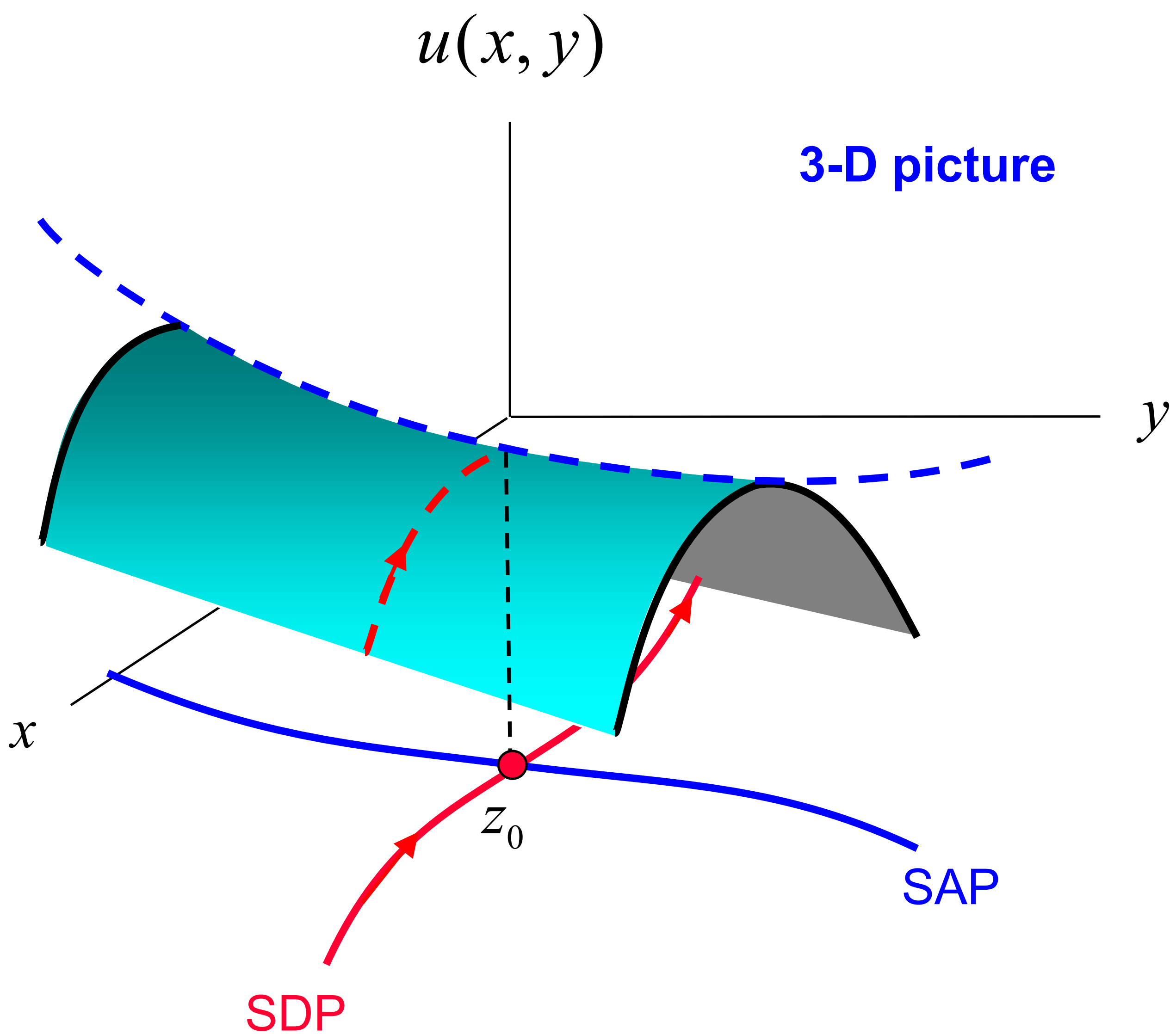
The SDP and SAP paths meet perpendicular at the saddle point.

Steepest-Descent Method (cont.)

The "landscape" of the function $u(z)$ near z_0



Steepest-Descent Method (cont.)



