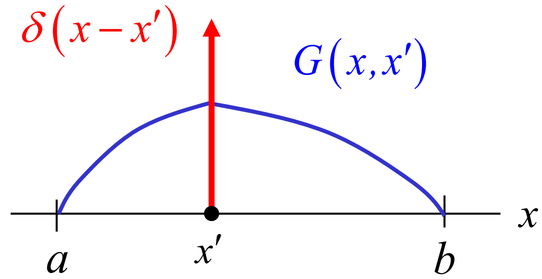


ECE 6382

Fall 2023

David R. Jackson

Notes 19

Green's Functions

Green's Functions

❖ The Green's function method is a powerful and systematic method for determining a solution to a problem with a known forcing function on the RHS.

❖ The Green's function is the solution to a "point" or "impulse" forcing function.

❖ It is similar to the idea of an "impulse response" in circuit theory (where we deal with time instead of space).

George Green (1793-1841)

Green's Mill in Sneinton (Nottingham), England, the mill owned by Green's father. The mill was renovated in 1986 and is now a science centre.

Green's Functions (cont.)

Consider the following second-order linear differential equation:

$$-\frac{1}{w(x)}\left(\frac{d}{dx}\left(P(x)\frac{d}{dx}\right)u(x)\right)+Q(x)u(x)=f(x)$$

or

$$\mathcal{L}u=f$$

(f is a "forcing" function.)

where

$$\mathcal{L}\equiv-\frac{1}{w(x)}\frac{d}{dx}\left(P(x)\frac{d}{dx}\right)+Q(x)$$

(self-adjoint)

Green's Functions (cont.)

Problem to be solved:

$$\mathcal{L}u = f$$

$$u(a) = 0,\ u(b) = 0 \quad \text{(Dirichlet BCs)}$$

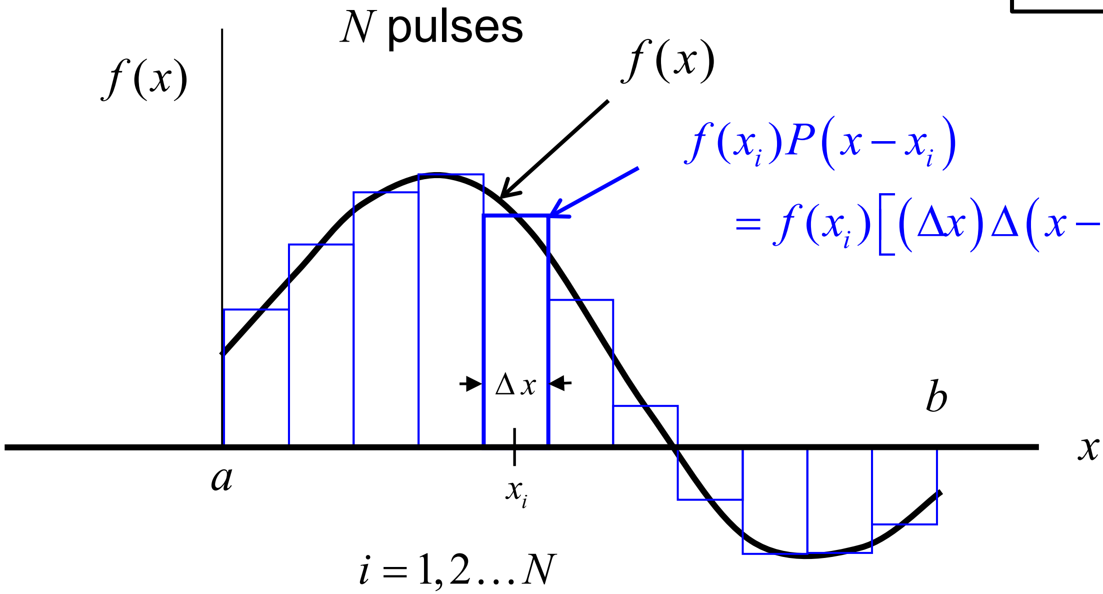
or

$$u'(a) = 0,\ u'(b) = 0 \quad \text{(Neuman BCs)}$$

Green's Functions (cont.)

- We can think of the forcing function $f(x)$ as being broken into many small rectangular pieces.
- Using superposition, we add up the solution from each small piece.
- Each small piece can be represented as a delta function in the limit as the width approaches zero.

