ECE 6382

$$W(x)=\begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_N \\ \phi_1' & \phi_2' & \cdots & \phi_N' \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(N-1)} & \phi_2^{(N-1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}$$

Fall 2023

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Notes 17

Linear Independence and Wronskians

Notes are from D. R. Wilton, Dept. of ECE

Linear Independence

Definition:

- A set of functions $\{\phi_n(x)\}, \; n = 1, 2, \ldots, N$ is *linearly independent* on an interval (a, b) of the real axis if the only solution to the set of homogeneous equations

$$\sum_{n=1}^{N} c_n \phi_n(x) \equiv 0$$

(identically qual to zero; i.e., equal to zero for all x in the interval)

is $c_n = 0, \; \forall n.$

- A set of functions that is not linearly independent is said to be linearly dependent.

Important point: The interval is important here. Functions can be independent in one region (interval) and dependent in another region.

Linear Independence (cont.)

Assume linear <u>dependence</u>:

$$\sum_{n=1}^{N} c_n \phi_n(x) \equiv 0 \; (\text{in a region}), \qquad c_n \not\equiv 0 \; (\text{not all coefficients are zero})$$

Assume $c_p \neq 0$

Then $$\phi_p = -\left(\frac{c_1}{c_p}\right)\phi_1 - \left(\frac{c_2}{c_p}\right)\phi_2 - \ldots - \left(\frac{c_{p-1}}{c_p}\right)\phi_{p-1} - \left(\frac{c_{p+1}}{c_p}\right)\phi_{p+1} - \ldots - \left(\frac{c_N}{c_p}\right)\phi_N$$

or $$\phi_p = a_1\phi_1 + a_2\phi_2 + \ldots a_{p-1}\phi_{p-1} + a_{p+1}\phi_{p+1} + \ldots a_N\phi_N$$

Conclusion:
If a set of functions is linearly dependent, then at least one of the functions can be written as a combination of the others.

Linear Independence (cont.)

Assume one function can be written as a <u>combination</u> of the others:

$$\phi_p = a_1\phi_1 + a_2\phi_2 + \ldots + a_p\phi_{p-1} + a_{p+1}\phi_{p+1} + \ldots + a_N\phi_N$$

(for <u>all</u> x in an interval)

Then

$$a_1\phi_1 + a_2\phi_2 + a_p\phi_{p-1} + (-1)\phi_p + a_{p+1}\phi_{p+1} + \ldots + a_N\phi_N = 0$$

$$c_p = -1 \quad \Rightarrow \quad c_n \neq 0 \ \forall n.$$

Conclusion:
If one function can be written as a combination of the others, then the functions are linearly dependent.

Linear Independence (cont.)

Linear dependence

At least one of the functions can be written as a combination of the others.

Note: When there are only two functions, linear dependence means that one function is a constant times the other one.

Linear Independence (cont.)

Examples

$(\sin x, \cos x)$ are linearly independent

$(x,\ x^2, x^3)$ are linearly independent

$(x,\ x^2, x^2+2x)$ are linearly dependent

$$x = x^2[-1/2] + (x^2+2x)[1/2]$$

$$x^2 = x[-2] + (x^2+2x)[1]$$

$$x^2+2x = x[2] + x^2[1]$$

$(x,\ x^2, 5x^2)$ are linearly dependent

$$x^2 = x[0] + 5x^2[1/5]$$

$$5x^2 = x[0] + x^2[5]$$

Note that we cannot write x as a combination of the other two functions.

Linear Independence (cont.)

Note: The classification of linear independence depends on the interval for x that is being considered.

Example :

$$f(x) \equiv \begin{cases} 1, & x \le 0 \\ \cos x, & x \ge 0 \end{cases}$$

$$g(x) \equiv \begin{cases} 2, & x \le 0 \\ x + 2, & x \ge 0 \end{cases}$$

- These two functions are linearly <u>independent</u> on any interval (a, b) that contains <u>positive</u> x values.
- These two functions are linearly <u>dependent</u> on any interval that contains <u>only negative</u> x values.

Wronskians and Linear Independence

The Wronskian allows us to test for linear independence on some interval.

$$W(x) \equiv \det\begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_N \\ \phi_1' & \phi_2' & \cdots & \phi_N' \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(N-1)} & \phi_2^{(N-1)} & \cdots & \phi_N^{(N-1)} \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_N \\ \phi_1' & \phi_2' & \cdots & \phi_N' \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(N-1)} & \phi_2^{(N-1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}$$

We look at whether or not the Wronskian vanishes identically over an interval.