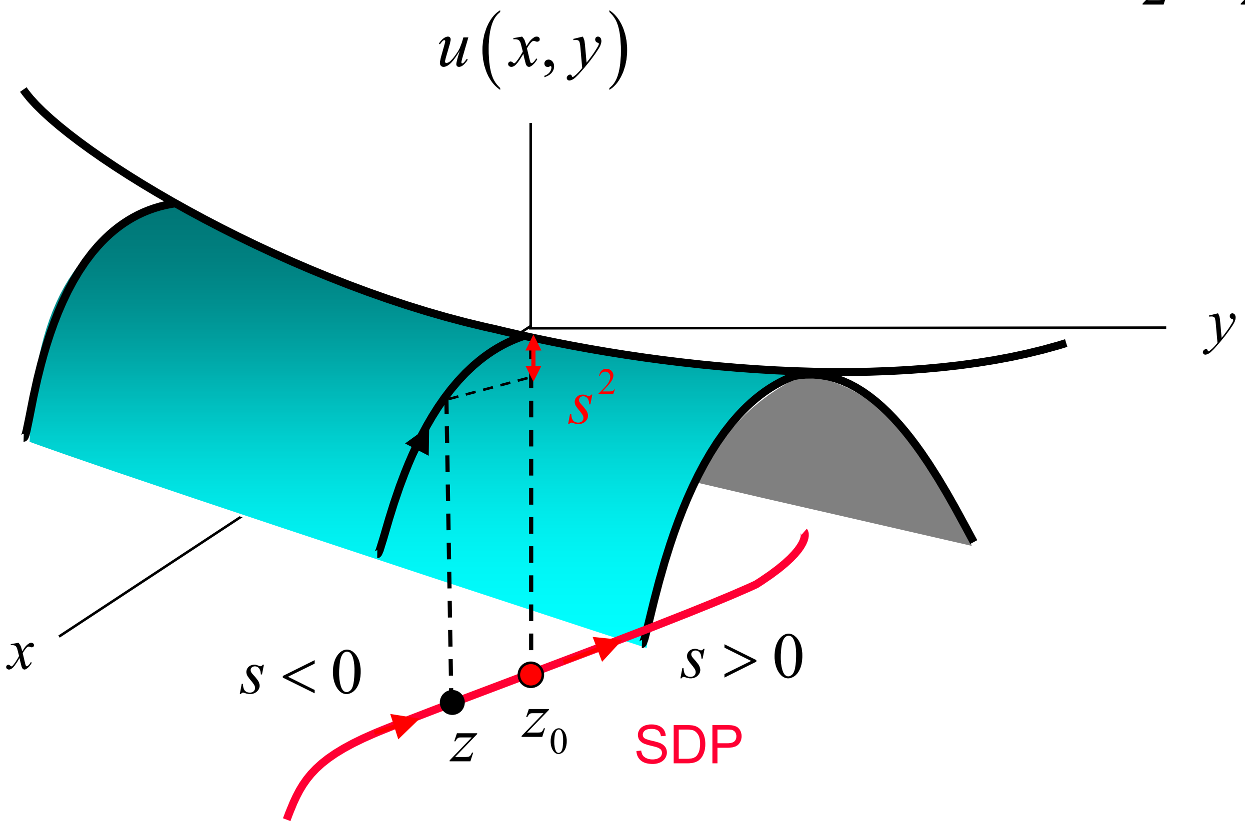
Steepest-Descent Method (cont.)

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \int\limits_{\mathrm{SDP}} e^{\Omega[u(z)-u(z_0)]}\, dz$$

Set $\quad s^2 = u(z_0) - u(z)$

This maps s to z:

$$z = z(s)$$



Steepest-Descent Method (cont.)

We then have

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \int_{-\infty}^{+\infty} e^{-\Omega s^2} \left(\frac{dz}{ds}\right) ds$$

or

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \left(\frac{dz}{ds}\right)_{s=0} \int_{-\infty}^{+\infty} e^{-\Omega s^2} ds$$

(leading term of the asymptotic expansion)

$$\int_{-\infty}^{+\infty} e^{-\Omega s^2} ds = \sqrt{\frac{\pi}{\Omega}}$$

Hence

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \left(\frac{dz}{ds}\right)_{s=0} \sqrt{\frac{\pi}{\Omega}}$$

Steepest-Descent Method (cont.)

To evaluate the derivative:

$$
\begin{aligned}
-s^2 &= u(z) - u(z_0) \\
&= g(z) - g(z_0)
\end{aligned}
$$

(Recall: v is constant along SDP.)

$$\Rightarrow \quad -2s\left(\frac{ds}{dz}\right) = g'(z)$$

At the SP this gives $0 = 0$.

Take one more derivative:

$$-2\left(\frac{ds}{dz}\right)\left(\frac{ds}{dz}\right) - 2s\frac{d^2 s}{dz^2} = g''(z)$$

Steepest-Descent Method (cont.)

At the saddle point ($s = 0$):

$$\left(\frac{ds}{dz}\right)_{z_0} = \left(\frac{g''(z_0)}{-2}\right)^{1/2}$$

so

$$\left(\frac{dz}{ds}\right)_{s=0} = \left(\frac{-2}{g''(z_0)}\right)^{1/2}$$

Hence, we have

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \left(\frac{-2}{g''(z_0)}\right)^{1/2} \sqrt{\frac{\pi}{\Omega}}$$