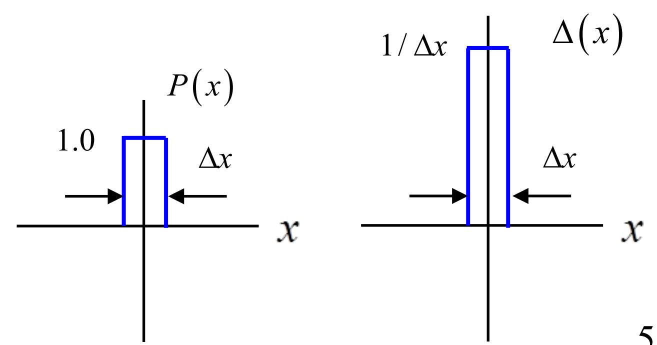
Note:
The $\Delta(x)$ function becomes a $\delta(x)$ function in the limit as $\Delta x \rightarrow 0$.



$$f(x_i)P(x-x_i) = f(x_i)\left[(\Delta x)\Delta(x-x_i)\right]$$

$$\Delta(x) \equiv \begin{cases} \dfrac{1}{\Delta x}, & x \in (\Delta x/2,\ \ \Delta x/2) \\ 0, & \text{otherwise} \end{cases}$$

$$P(x) = (\Delta x)\Delta(x)$$



Green's Functions (cont.)

From superposition:

$$u(x) \approx \sum_{i=1}^{N} \left[f(x_i) \Delta x \right] G_\Delta(x, x_i)$$

$G_\Delta(x, x_i) \equiv$ solution of DE from single pulse $\Delta(x - x_i)$ centered at x_i

Let $\Delta x \rightarrow 0$

$$\sum_{i=1}^{N} f(x_i)\, G_\Delta(x, x_i)\, \Delta x \rightarrow \int_a^b f(x') G(x, x') dx'$$

$G(x, x') \equiv$ solution from $\delta(x - x')$ centered at x'

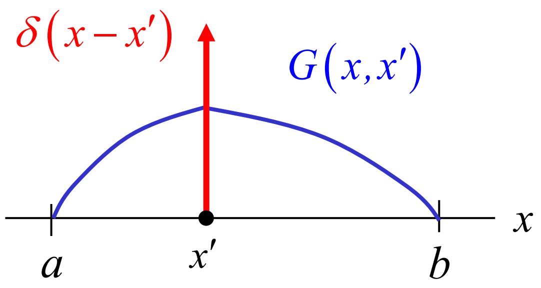
Green's Functions (cont.)

The <u>Green's function</u> $G(x, x')$ is defined as the solution with a delta-function at $x = x'$ for the RHS.

$$\mathcal{L}G(x,x') = \delta(x - x')$$

The solution to the original differential equation $\mathcal{L}u = f$ (from superposition) is then

$$u(x) = \int_a^b f(x')G(x,x')dx'$$

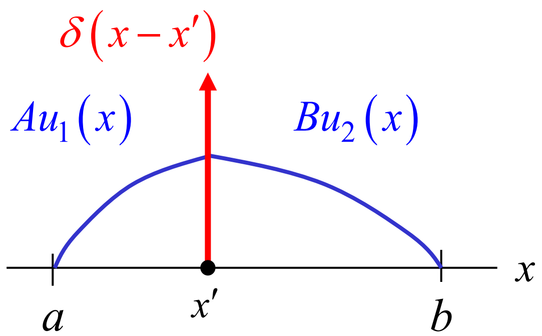


Green's Functions (cont.)

There are two general methods for constructing Green's functions.

Method 1:
Find the solution to the homogenous equation to the left and right of the delta function, and then enforce boundary conditions at the location of the delta function.

$$G(x,x') = \begin{cases} Au_1(x), & x \le x' \\ Bu_2(x), & x \ge x' \end{cases}$$



The functions u_1 and u_2 are solutions of the *homogenous equation*.

Note: u_1 satisfies the left BC, u_2 satisfies the right BC.

- The Green's function is assumed to be continuous.
- The derivative of the Green's function is allowed to be discontinuous.

Green's Functions (cont.)

Method 2:
Use the method of eigenfunction expansion.

