$$\mathcal{L}u = \lambda u$$

ECE 6382

Fall 2023

David R. Jackson

Notes 18

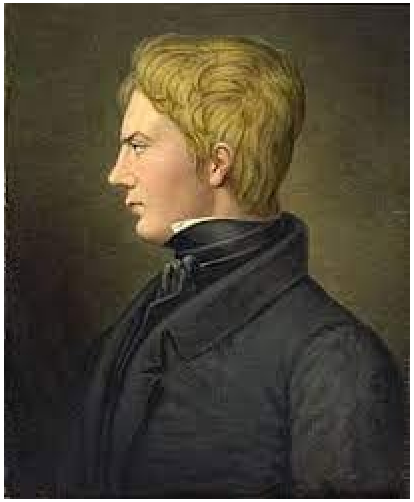
Sturm-Liouville Theory

Notes are from D. R. Wilton, Dept. of ECE

Sturm-Liouville Theory

- We first illustrate Sturm-Liouville theory for solutions to second-order differential equations.

- We then apply the theory to matrices (linear algebra).



Jacques Charles François Sturm

Joseph Liouville

Second-Order Linear Differential Equations (SOLDE)

- A SOLDE has the form

$$p_0(x)\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_2(x)y=f(x)$$

- If $f(x)=0$, the equation is said to be "homogeneous".

- The inhomogeneous equation can be solved once we know the solution to the homogeneous equation using the method of Green's functions (discussed later).

- Boundary conditions (BC) are usually of the form

$$y(a)=y(b)=0 \quad \text{(Dirichlet)}$$

$$y'(a)=y'(b)=0 \quad \text{(Neumann)}$$

Sturm-Liouville Form

- If we multiply the general differential equation

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = f(x)$$

by the integrating factor $w(x) = -\dfrac{e^{\int^x \frac{p_1(t)}{p_0(t)}dt}}{p_0(x)}$ we have :

$$-e^{\int^x \frac{p_1(t)}{p_0(t)}dt} y'' - \frac{p_1(x)}{p_0(x)} e^{\int^x \frac{p_1(t)}{p_0(t)}dt} y' + p_2(x)w(x)y = w(x)f(x)$$

$$\Rightarrow -\frac{d}{dx}\left(e^{\int^x \frac{p_1(t)}{p_0(t)}dt} y' \right) + p_2(x)w(x)y = w(x)f(x)$$

- Dividing this result by $w(x)$ yields

$$-\frac{1}{w(x)}\frac{d}{dx}\left[P(x)\frac{dy(x)}{dx}\right] + Q(x)y(x) = f(x)$$

where $\quad P(x) \equiv e^{\int^x \frac{p_1(t)}{p_0(t)}dt}, \; Q(x) \equiv p_2(x)$

Sturm-Liouville Operator

This is called the Sturm-Liouville or self-adjoint form of the differential equation:

$$-\frac{1}{w(x)}\frac{d}{dx}\left[P(x)\frac{dy(x)}{dx}\right]+Q(x)y(x)=f(x)$$

or (using u instead of y):

$$\mathcal{L}u = f$$

Where $\mathcal{L}$ is the (self-adjoint*) "Sturm-Liouville" operator:

$$\mathcal{L} \equiv -\frac{1}{w(x)}\frac{d}{dx}\left[P(x)\frac{d}{dx}\right]+Q(x)$$

* Discussed later

Note:
The operator $\mathcal{L}$ is assumed to be real here (w, P, Q are real). The solution u does not have to be real (because f is allowed to be complex).

Inner Product Definition

An <u>inner product</u> between two functions is defined:

- We define an *inner product* as

$$< u, v > \equiv \int_a^b u(x)\, v^*(x)\, w(x)\, dx$$

where $w(x)$ is called a *weight* function.

- Although the weight function is arbitrary, we will choose it to be the same as the integrating function $w(x)$ in the Sturm-Liouville equation. This will give us the nice "self-adjoint" properties, as we will see.

