ECE 6382

Fall 2023

David R. Jackson

George N. Watson

Notes 16

Watson's Lemma

Watson's Lemma

- Watson's Lemma gives us a way to obtain the complete asymptotic expansion of integrals on the real axis that have an exponential type of kernel.

$$I(\Omega) = \int_0^\infty f(s)\, e^{-\Omega s}\, ds$$

Note:
As Ω gets large, the main contribution to the integral comes from the region near $s = 0$.

Note: $f(s)$ does not have to be analytic at $s = 0$.

More generally:

$$I(\Omega) = \int_0^\infty f(s)\, e^{-\Omega s^n}\, ds$$

This form can be converted to the form above – please see extension given later.

N. Bleistein and R. Handlesman, *Asymptotic Expansions of Integrals*, Holt, Rinehart, and Winston, 1975 (reprinted by Dover, 2010).

Watson's Lemma (cont.)

$$I(\Omega) = \int_0^{\infty} f(s)\, e^{-\Omega s} ds$$

Assumption:
$f(s) = O(e^{as})$
as $s \to \infty$
(for some a)

Assume:

$$f(s) \sim \sum_{n=0}^{\infty} a_n s^{\alpha_n} \quad \text{as} \quad s \to 0$$

Note:
This may or may not be a converging series.

Note: The numbers α_n are real numbers (not necessarily integers). Branch cuts are allowed!

This assumption means:

$$f(s) - \sum_{n=0}^{N} a_n s^{\alpha_n} = o\left(s^{\alpha_N}\right) \quad \text{as} \quad s \to 0 \quad (\text{for all } N)$$

Watson's Lemma (cont.)

Statement of Watson's Lemma:

$$I(\Omega) = \int_0^\infty f(s)\, e^{-\Omega s}\, ds$$

If

$$f(s) \sim \sum_{n=0}^{\infty} a_n s^{\alpha_n} \quad \text{as} \quad s \to 0$$

Then

$$I(\Omega) \sim \sum_{n=0}^{\infty} a_n \left(\int_0^\infty s^{\alpha_n} e^{-\Omega s}\, ds \right) \qquad \text{as} \quad \Omega \to \infty$$

In words: We can plug in the complete asymptotic series for $f(s)$ into the integral and integrate term-by-term to generate the complete asymptotic expansion of the integral.

Please see the Appendix for a proof of Watson's Lemma.

Watson's Lemma (cont.)

$$I(\Omega) = \int_0^{\infty} f(s)\, e^{-\Omega s}\, ds$$

We then have:

$$I(\Omega) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha_n + 1)}{\Omega^{\alpha_n + 1}} \quad \text{as} \quad \Omega \to \infty$$

if

$$f(s) \sim \sum_{n=0}^{\infty} a_n s^{\alpha_n} \quad \text{as} \quad s \to 0$$

Use $t = \Omega s,\ dt = \Omega\, ds$

$$\int_0^{\infty} e^{-\Omega s} s^{\alpha_n} ds = \int_0^{\infty} e^{-t} \left(\frac{t}{\Omega}\right)^{\alpha_n} \frac{1}{\Omega} dt$$

$$= \frac{1}{\Omega^{\alpha_n+1}} \int_0^{\infty} e^{-t} t^{\alpha_n} dt$$

$$= \frac{1}{\Omega^{\alpha_n+1}} \int_0^{\infty} e^{-t} t^{(\alpha_n+1)-1} dt$$

$$= \frac{1}{\Omega^{\alpha_n+1}} \Gamma(\alpha_n + 1)$$

Watson's Lemma (cont.)

Summary

$$I(\Omega) = \int_0^\infty f(s)\, e^{-\Omega s}\, ds$$

$$f(s) \sim \sum_{n=0}^{\infty} a_n s^{\alpha_n} \quad \text{as} \quad s \to 0$$

Then we have:

$$I(\Omega) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha_n + 1)}{\Omega^{\alpha_n + 1}} \quad \text{as} \quad \Omega \to \infty$$

Watson's Lemma (cont.)

Extension:

$$I(\Omega)=\int_0^{\infty} f(s)\,e^{-\Omega s^n}\,ds \qquad n=2,3,4\ldots$$

Use: $\quad t=s^n,\quad dt=ns^{n-1}ds$

$$\Rightarrow\quad I(\Omega)=\frac{1}{n}\int_0^{\infty} f(s(t))\frac{1}{t^{(n-1)/n}}e^{-\Omega t}\,dt$$

or

$$I(\Omega)=\int_0^{\infty} g(t)\,e^{-\Omega t}\,dt$$

Hence, we need only to consider the case $n=1$.