Josef Hoëné-Wronski

Note: The fact that the Wronskian vanishes at some points does not tell us anything useful.

Wronskians and Linear Independence (cont.)

Assume *linear dependence*:

For some value of p: $\phi_p = a_1\phi_1 + a_2\phi_2 + \ldots a_{p-1}\phi_{p-1} + a_{p+1}\phi_{p+1} + \ldots a_N\phi_N$ (for <u>all</u> x in an interval)

Then, taking up to N - 1 derivatives, we have:

$$\phi_p = a_1\phi_1 + a_2\phi_2 + \ldots a_{p-1}\phi_{p-1} + a_{p+1}\phi_{p+1} + \ldots a_N\phi_N$$
$$\phi_p' = a_1\phi_1' + a_2\phi_2' + \ldots a_{p-1}\phi_{p-1}' + a_{p+1}\phi_{p+1}' + \ldots a_N\phi_N'$$
$$\phi_p'' = a_1\phi_1'' + a_2\phi_2'' + \ldots a_{p-1}\phi_{p-1}'' + a_{p+1}\phi_{p+1}'' + \ldots a_N\phi''$$
$$\vdots$$
$$\phi_p^{(N-1)} = a_1\phi_1^{(N-1)} + a_2\phi_2^{(N-1)} + \ldots a_{p-1}\phi_{p-1}^{(N-1)} + a_{p+1}\phi_{p+1}^{(N-1)} + \ldots a_N\phi_N^{(N-1)}$$

The pth column of the Wronskian matrix is thus a combination of the other columns, so the determiant is zero (from linear algrebra).

$$\left[\underline{W}_p(x)\right] = a_1\left[\underline{W}_1(x)\right] + a_2\left[\underline{W}_2(x)\right] + \ldots \quad \Rightarrow \quad W(x) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_N \\ \phi_1' & \phi_2' & \cdots & \phi_N' \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(N-1)} & \phi_2^{(N-1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix} = 0$$

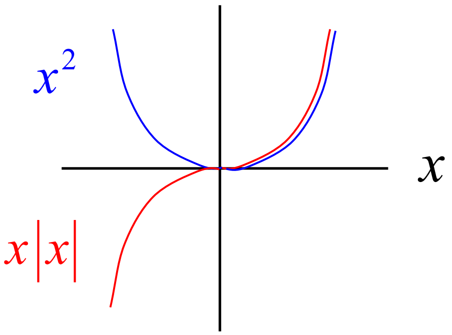
↑ pth column vector

Hence **Linear dependence $\Rightarrow W(x) \equiv 0$**

Wronskians and Linear Independence (cont.)

- Interestingly,

$$W(x) \equiv 0 \quad \not\Rightarrow \quad \text{Linear dependence}$$



Example (Peano):

$$\left(x^2,\ x|x|\right)$$

$W(x) \equiv 0$, but we have linear independence on any interval (a,b) that includes the origin!

Giuseppe Peano

- However, if the functions are *analytic* (with $x \to z$), then

$$W(x) \equiv 0 \quad \Rightarrow \quad \text{linear dependence}$$

(proof by Peano).

Note : $W(0) \equiv 0 \ \left(\phi_1(0) = \phi_2(0) = 0,\ \phi_1'(0) = \phi_2'(0) = 0\ \right)$

Wronskians and Linear Independence (cont.)

Summary

For arbitrary functions:

$$\text{Linear dependence} \quad \Rightarrow \quad W(x) \equiv 0$$

For analytic functions:

$$\text{Linear dependence} \quad \Leftrightarrow \quad W(x) \equiv 0$$

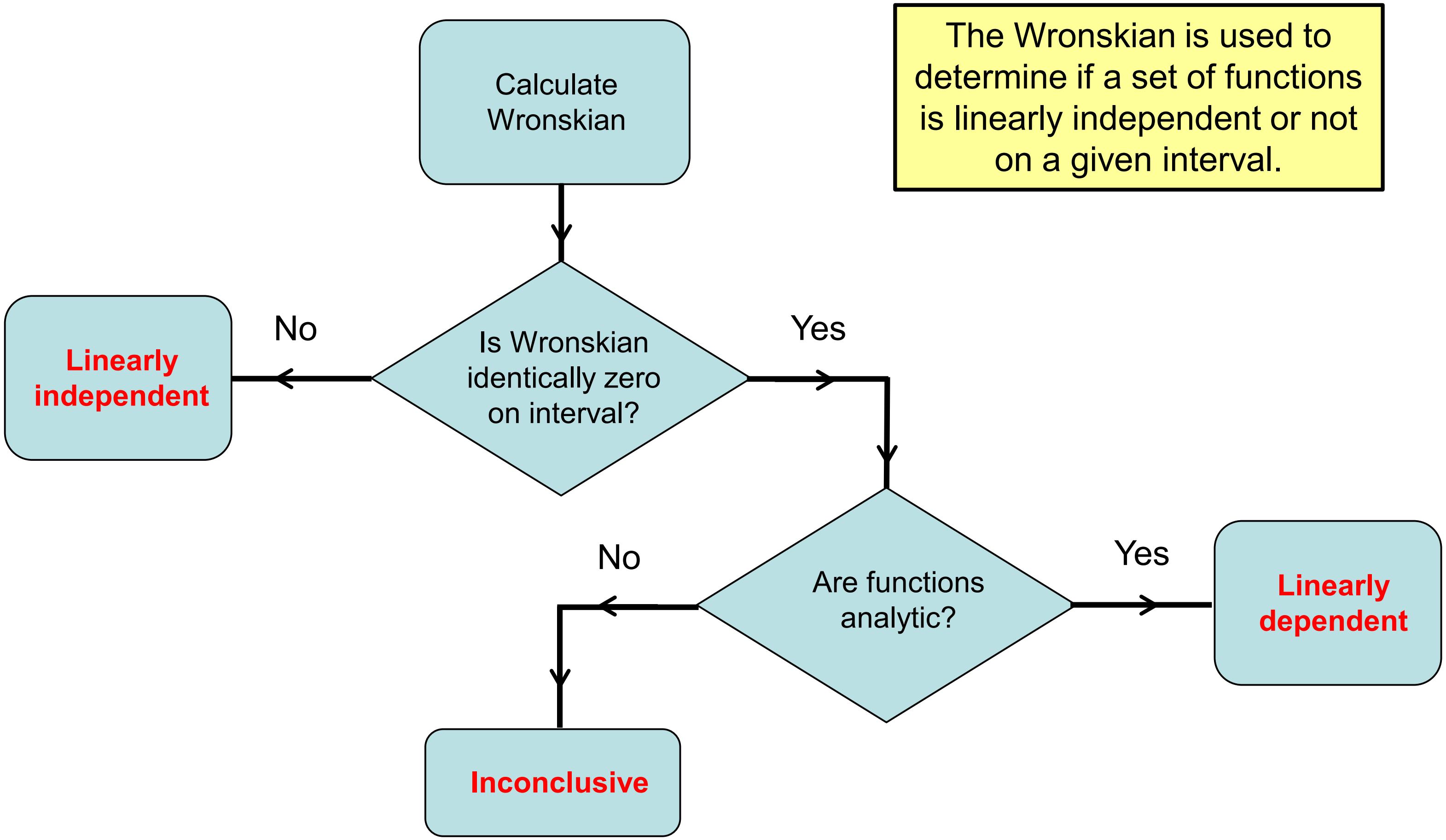
Note : For arbitraary functions : $W(x) \not\equiv 0 \quad \Rightarrow$ Linear independence

(This is the converse of the first statement.)

Wronskians and Linear Independence (cont.)

Wronskian Flowchart

The Wronskian is used to determine if a set of functions is linearly independent or not on a given interval.



Wronskians and Linear Independence (cont.)

Note on Analytic Functions

$$\text{Linear dependence} \quad \Leftrightarrow \quad W(x) \equiv 0$$

This is logically equivalent to:

$$\text{Linear independence} \quad \Leftrightarrow \quad W(x) \not\equiv 0$$

Hence, for analytic functions we have:

$$W(x) \not\equiv 0 \quad \Leftrightarrow \quad \text{independence}$$

$$W(x) \equiv 0 \quad \Leftrightarrow \quad \text{dependence}$$

Wronskians and Linear Independence (cont.)