The Adjoint Problem

The adjoint operator $\mathcal{L}^{\dagger}$ is defined from

$$< \mathcal{L}u, v > \; = \; < u, \mathcal{L}^{\dagger} v >$$

For the Sturm-Liouville operator $< \mathcal{L}u, v > \; = \; < u, \mathcal{L}v >$ so $\mathcal{L}^{\dagger} = \mathcal{L}.$

(proof given next)

Hence, the Sturm-Liouville operator is said to be self-adjoint:

$$\mathcal{L} = \mathcal{L}^{\dagger} = -\frac{1}{w(x)} \frac{d}{dx}\left[P(x) \frac{d}{dx} \right] + Q(x)$$

Note:
Self-adjoint operators have nice properties for eigenvalue problems, which is discussed a little later.

Proof of Self-Adjoint Property

- Consider the inner product between the two functions $\mathcal{L}u$ and v:

$$< \mathcal{L}u, v > = \int_a^b v^*(x)\, w(x) \mathcal{L}u(x) dx$$

Recall: $< u, v > \equiv \int_a^b u(x)\, v^*(x)\, w(x)\, dx$

$$= \int_a^b v^*(x) w(x) \left[-\frac{1}{w(x)} \frac{d}{dx}\left(P(x) \frac{d}{dx} \right) + Q(x) \right] u(x) dx$$

$$= \int_a^b v^*(x) \left[-\frac{d}{dx}\left(P(x) \frac{d}{dx} \right) + Q(x) w(x) \right] u(x) dx$$

The first term inside the square brackets is first integrated by parts, twice:

$$-\int_a^b v^*(x) \frac{d}{dx}\left[P(x) \frac{du(x)}{dx} \right] dx = -\left[v^*(x) P(x) \frac{du(x)}{dx} \right]\Bigg|_a^b + \int_a^b \left(P(x) \frac{du(x)}{dx} \right) \frac{dv^*(x)}{dx} dx$$

$$= -\left[v^*(x) P(x) \frac{du(x)}{dx} \right]\Bigg|_a^b + \int_a^b \left(P(x) \frac{dv^*(x)}{dx} \right) \frac{du(x)}{dx} dx$$

$$= \left[-v^*(x) P(x) \frac{du(x)}{dx} + P(x) \frac{dv^*(x)}{dx} u(x) \right]\Bigg|_a^b - \int_a^b u(x) \frac{d}{dx}\left[P(x) \frac{dv^*(x)}{dx} \right] dx$$

Proof of Self-Adjoint Property (cont.)

Hence, we have:

$$< \mathcal{L}u, v > \; = \left[-v^*(x)P(x)\frac{du(x)}{dx} + P(x)\frac{dv^*(x)}{dx}u(x) \right]\Bigg|_a^b - \int_a^b u(x)\frac{d}{dx}\left[P(x)\frac{dv^*(x)}{dx} \right] dx$$

$$+ \int_a^b v^*(x)Q(x)u(x)w(x)\,dx$$

Multiply and divide by $w(x)$, combine with last term.

or

$$< \mathcal{L}u, v > \; = \left[-v^*(x)P(x)\frac{du(x)}{dx} + P(x)\frac{dv^*(x)}{dx}u(x) \right]\Bigg|_a^b$$

$$+ \int_a^b u(x)\,w(x)\left[-\frac{1}{w(x)}\frac{d}{dx}\left[P(x)\frac{d}{dx} \right] + Q(x) \right] v^*(x)dx$$

Proof of Self-Adjoint Property (cont.)

$$< \mathcal{L}u, v > \; = \left[-v^*(x) P(x) \frac{du(x)}{dx} + P(x) \frac{dv^*(x)}{dx} u(x) \right] \Bigg|_a^b$$

$$+ \int_a^b u(x) \left[-\frac{1}{w(x)} \frac{d}{dx} \left[P(x) \frac{d}{dx} \right] + Q(x) \right] v^*(x) \; w(x) dx$$

This can thus be written as:

$$< \mathcal{L}u, v > \; = \; J(u,v) \Big|_a^b \; + \; < u, \mathcal{L}v >$$

Note :

$\mathcal{L}v^* = (\mathcal{L}v)^*$

$(\mathcal{L} = \text{real operator})$

where

$$J(u,v) \equiv P(x) \left(-v^*(x) \frac{du(x)}{dx} + \frac{dv^*(x)}{dx} u(x) \right)$$

From boundary conditions we have: $J(u,v) \Big|_a^b = 0$

$y(a) = y(b) = 0$ (Dirichlet)

$y'(a) = y'(b) = 0$ (Neumann)

$$\Rightarrow \; < \mathcal{L}u, v > = \int_a^b u(x) \left(\mathcal{L}v(x) \right)^* \; w(x) dx = < u, \mathcal{L}v >$$

(proof complete)

Eigenvalue Problems

We often encounter an eigenvalue problem of the form

$$\mathcal{L}u = \lambda u$$

(The operator $\mathcal{L}$ can be the Sturm-Liouville operator, or any other operator here.)