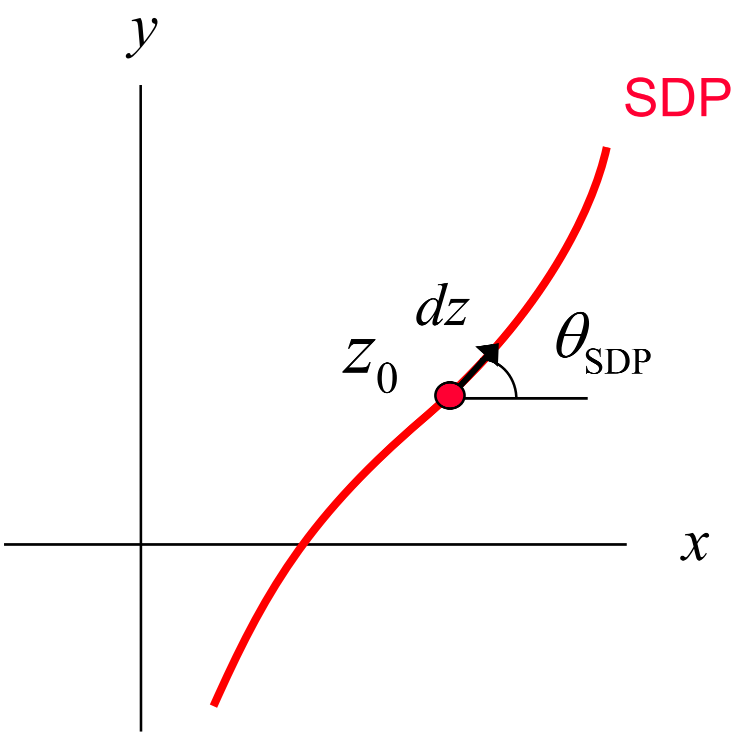
Steepest-Descent Method (cont.)

There is an ambiguity in sign for the square root:

$$\left(\frac{dz}{ds}\right)_{s=0} = \left(\frac{-2}{g''(z_0)}\right)^{1/2}$$

To avoid this ambiguity, define

$$\theta_{\mathrm{SDP}} \equiv \arg\left(\frac{dz}{ds}\right)_{s=0}$$



Steepest-Descent Method (cont.)

The derivative term is therefore

$$\left(\frac{dz}{ds}\right)_{s=0} = \sqrt{\frac{2}{\left|g''(z_0)\right|}}\, e^{i\theta_{\text{SDP}}}$$

Hence

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{\left|g''(z_0)\right|}}\, e^{i\theta_{\text{SDP}}}$$

Steepest-Descent Method (cont.)

To find θ_{SDP} :

Denote again:

$$g''(z_0) = R\,e^{i\alpha}$$

$$\alpha = \arg\left(g''(z_0)\right)$$

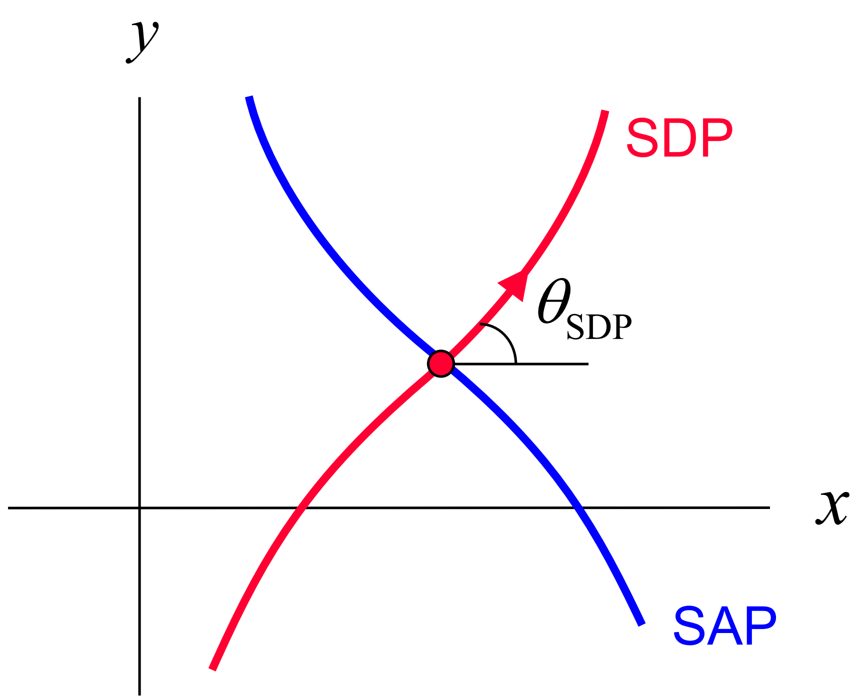
$$z - z_0 = r\,e^{i\theta_{\text{SDP}}}$$

$$g(z) - g(z_0) \approx \frac{1}{2} g''(z_0)(z - z_0)^2 = \frac{1}{2}\left(Rr^2\right)e^{i(\alpha + 2\theta_{\text{SDP}})}$$

$$\alpha + 2\theta_{\text{SDP}} = \pm\pi$$

$$\theta_{\text{SDP}} = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

Steepest-Descent Method (cont.)



$$\theta_{\text{SDP}} = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

$$\alpha = \arg\left(g''\left(z_0\right)\right)$$

The two sign choices correspond to going one way (e.g., “up”) or the other way (e.g., “down”) on the SDP.

The direction of integration determines The sign.

The “user” must determine this!

Steepest-Descent Method (cont.)