Example

Show that $\phi_1(x) = e^{ikx},\ \phi_2(x) = e^{-ikx},\ \phi_3(x) = \sin kx$ are *linearly dependent*.

Note: The three functions are analytic.

Note: (a,b) is arbitrary here.

$$\begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{vmatrix} = \begin{vmatrix} e^{ikx} & e^{-ikx} & \sin kx \\ ike^{ikx} & -ike^{-ikx} & k\cos kx \\ -k^2e^{ikx} & -k^2e^{-ikx} & -k^2\sin kx \end{vmatrix}$$

Obervation : Last row is $\left(-k^2\right) \times$ first row

⇨ The determinant is identically zero.

⇨ Linear dependence

Note:
The three functions are analytic, so $W \equiv 0$ is equivalent to linear dependence.

Wronskians and Linear Independence (cont.)

Example

Show that any *pair* of these four functions are *linearly independent* :

$$\phi_1(x) = e^{ikx}, \ \phi_2(x) = e^{-ikx}, \ \phi_3(x) = \sin kx, \ \phi_4(x) = \cos kx$$

Note: (a,b) is arbitrary here.

Illustrate using $\phi_1, \ \phi_3$:

$$\begin{vmatrix} \phi_1 & \phi_3 \\ \phi_1' & \phi_3' \end{vmatrix} = \begin{vmatrix} e^{ikx} & \sin kx \\ ike^{ikx} & k\cos kx \end{vmatrix} = k\, e^{ikx} \cos kx - ike^{ikx} \sin kx$$

$$\Rightarrow \begin{vmatrix} \phi_1 & \phi_3 \\ \phi_1' & \phi_3' \end{vmatrix} = k\, e^{ikx} \left(\cos kx - i \sin kx \right) = k\, e^{ikx} \left(e^{-ikx} \right) = k$$

$$\Rightarrow \begin{vmatrix} \phi_1 & \phi_3 \\ \phi_1' & \phi_3' \end{vmatrix} \not\equiv 0 \ (\text{anywhere})$$ $\Rightarrow$ Linear independence on any interval

Note: These functions are analytic, but here the conclusion does not depend on this.

Wronskians for SOLDE*s

- Assume $y_1(x),\ y_2(x)$ are two solutions of

$$\mathcal{L}y = y'' + p(x)y' + q(x)y = 0$$

*SOLDE: Second-Order Linear Differential Equation

We have :

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W' = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1''$$

$$= y_1\left(-p y_2' - \cancel{q y_2}\right) - y_2\left(-p y_1' - \cancel{q y_1}\right)$$

$$= -p\left(y_1 y_2' - y_2 y_1'\right) = -pW$$

$$\Rightarrow \quad \frac{W'}{W} = -p(x)$$

Also, we have

$$\frac{W'}{W} = \frac{d}{dx}\left(\ln W(x)\right) \quad \left(\text{from the chain rule}\right)$$

Hence

$$\frac{d}{dx}\left(\ln W(x)\right) = -p(x)$$

Wronskians for SOLDEs (cont.)

From the last slide: $\frac{d}{dx}\left(\ln W(x)\right) = -p(x)$

Integrate from $x = a$ to $x = x$ (a is arbitrary):

$$\ln W(x) - \ln W(a) = \ln\left(\frac{W(x)}{W(a)}\right) = -\int_a^x p(x)dx$$

$$\Rightarrow W(x) = W(a) \underbrace{e^{-\int_a^x p(x)dx}}_{>0}$$

- The Wronskian either vanishes for *all* x ($W(a) = 0$) or for *no* x ($W(a) \neq 0$)!

Recall: If the functions are analytic:

$W(x) \not\equiv 0 \Leftrightarrow$ independence

$W(x) \equiv 0 \Leftrightarrow$ dependence

Conclusion:

If the two solutions to a SOLDE are analytic, then they must be either linearly dependent or linearly independent everywhere on the x axis.

Wronskians for SOLDEs (cont.)

Example:

$$y''(x) + k^2 y(x) = 0$$

General solution:

$$y(x) = A\sin(kx) + B\cos(kx)$$ (analytic functions)

$$\phi_1(x) = \sin(kx)$$
$$\phi_2(x) = \cos(kx)$$

Linearly independent <u>everywhere</u>

$$\phi_1(x) = 3\sin(kx)$$
$$\phi_2(x) = 10\sin(kx)$$

Linearly dependent <u>everywhere</u>