- The eigenvalue problem (with boundary conditions) is usually only satisfied for specific eigenvalues:

$$\lambda = \lambda_n, \quad n = 1, 2, \ldots$$

$u(a) = u(b) = 0$ (Dirichlet)

$u'(a) = u'(b) = 0$ (Neumann)

- For each distinct eigenvalue, there corresponds an eigenfunction $u = u_n$ that satisfies the eigenvalue equation.

Property of Eigenvalues

Property 1

The eigenvalues corresponding to a self-adjoint operator are real.

Proof:

$$\mathcal{L}u = \lambda u$$

$$\Rightarrow \int_a^b (\mathcal{L}u)\, u^*\, w\, dx = \lambda \int_a^b u\, u^*\, w\, dx$$

$$\Rightarrow \langle \mathcal{L}u, u \rangle = \lambda \langle u, u \rangle$$

$$\Rightarrow \langle u, \mathcal{L}^\dagger u \rangle = \lambda \langle u, u \rangle$$

$$\Rightarrow \langle u, \mathcal{L}u \rangle = \lambda \langle u, u \rangle$$

$$\mathcal{L}u = \lambda u$$

$$\Rightarrow (\mathcal{L}u)^* = \lambda^* u^*$$

$$\Rightarrow \int_a^b (\mathcal{L}u)^*\, u\, w\, dx = \lambda^* \int_a^b u^* u\, w\, dx$$

$$\Rightarrow \langle u, \mathcal{L}u \rangle = \lambda^* \langle u, u \rangle$$

Hence: $\lambda = \lambda^*$

(proof complete)

Orthogonality of Eigenfunctions

Property 2

The eigenfunctions corresponding to a self-adjoint operator equation are orthogonal* if the eigenvalues are distinct.

*Orthogonal means that the inner product is zero.

Consider two different solutions of the eigenvalue problem corresponding to distinct eigenvalues:

$$\mathcal{L}u_m = \lambda_m u_m \qquad \mathcal{L}u_n = \lambda_n u_n$$

$$\lambda_n \neq \lambda_m$$

$$\int_a^b \mathcal{L}u_m \, u_n^* \, w \, dx = \lambda_m \int_a^b u_m u_n^* \, w \, dx$$

$$\int_a^b (\mathcal{L}u_n)^* \, u_m \, w \, dx = \lambda_n^* \int_a^b u_n^* u_m w \, dx$$

Subtract

$$\int_a^b \left((\mathcal{L}u_m) u_n^* - u_m (\mathcal{L}u_n)^* \right) w \, dx = \left(\lambda_m - \lambda_n^* \right) \int_a^b u_m u_n^* \, w \, dx$$

Orthogonality of Eigenfunctions (cont.)

The LHS is:

$$\int_a^b \left((\mathcal{L}u_m) u_n^* - u_m (\mathcal{L}u_n)^* \right) w\,dx = \; <\mathcal{L}u_m, u_n> - <u_m, \mathcal{L}u_n>$$

$$= <u_m, \mathcal{L}^\dagger u_n> - <u_m, \mathcal{L}u_n> \quad (\text{from the definition of adjoint})$$

$$= \; <u_m, \mathcal{L}u_n> - <u_m, \mathcal{L}u_n> \; (\text{from the self - adjoint property})$$

$$= 0$$

Hence, for the RHS we have

$$\left(\lambda_m - \lambda_n^*\right) \int_a^b u_m u_n^* \, w\,dx = 0 \quad \Rightarrow \quad \int_a^b u_m u_n^* \, w\,dx = 0 \;\; (\text{orthogonality})$$

since $\lambda_n^* = \lambda_n, \quad \lambda_m \neq \lambda_n$

(proof complete)

Summary of Eigenvalue Properties

Assume an eigenvalue problem with a self-adjoint operator:

$$\mathcal{L}u = \lambda u, \quad \mathcal{L} = \mathcal{L}^{\dagger}$$

$$\downarrow$$

$$<\mathcal{L}u, v> \; = \; <u, \mathcal{L}v>$$

Then we have:

- ❖ The eigenvalues are real.
- ❖ The eigenfunctions corresponding to distinct eigenvalues are orthogonal.