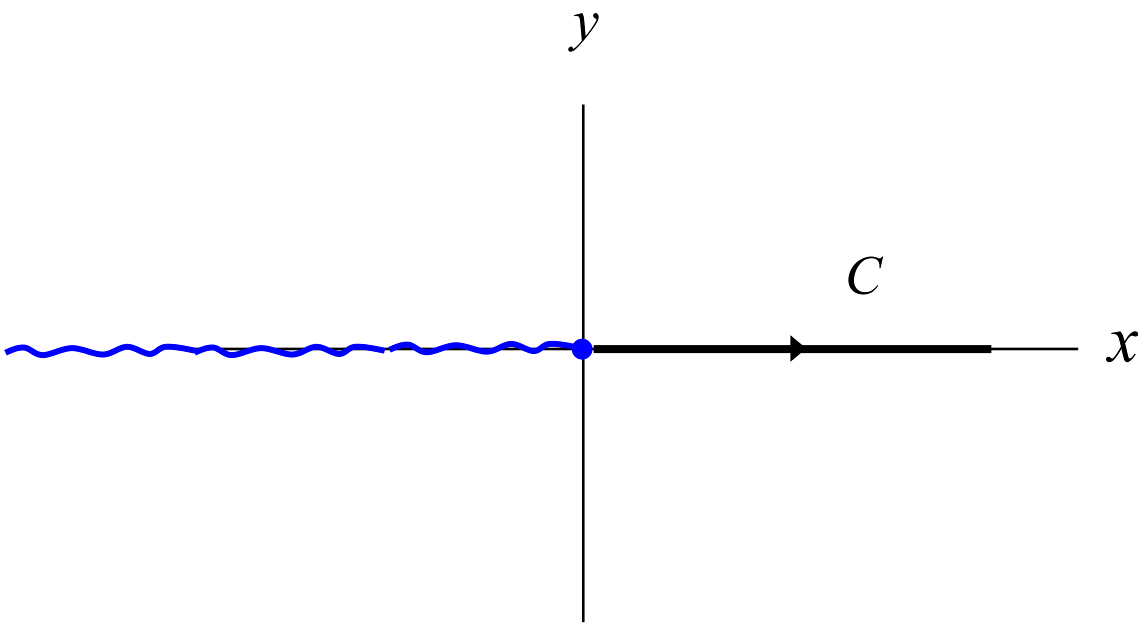
where

$$g(t)\equiv\frac{1}{n}f(s(t))\frac{1}{t^{(n-1)/n}}$$

Watson's Lemma (cont.)

Example:

$$I(\Omega) = \int_0^\infty \sin\left(\sqrt{s}\right) e^{-\Omega s^2} ds$$



Watson's Lemma (cont.)

Example (cont.)

$$I(\Omega) = \int_0^\infty \sin\left(\sqrt{s}\right) e^{-\Omega s^2} ds \qquad t = s^2 \;\; (n = 2)$$

We first convert the integral into the "standard form" using $n = 2$ in our previous formula:

$$I(\Omega) = \int_0^\infty f(s) e^{-\Omega s^2} ds \quad \Rightarrow \quad I(\Omega) = \int_0^\infty g(t) e^{-\Omega t} dt$$

where

$$g(t) \equiv \frac{1}{n} f(s(t)) \frac{1}{t^{(n-1)/n}}$$

$$\Rightarrow g(t) = \frac{1}{2} \sin\left(\sqrt{t^{1/2}}\right) \frac{1}{t^{1/2}}$$

Watson's Lemma (cont.)

Example (cont.)

We then have the "standard" form (with $t \to s,\ g \to f$)

$$I(\Omega) = \int_0^\infty f(s)\, e^{-\Omega s}\, ds$$

$$f(s) = \frac{1}{2}\frac{1}{s^{1/2}}\sin\left(s^{1/4}\right)$$

Note: $\sin(z) = \sum_{\substack{n=0 \\ \text{odd}}}^{\infty} \frac{(-1)^{(n-1)/2}}{n!} z^n$

so

$$f(s) = \frac{1}{2}\frac{1}{s^{1/2}} \sum_{\substack{n=0 \\ \text{odd}}}^{\infty} \frac{(-1)^{(n-1)/2}}{n!} \left(s^{1/4}\right)^n$$

$$\Rightarrow \quad f(s) \sim \sum_{n=0}^{\infty} a_n s^{\alpha_n}$$

$$a_n = \frac{1}{2}\frac{(-1)^{(n-1)/2}}{n!} \quad \& \quad \alpha_n = \frac{n}{4} - \frac{1}{2} \qquad (n \text{ odd})$$

Watson's Lemma (cont.)