Summary

$$I(\Omega) = \int\limits_C f(z)\, e^{\Omega g(z)}\, dz$$

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{\left|g''(z_0)\right|}}\, e^{i\theta_{\mathrm{SDP}}}$$

$$\theta_{\mathrm{SDP}} = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

You must determine which sign is correct by looking at the direction along the SDP.

$$\alpha = \arg\left(g''(z_0)\right)$$

Example

$$J_0(\Omega) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(\Omega \cos z)\, dz$$

$$J_0(\Omega) = \frac{1}{\pi} \operatorname{Re} I(\Omega)$$

where

$$I(\Omega) \equiv \int_{-\pi/2}^{\pi/2} e^{\Omega(i \cos z)} dz$$

Hence, we identify:

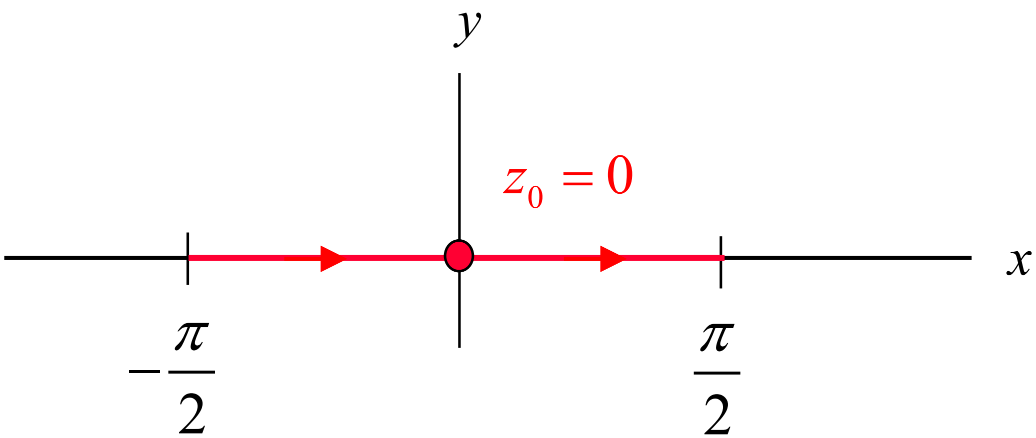
$$g'(z_0) = -i \sin z_0 = 0$$

$$f(z) = 1$$

$$g(z) = i \cos z$$

$$\Rightarrow \quad z_0 = 0 \pm n\pi$$

Example (cont.)



$$g(z) = i\cos z$$

$$g''(z_0) = -i\cos z_0 = -i$$

$$\alpha = \arg g''(z_0) = -\frac{\pi}{2}$$

$$\theta_{\mathrm{SDP}} = -\frac{\alpha}{2} \pm \frac{\pi}{2} = \frac{\pi}{4} \pm \frac{\pi}{2}$$

$$\boxed{\theta_{\mathrm{SDP}} = -\frac{\pi}{4} \text{ or } \frac{3\pi}{4}}$$

Example (cont.)

Identify the SDP and SAP:

$$g(z) = i\cos(x+iy)$$
$$= i\left[\cos x \cosh y - i \sin x \sinh y\right]$$

$$\Rightarrow \begin{cases} u(x,y) = \sin x \sinh y \\ v(x,y) = \cos x \cosh y \end{cases}$$

SDP and SAP: $v(z) = v(z_0) = \text{constant}$

$$\Rightarrow \cos x \cosh y = \text{constant} = \cos(0)\cosh(0)$$
$$= 1$$

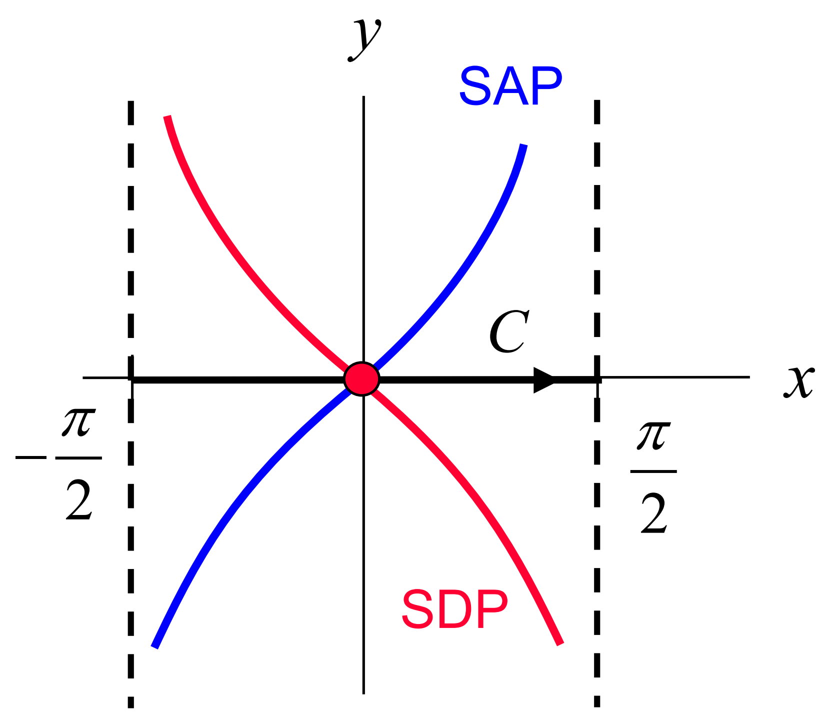
Example (cont.)