That is,

a) $\lambda_m^* = \lambda_m$

b) $\langle u_m, u_n \rangle = \int_a^b u_m u_n^* \, w \, dx = 0, \quad \lambda_m \neq \lambda_n$

Example

Orthogonality of Bessel functions

Derive this identity:

$$\int_0^1 J_\nu(p_{\nu m}x)J_\nu(p_{\nu n}x)\,x\,dx = 0, \qquad m \neq n$$

$$p_{\nu m} = m^{th} \text{ root of } J_\nu(x)$$

$$p_{\nu n} = n^{th} \text{ root of } J_\nu(x)$$

Example (cont.)

Recall:

$$\int_a^b u_m(x)\,u_n^*(x)\,w(x)\,dx = 0 \quad \text{(for a Sturm - Liouville problem)}$$

if $u_{m,n}(a) = u_{m,n}(b) = 0$ (Dirchlet boundary conditions)

Consider: $u_m(x) = J_\nu(p_{\nu m} x), \quad u_n(x) = J_\nu(p_{\nu n} x)$

Choose: $a = 0,\ b = 1$

$$J_\nu(p_{\nu i} 0) = 0, \quad J_\nu(p_{\nu i} 1) = 0 \qquad (i = m, n)$$

(The first equation is true for $\nu \neq 0$. The case $\nu = 0$ can be considered as a limiting case.)

$$\Rightarrow \quad \int_0^1 J_\nu(p_{\nu m} x)\,J_\nu(p_{\nu n} x)\,w(x)\,dx = 0, \qquad m \neq n$$

What is $w(x)$?

(This will be true if if u_m and u_n correspond to a Sturm-Liouville eigenvalue problem.)

Example (cont.)

$$\int_0^1 u_m(x)\,u_n(x)\,w(x)\,dx = 0, \qquad m \neq n$$

$$\left(u_m(x) = J_\nu(p_{\nu m} x), \quad u_n(x) = J_\nu(p_{\nu n} x) \right)$$

This orthogonality will be true if if u_m and u_n correspond to a Sturm-Liouville eigenvalue problem, with the associated function $w(x)$.

What is $w(x)$?

We need to identify the appropriate Sturm-Liouville eigenvalue problem that $u(x)$ satisfies:

$$-\frac{1}{w(x)}\frac{d}{dx}\left[P(x)\frac{du(x)}{dx}\right] + Q(x)u(x) = \lambda u$$

Example (cont.)

Bessel equation: $t^2 y'' + t y' + \left(t^2 - \nu^2\right) y = 0, \quad y(t) = J_\nu(t)$

Use:

$$t = p_{\nu i} x, \quad dt = p_{\nu i}\, dx$$

$$\Rightarrow \frac{\partial}{\partial t} = \frac{1}{p_{\nu i}} \frac{\partial}{\partial x}$$

$y(t)$ is then denoted as $u(x)$

$\left(u(x) = u_i(x) = J_\nu\left(p_{\nu i} x\right),\ i = 1,2 \right)$

⇩

$$x^2 u'' + x u' + \left(p_{\nu i}^2 x^2 - \nu^2\right) u = 0$$

Note:
The primes now mean differentiating with respect to x.

⇩

$$u'' + \frac{1}{x} u + \left(p_{\nu i}^2 - \frac{\nu^2}{x^2} \right) u = 0$$

Example (cont.)

Rearrange to put into Sturm-Liouville eigenvalue form:

$$u'' + \frac{1}{x}u + \left(p_{\nu i}^2 - \frac{\nu^2}{x^2} \right) u = 0$$

$$\Downarrow$$

$$-u'' - \frac{1}{x}u' + \left(\frac{\nu^2}{x^2} \right) u = p_{\nu i}^2 u$$

$$\Downarrow$$

$$-u'' - \frac{1}{x}u' + \left(\frac{\nu^2}{x^2} \right) u = \lambda u, \qquad u(x) \equiv J_\nu (p_{\nu i} x), \qquad \lambda \equiv p_{\nu i}^2$$

$$\Downarrow$$

$$\left(-\frac{1}{x}\frac{d}{dx}\left[x\frac{d}{dx} \right] + \frac{\nu^2}{x^2} \right) u = \lambda u$$

Example (cont.)