Example (cont.)

So we now have:

$$I(\Omega) = \int_0^{\infty} f(s)\, e^{-\Omega s}\, ds$$

where

$$f(s) \sim \sum_{n=0}^{\infty} a_n s^{\alpha_n}$$

$$a_n = \frac{1}{2}\frac{(-1)^{(n-1)/2}}{n!}$$

$$\alpha_n = \frac{n}{4} - \frac{1}{2}$$

$n =$ odd

We then plug into:

$$I(\Omega) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha_n + 1)}{\Omega^{\alpha_n + 1}} \quad \text{as } \Omega \to \infty$$

Watson's Lemma (cont.)

Example (cont.)

We then have the final result:

$$I(\Omega) = \int_0^\infty \sin\left(\sqrt{s}\right) e^{-\Omega s^2} ds$$

$$I(\Omega) \sim \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\frac{1}{\Omega^{3/4}} - \frac{1}{12}\Gamma\left(\frac{5}{4}\right)\frac{1}{\Omega^{5/4}} + \frac{1}{120}\Gamma\left(\frac{7}{4}\right)\frac{1}{\Omega^{7/4}} + \ldots$$

as $\Omega \to \infty$

Appendix

Proof of Watson's Lemma:

$$f(s) \sim \sum_{n=0}^{\infty} a_n s^{\alpha_n} \quad \Rightarrow \quad f(s) = \sum_{n=0}^{N} a_n s^{\alpha_n} + o\left(s^{\alpha_N}\right) \quad \text{as} \quad s \to 0$$

Therefore

$$I(\Omega) = \int_0^{\infty} f(s) e^{-\Omega s} ds = \int_0^{\infty} \sum_{n=0}^{N} a_n s^{\alpha_n} e^{-\Omega s} ds + \int_0^{\infty} o\left(s^{\alpha_N}\right) e^{-\Omega s} ds$$

$$= \sum_{n=0}^{N} a_n \left(\int_0^{\infty} s^{\alpha_n} e^{-\Omega s} ds \right) + \int_0^{\infty} o\left(s^{\alpha_N}\right) e^{-\Omega s} ds$$

$$= \sum_{n=0}^{N} a_n \frac{\Gamma(\alpha_n + 1)}{\Omega^{\alpha_n + 1}} + \int_0^{\infty} o\left(s^{\alpha_N}\right) e^{-\Omega s} ds$$

Appendix (cont.)

We need to show:

$$\int_0^{\infty} o\left(s^{\alpha_N}\right) e^{-\Omega s} ds = o\left(\frac{1}{\Omega^{\alpha_N+1}}\right)$$

From the definition of "small o" we have the following statement:

For <u>any</u> ε we can choose a Δ small enough so that

$$o\left(s^{\alpha_N}\right) < \varepsilon s^{\alpha_N}, \quad s < \Delta$$

$$\Rightarrow \int_0^{\infty} o\left(s^{\alpha_N}\right) e^{-\Omega s} ds < \int_0^{\Delta} \varepsilon\, s^{\alpha_N} e^{-\Omega s} ds + A\int_{\Delta}^{\infty} e^{as} e^{-\Omega s} ds \quad (\text{for some } A)$$

$$< \varepsilon \int_0^{\infty} s^{\alpha_N} e^{-\Omega s} ds + A\int_{\Delta}^{\infty} e^{-(\Omega-a)s} ds$$

$$= \varepsilon \frac{\Gamma\left(\alpha_N+1\right)}{\Omega^{\alpha_N+1}} + \frac{A}{\Omega-a} e^{-(\Omega-a)\Delta}$$

Assumption: $f(s) = O\left(e^{as}\right)$ as $s \to \infty$ (for some a)

Appendix (cont.)

For a large enough Ω, we have

$$\frac{A}{\Omega - a} e^{-(\Omega - a)\Delta} < \varepsilon \frac{\Gamma(\alpha_N + 1)}{\Omega^{\alpha_N + 1}}$$

$$\Rightarrow \quad \int_0^\infty o\left(s^{\alpha_N}\right) e^{-\Omega s} ds < 2\varepsilon \frac{\Gamma(\alpha_N + 1)}{\Omega^{\alpha_N + 1}}$$

Since this is true for any ε, we have

$$\int_0^\infty o\left(s^{\alpha_N}\right) e^{-\Omega s} ds = o\left(\frac{1}{\Omega^{\alpha_N + 1}}\right)$$