SDP and SAP:

$$\cos x \cosh y = 1$$

Examination of the u function reveals which of the two paths is the SDP.

$$u(x,y) = \sin x \ \sinh y$$



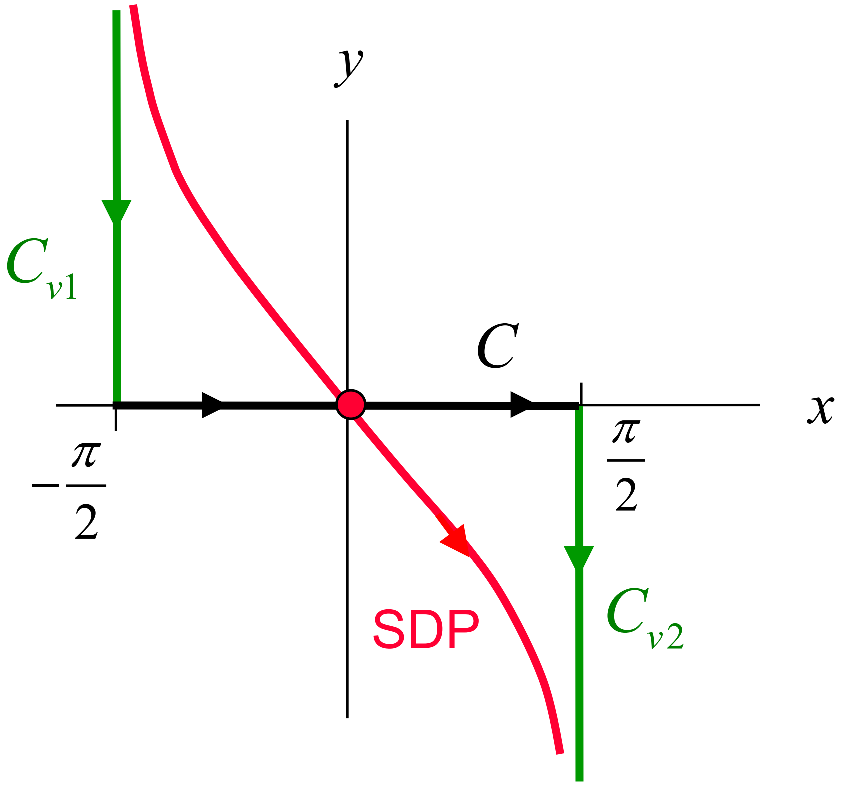
Example (cont.)

Vertical paths are added so that the path now has limits at infinity.

$$\text{SDP} = C + C_{v1} + C_{v2}$$

It is now clear which choice is correct for the departure angle:

$$\theta_{\text{SDP}} = -\frac{\pi}{4}$$



Example (cont.)

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{|g''(z_0)|}}\, e^{i\theta_{\mathrm{SDP}}}$$

(If we ignore the contributions of the vertical paths.)

Hence,

$$I(\Omega) \sim (1)\, e^{\Omega i \cos(0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{|-i|}}\, e^{-i\frac{\pi}{4}}$$

$$I(\Omega) \sim \sqrt{\frac{2\pi}{\Omega}}\, e^{i\left(\Omega - \frac{\pi}{4}\right)}$$

so

$$J_0(\Omega) \sim \frac{1}{\pi} \operatorname{Re}\left\{ \sqrt{\frac{2\pi}{\Omega}}\, e^{i\left(\Omega - \frac{\pi}{4}\right)} \right\}$$

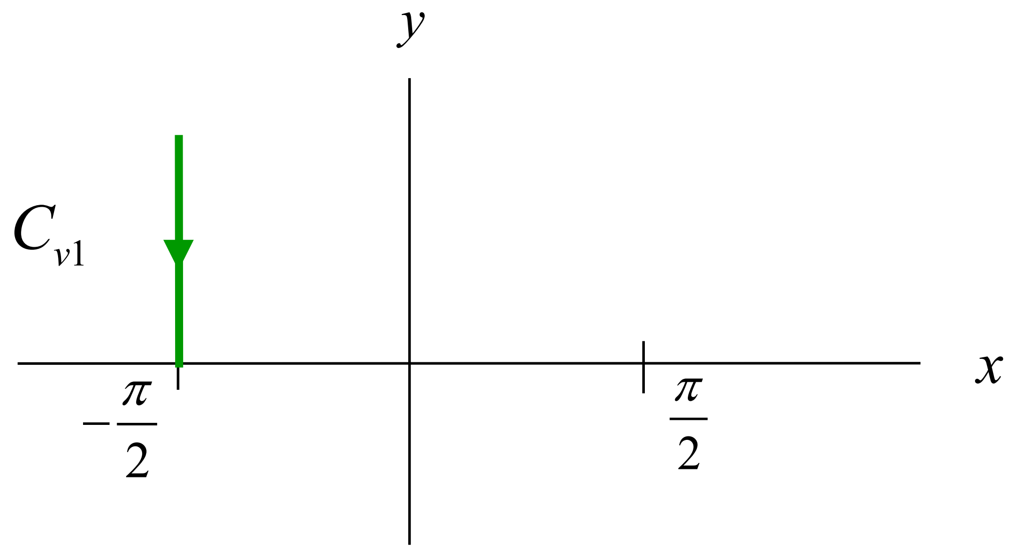
Example (cont.)

Hence, we have

$$J_0(\Omega) \sim \sqrt{\frac{2}{\pi\Omega}} \cos\left(\Omega - \frac{\pi}{4}\right)$$

Example (cont.)