Hence, we have:

$$\left(-\frac{1}{x}\frac{d}{dx}\left[x\frac{d}{dx}\right]+\frac{\nu^2}{x^2}\right)u=\lambda u, \qquad u\equiv J_\nu\left(p_{\nu i}x\right), \qquad \lambda\equiv p_{\nu i}^2$$

Compare with our standard Sturm-Liouville eigenvalue form:

$$\left(-\frac{1}{w(x)}\frac{d}{dx}\left[P(x)\frac{d}{dx}\right]+Q(x)\right)u=\lambda u$$

We can now see that the $u(x)$ functions come from a Sturm-Liouville problem, and we can identify:

$$w\left(x\right)=x, \quad P\left(x\right)=x, \quad Q\left(x\right)=\frac{\nu^2}{x^2}$$

Example (cont.)

Hence, we have:

$$\int_a^b u_m(x)\,u_n^*(x)\,w(x)\,dx = 0$$

$$\Downarrow$$

$$\int_0^1 J_\nu\left(p_{\nu m}x\right)J_\nu\left(p_{\nu n}x\right)x\,dx = 0, \qquad m \neq n$$

where $J_\nu\left(p_{\nu i}\right) = 0 \quad \left(p_{\nu i} = i^{th}\text{ root of Bessel function } J_\nu\right)$

Adjoint in Linear Algebra

An inner product between two vectors is defined as:

$$\langle a, b \rangle \equiv \underline{a} \cdot \underline{b}^* = \sum_i a_i b_i^*$$ (There is no "weight" function here.)

An adjoint of a square complex matrix [A] is defined from:

$$\langle Au, v \rangle = \langle u, A^\dagger v \rangle$$

Self-adjoint means:

$$A = A^\dagger$$

Adjoint in Linear Algebra (cont.)

Theorem For a complex square matrix $[A]$, the adjoint is given by

$$\left[A^{\dagger}\right]=\left[A^{t*}\right]$$ (i.e., the conjugate of the transpose)

Proof :

Note : $\left[A^{H}\right]\equiv\left[A^{t*}\right]$ (the Hermetian transpose)

To show this, we need to show :

$$\langle Au,v\rangle=\langle u,A^{t*}v\rangle$$

where $\langle a,b\rangle=\underline{a}\cdot\underline{b}^{*}=\sum_i a_i b_i^{*}$

To show this :

$$\langle Au,v\rangle=\sum_i\sum_j\left(A_{ij}u_j\right)v_i^{*}=\sum_i\sum_j u_j A_{ij}v_i^{*}$$

$$\langle u,A^{t*}v\rangle=\sum_i\sum_j u_i\left(A_{ij}^{t}v_j^{*}\right)=\sum_i\sum_j u_i A_{ji}v_j^{*}=\sum_i\sum_j u_j A_{ij}v_i^{*}$$ (relabeling i and j)

Adjoint in Linear Algebra (cont.)

For a complex matrix we have established that

$$\left[A^{\dagger}\right]=\left[A^{t*}\right]$$

Therefore, if a complex matrix is <u>self-adjoint</u>, this means that:

$$[A]=\left[A^{t*}\right]=\left[A^{H}\right]$$

(The matrix is then also called Hermetian.)

Note: For a <u>real</u> matrix, self-adjoint means that the matrix is <u>symmetric</u>.

Orthogonality in Linear Algebra

Because a Hermetian matrix is self-adjoint, we have the following properties:

- ❖ The eigenvalues of a Hermetian matrix are real.
- ❖ The eigenvectors of a Hermetian matrix corresponding to distinct eigenvalues are orthogonal.
- ❖ The eigenvectors of a Hermetian matrix corresponding to the same eigenvalue can be chosen to be orthogonal (proof omitted).

For a real matrix, these properties apply to a symmetric matrix.