Examine the path C_{v1} (the path C_{v2} is similar).



$$I_{v1} = \int_{-\pi/2+i\infty}^{-\pi/2} e^{\Omega(i\cos z)}\, dz$$

Let $z = -\frac{\pi}{2} + iy$

Example (cont.)

$$I_{v1} = i\int_{\infty}^{0} e^{i\Omega\cos\left(-\frac{\pi}{2}+iy\right)}dy = i\int_{\infty}^{0} e^{-\Omega\sinh y}dy = -i\int_{0}^{\infty} e^{-\Omega\sinh y}dy$$

since $\cos\left(-\frac{\pi}{2}+iy\right) = \sin(iy) = i\sinh y$

Since Ω is becoming very large, we can write:

$$I_{v1} \sim -i\int_{0}^{\infty} e^{-\Omega y}\,dy \quad \left(\sinh y \approx y \text{ for } y \approx 0\right)$$

$$\sim -i\left(\frac{-1}{\Omega}\right)\left(e^{-\Omega y}\right)\Big|_{0}^{\infty}$$

Hence, we have

$$\boxed{I_{v1} \sim -i\left(\frac{1}{\Omega}\right)}$$

Example (cont.)

Hence,

$$I_{v1} = O\left(\frac{1}{\Omega}\right) = o\left(\frac{1}{\sqrt{\Omega}}\right)$$

Hence, I_{v1} is a much smaller term than what we obtain from the method of steepest descent, when Ω gets large, and we can ignore it.

Example

The Gamma (generalized factorial) function:

$$\Gamma(x) \equiv \int_0^\infty e^{-t} t^{x-1} dt, \qquad x > 0$$

$\Gamma(x) = (x-1)! =$ Gamma function (generalized factorial function)

$$\Gamma(\Omega+1) = \int_0^\infty e^{-t} t^{\Omega} dt = \int_0^\infty e^{-t} \left(e^{\ln t}\right)^{\Omega} dt = \int_0^\infty e^{-t} e^{\Omega \ln t} dt$$

Note: There is no saddle point!

$(g(t) = \ln(t))$

Let $t = \Omega z$

$$\Gamma(\Omega+1) = e^{\Omega \ln \Omega} \int_0^\infty e^{-\Omega z} e^{\Omega \ln z}\, \Omega\, dz = \Omega^{\Omega} \int_0^\infty e^{-\Omega z} e^{\Omega \ln z}\, \Omega\, dz$$

Example (cont.)

$$\Gamma(\Omega+1) = \Omega^{\Omega+1} \int_0^\infty e^{-\Omega z} e^{\Omega \ln z}\, dz$$

or

$$\Gamma(\Omega+1) = \Omega^{\Omega+1} \int_0^\infty e^{\Omega(\ln z - z)}\, dz$$

$$z_0 = 1$$

$$f(z) = \Omega^{\Omega+1}$$

$$g(z_0) = -1$$

$$g(z) = \ln(z) - z$$

$$g''(z_0) = -\frac{1}{z_0^2} = -1$$

$$g'(z) = \frac{1}{z} - 1$$

$$\alpha = \arg\left(g''(z_0)\right) = \pi$$

$$g''(z) = -\frac{1}{z^2}$$

$$\theta_{\mathrm{SDP}} = -\frac{\alpha}{2} \pm \frac{\pi}{2} = -\frac{\pi}{2} \pm \frac{\pi}{2} = 0, -\pi$$

Example (cont.)

Recall the recipe:

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{\left|g''(z_0)\right|}}\, e^{i\theta_{\mathrm{SDP}}}$$

$$\theta_{\mathrm{SDP}} = 0$$

Note:
The SDP is the positive real axis (see the derivation below). The departure angle is zero, not π, since we are integrating from 0 to ∞.

Hence

$$\Gamma(\Omega+1) \sim \Omega^{\Omega+1} e^{-\Omega} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{\left|-1\right|}}\, e^{i0}$$

$g(z) = \ln(z) - z$

Let $z = re^{i\theta}$

Then $v(x,y) = \theta - y$

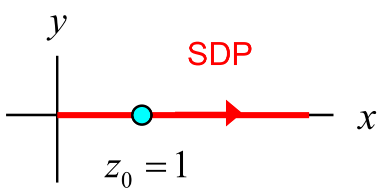
SDP / SAP : $\theta - y = 0$

$\Rightarrow$ real axis is SDP or SAP.

or

$$\Omega! = \Gamma(\Omega+1) \sim \sqrt{2\pi\Omega}\, \Omega^{\Omega} e^{-\Omega}$$

This is Sterling's formula (leading term).



Note : On the positive real axis, $g(z) = u(z) = u(x) = \ln(x) - x$. Also, $u(x)$ is *maximum* at $x_0 = 1$ (the real axis is the SDP).

Complete Asymptotic Expansion

We can obtain the <u>complete asymptotic expansion</u> of the integral with the steepest-descent method (including as many higher-order terms as we wish).

$$I(\Omega)=\int_a^b f(z)\,e^{\Omega g(z)}\,dz$$

The path is assumed to be deformed to the SDP.