Diagonalizing a Matrix

If the eigenvectors of an $N \times N$ matrix $[A]$ are linearly independent, the matrix can be "diagonalized" as follows:

$$[A] = [e][D][e]^{-1}$$

(proof on next slide)

$$[D] = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_N \end{bmatrix}$$

$$[e] = \big[[e_1] \quad [e_2] \quad [\cdots] \quad [e_N] \big] = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1N} \\ e_{21} & e_{22} & \cdots & e_{2N} \\ e_{31} & e_{32} & \cdots & e_{3N} \\ e_{41} & e_{42} & \cdots & e_{4N} \end{bmatrix}$$

$$[e_n] = \begin{bmatrix} e_{1n} \\ e_{2n} \\ \vdots \\ e_{Nn} \end{bmatrix} = n^{\text{th}}$$ eigenvector (a column vector) corresponding to eigenvalue λ_n

Diagonalizing a Matrix (cont.)

Proof of diagonalization property:

$$[e][D] = \left[\lambda_1 [e_1] \quad \lambda_2 [e_2] \quad \cdots \; \lambda_N [e_N] \right]$$ (from properties of a diagonal matrix)

$$[A][e] = \left[\lambda_1 [e_1] \quad \lambda_2 [e_2] \quad \cdots \; \lambda_N [e_N] \right]$$ (from properties of eigenvectors)

(Please see next slide.)

Hence, we have

$$[A][e] = [e][D]$$

so that

$$[A] = [e][D][e]^{-1}$$

Note:
The inverse will exist since the columns of the matrix $[e]$ are linearly independent, by assumption.

Diagonalizing a Matrix (cont.)

$$[e][D] = \left[\ \lambda_1[e_1] \quad \lambda_2[e_2] \quad \cdots \ \lambda_N[e_N]\ \right] \qquad \left(\text{from properties of a diagonal matrix}\right)$$

$$[e][D] = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 e_{11} & \lambda_2 e_{12} & \lambda_3 e_{13} & \lambda_3 e_{14} \\ \lambda_1 e_{21} & \lambda_2 e_{22} & \lambda_3 e_{23} & \lambda_3 e_{24} \\ \lambda_1 e_{31} & \lambda_2 e_{32} & \lambda_3 e_{33} & \lambda_3 e_{34} \\ \lambda_1 e_{41} & \lambda_2 e_{42} & \lambda_3 e_{43} & \lambda_3 e_{44} \end{bmatrix} = \left[\ \lambda_1[e_1] \quad \lambda_2[e_2] \quad \lambda_2[e_2] \quad \lambda_4[e_4]\ \right]$$

$$[A][e] = \left[\ \lambda_1[e_1] \quad \lambda_2[e_2] \quad \cdots \ \lambda_N[e_N]\ \right] \qquad \left(\text{from properties of eigenvectors}\right)$$

$$[A][e] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix} = \left[\ A[e_1] \quad A[e_2] \quad A[e_3] \quad A_4[e_4]\ \right] = \left[\ \lambda_1[e_1] \quad \lambda_2[e_2] \quad \lambda_3[e_3] \quad \lambda_4[e_4]\ \right]$$

Note: We visualize here for a 4×4 matrix.

Diagonalizing a Matrix (cont.)

If the matrix $[A]$ is Hermetian (self-adjoint), then the eigenvectors are orthogonal, and hence linearly independent. We then have

$$[A]=[e][D][e]^{-1}$$

A Hermetian matrix is always diagonalizable!

If the eigenvectors are scaled so that they have unit magnitude, then we also have:

$$[M]\equiv\left[e^{t*}\right][e]=[I]\quad(\text{identity matrix})$$

This follows from the orthogonality property of the eigenvectors:

$$[M]_{mn}=\underline{e}_m^*\cdot\underline{e}_n=0\quad(m\neq n)$$

(The m^{th} row of $[M]$ is the transpose of e_m^*.)

Note: For the diagonal elements, $[M]_{mm}=1$ if the eigenvectors have been scaled so that

$$\underline{e}_m^*\cdot\underline{e}_m=1$$

Diagonalizing a Matrix (cont.)

Hence, for a Hermetian matrix with scaled (unit-magnitude) eigenvectors we then have:

$$[e]^{-1} = \left[e^{t*}\right] \quad \left(\text{the eigenvalue matrix is "unitary"}\right)$$

Therefore, for a Hermetian (self-adjoint) matrix with scaled (unit-magnitude) eigenvectors we have:

$$[A] = [e][D]\left[e^{t*}\right]$$