Change of variables: $dz=\left(\dfrac{dz}{ds}\right)ds$

Define:

$$s^2 \equiv \left[g(z_0)-g(z)\right]_{\mathrm{SDP}} \qquad h(s)\equiv f(z(s))\left(\frac{dz(s)}{ds}\right)$$

Then we have

$$I(\Omega)=e^{\Omega g(z_0)}\int_{s_a}^{s_b} h(s)\,e^{-\Omega s^2}\,ds \qquad \begin{array}{l} s_a<0 \\ s_b>0 \end{array}$$

Complete Asymptotic Expansion (cont.)

We can then write

$$I(\Omega) \sim e^{\Omega g(z_0)} \int_{-\infty}^{\infty} h(s)\, e^{-\Omega s^2} ds$$

(Extending the limits to infinity gives an <u>exponentially small error</u>: This does not affect the asymptotic expansion.)

We can asymptotically evaluate this using <u>Watson's lemma</u>.

(Please see the next slide.)

Complete Asymptotic Expansion (cont.)

Watson's Lemma*

$$I(\Omega) = e^{\Omega g(z_0)} \int_{-\infty}^{\infty} h(s) e^{-\Omega s^2} ds$$

Assume

$$h(s) \sim \sum_{n=0,1,2,3\ldots} a_n s^n \quad \text{as} \quad s \to 0$$

This means :

$$h(s) - \sum_{n=0}^{N} a_n s^n = o(s^N)$$

as $s \to 0$

Note: We can use an equal sign if h is analytic at $s = 0$, which is usually the case.

Then

$$I(\Omega) \sim e^{\Omega g(z_0)} \sum_{n=0,1,2,3\ldots} a_n \int_{-\infty}^{\infty} s^n e^{-\Omega s^2} ds$$

Note: The integral is zero for $n =$ odd (the integrand is then an odd function).

* A different form of Watson's lemma is discussed in Notes 16, where s goes from zero to infinity, and n does not have to be an integer.

Complete Asymptotic Expansion (cont.)

Performing the integration,

$$\underbrace{\int_{-\infty}^{\infty} s^n e^{-\Omega s^2} ds}_{n=\text{even}} = \frac{1}{\Omega^{\frac{(n+1)}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

where

$$\Gamma(x) = (x-1)!$$

$$\Gamma(x) \equiv \int_0^{\infty} e^{-t} t^{x-1} dt$$

Use $t = \Omega s^2, \; dt = \Omega(2s)ds$

$$\underbrace{\int_{-\infty}^{\infty} e^{-\Omega s^2} s^n ds}_{n=\text{even}} = 2\int_0^{\infty} e^{-t}\left(\frac{t}{\Omega}\right)^{n/2}\left(\frac{1}{\Omega\left(2\sqrt{\frac{t}{\Omega}}\right)}\right)dt$$

$$= \frac{1}{\Omega^{\frac{(n+1)}{2}}}\int_0^{\infty} e^{-t} t^{(n-1)/2} dt$$

$$= \frac{1}{\Omega^{\frac{(n+1)}{2}}}\Gamma\left(\frac{n+1}{2}\right)$$

We then have

$$I(\Omega) \sim e^{\Omega g(z_0)} \sum_{n=0,2,4\ldots} \frac{a_n}{\Omega^{\frac{(n+1)}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

Complete Asymptotic Expansion (cont.)

Summary

$$I(\Omega) = \int_a^b f(z)\, e^{\Omega g(z)}\, dz$$

$$I(\Omega) \sim e^{\Omega g(z_0)} \sum_{\substack{n=0,2,4\ldots \\ \text{(even)}}} \frac{a_n}{\Omega^{\frac{(n+1)}{2}}}\, \Gamma\!\left(\frac{n+1}{2}\right)$$

where

$$h(s) \sim \sum_{n=0,1,2,3\ldots} a_n s^n \quad \text{as} \quad s \to 0$$

$$s^2 = \left[g(z_0) - g(z) \right]_{\mathrm{SDP}} \qquad h(s) = f(z(s)) \left(\frac{dz(s)}{ds} \right)$$

Complete Asymptotic Expansion (cont.)

Note: Keeping only the leading term gives the usual steepest-descent result:

$$I(\Omega) \sim e^{\Omega g(z_0)} \frac{a_0}{\sqrt{\Omega}} \sqrt{\pi} \qquad \left(\text{Recall}: \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \right)$$

We have

$$h(s) \sim a_0 \quad \text{as} \quad s \to 0 \qquad \left(\text{so } a_0 = h(0) \right)$$

$$h(0) = f(z(0)) \left(\frac{dz}{ds} \right)_{s=0} = f(z_0) \left(\frac{dz}{ds} \right)_{s=0} = f(z_0) \sqrt{\frac{2}{|g''(z_0)|}} \, e^{i\theta_{\text{SDP}}}$$

$$\Rightarrow \quad I(\Omega) \sim e^{\Omega g(z_0)} \frac{1}{\sqrt{\Omega}} \sqrt{\pi} f(z_0) \sqrt{\frac{2}{|g''(z_0)|}} \, e^{i\theta_{\text{SDP}}}$$

(our previous result)

Complete Asymptotic Expansion (cont.)

Example:

$$I(\Omega) = \int_{-1}^{1} \cos(z) e^{-\Omega z^2} \, dz$$

$$f(z) = \cos(z)$$

$$g(z) = -z^2$$

We have $g'(z) = -2z$

$$\Rightarrow \quad z_0 = 0$$

$$g(z) = -z^2 = -(x+iy)^2 = (-x^2 + y^2) + i(-2xy)$$

$$\Rightarrow \quad v(x,y) = -2xy$$

SDP : $v(x,y) = v(x_0, y_0) = 0 \quad \Rightarrow \quad xy = 0$

SDP: $y = 0$ (real axis)
SAP: $x = 0$ (imaginary axis)

Complete Asymptotic Expansion (cont.)

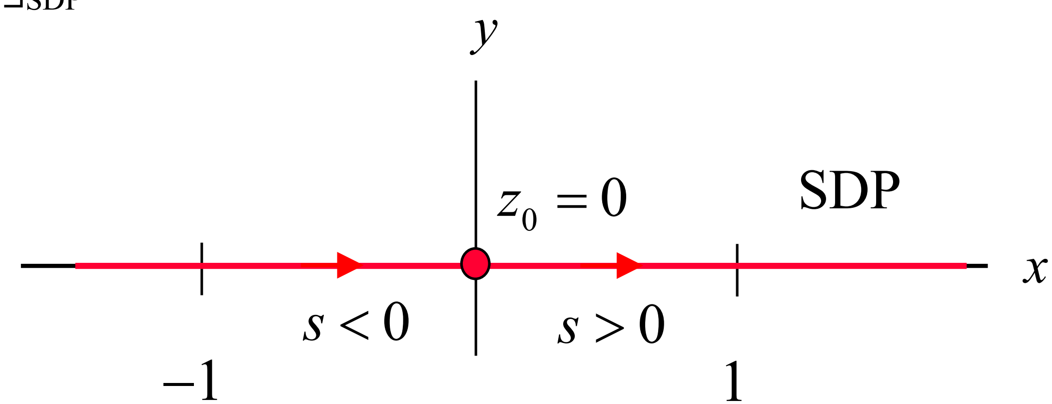
$$h(s) = f(z(s))\left(\frac{dz(s)}{ds}\right)$$

Also, we have

$$\begin{aligned} s^2 &= \left[g(z_0) - g(z)\right]_{\mathrm{SDP}} \\ &= \left[-z^2\Big|_{z=0} + z^2\right]_{\mathrm{SDP}} \\ &= \left[0 + z^2\right] \\ &= z^2 \end{aligned}$$

$u(x,y) = \left(-x^2 + y^2\right) = -x^2$ (on real axis)

$\Rightarrow$ real axis is SDP



Hence

$s = \pm z$ (choose + sign : see figure)

We then have

$z = s$ $\Rightarrow$ $f(z(s)) = \cos(z(s)) = \cos(s)$ & $\dfrac{dz}{ds} = 1$

Complete Asymptotic Expansion (cont.)

Hence

$$h(s) = f\left(z(s)\right)\left(\frac{dz(s)}{ds}\right)$$
$$= \cos(s)(1)$$
$$= \cos(s)$$

Recall:

$$h(s) \sim \sum_{n=0,1,2,3\ldots} a_n s^n \quad \text{as} \quad s \to 0$$

$$h(s) = \cos(s) \sim 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \ldots \quad \Rightarrow \quad a_0 = 1,\ a_2 = -\frac{1}{2},\ a_4 = \frac{1}{24},\ \text{etc.}$$

Complete Asymptotic Expansion (cont.)

The complete asymptotic expansion is

$$I(\Omega) \sim e^{\Omega g(z_0)} \sum_{n=0,2,4\ldots} \frac{a_n}{\Omega^{\frac{(n+1)}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

Hence, we have:

$$I(\Omega) \sim e^{\Omega(0)} \left(\frac{a_0}{\sqrt{\Omega}} \Gamma\left(\frac{1}{2}\right) + \frac{a_2}{\Omega^{3/2}} \Gamma\left(\frac{3}{2}\right) + \ldots \right)$$

so that

$a_0 = 1,\ a_2 = -\frac{1}{2},\ a_4 = \frac{1}{24},$ etc.

$$I(\Omega) \sim \frac{1}{\sqrt{\Omega}} \Gamma\left(\frac{1}{2}\right) + \frac{-1/2}{\Omega^{3/2}} \Gamma\left(\frac{3}{2}\right) + \frac{+1/24}{\Omega^{5/2}} \Gamma\left(\frac{5}{2}\right) + \ldots$$

where

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}, \ldots$$

(Please see the notes on the Gamma function.)

Complete Asymptotic Expansion (cont.)

Hence, we have:

$$I(\Omega) = \int_{-1}^{1} \cos(z)\, e^{-\Omega z^2}\, dz \sim \sqrt{\frac{\pi}{\Omega}} - \frac{1}{4}\frac{\sqrt{\pi}}{\Omega^{3/2}} + \frac{1}{32}\frac{\sqrt{\pi}}{\Omega^{5/2}} + \ldots$$

as $\Omega \to \infty$

Note:
In a similar manner, we could (in principle) obtain higher-order terms in the asymptotic expansion of the Bessel function or the Gamma function.