Eigenvalue problem:

$$\mathcal{L}\psi_n = \lambda_n \psi_n$$

$$\psi_n(a) = 0, \;\; \psi_n(b) = 0 \quad \text{(Dirichlet BCs)}$$

or

$$\psi_n'(a) = 0, \;\; \psi_n'(b) = 0 \quad \text{(Neuman BCs)}$$

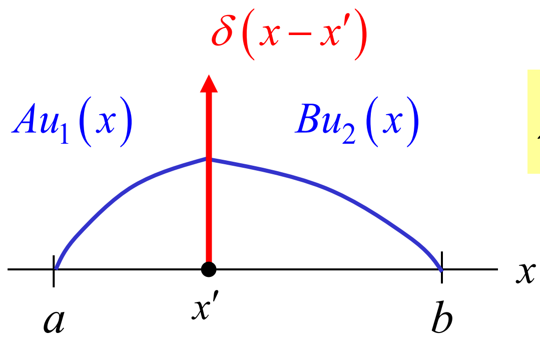
We then have:

$$G(x, x') = \sum_n a_n \psi_n(x)$$

Note:
The eigenfunctions are orthogonal ($\mathcal{L}$ is self-adjoint).

Green's Functions (cont.)

Method 1



$$\mathcal{L}G(x,x') = \delta(x-x')$$

$$G(x,x') = \begin{cases} Au_1(x), & x \le x' \\ Bu_2(x), & x \ge x' \end{cases}$$

Integrate both sides of the above DE over the delta function:

$$\lim_{\varepsilon\to 0}\int_{x'-\varepsilon}^{x'+\varepsilon} \mathcal{L}G\,dx = \lim_{\varepsilon\to 0}\int_{x'-\varepsilon}^{x'+\varepsilon}\left(-\frac{1}{w(x)}\frac{d}{dx}\left(P(x)\frac{dG}{dx}\right)+Q(x)G\right)dx$$

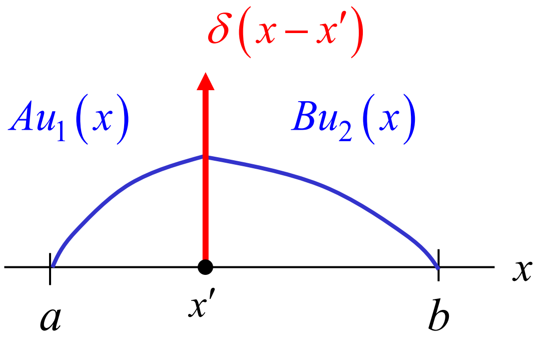
$$= -\frac{1}{w(x')}\left(\frac{P(x'^{+})\,dG(x'^{+},x')}{dx} - \frac{P(x'^{-})\,dG(x'^{-},x')}{dx}\right) = \lim_{\varepsilon\to 0}\int_{x'-\varepsilon}^{x'+\varepsilon}\delta(x-x')\,dx = 1$$

$$\Rightarrow \quad \left(\frac{dG(x'^{+},x')}{dx} - \frac{dG(x'^{-},x')}{dx}\right) = -\frac{w(x')}{P(x')}$$

$$\Rightarrow \quad Bu_2'(x') - Au_1'(x') = -\frac{w(x')}{P(x')}$$

Green's Functions (cont.)

Method 1 (cont.)



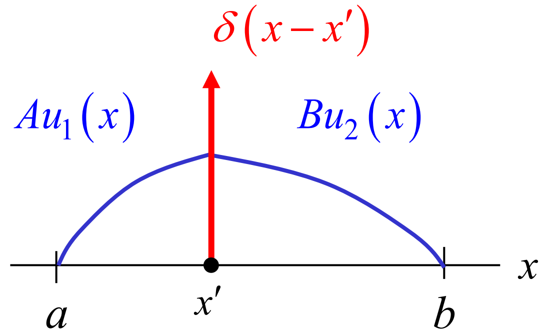
$$G(x,x') = \begin{cases} Au_1(x), & x \le x' \\ Bu_2(x), & x \ge x' \end{cases}$$

Also, we have (from continuity of the Green's function):

$$Au_1(x') = Bu_2(x')$$

Green's Functions (cont.)

Method 1 (cont.)



We then have:

$$\begin{bmatrix} -u_1'(x') & u_2'(x') \\ u_1(x') & -u_2(x') \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -\dfrac{w(x')}{P(x')} \\ 0 \end{bmatrix}$$

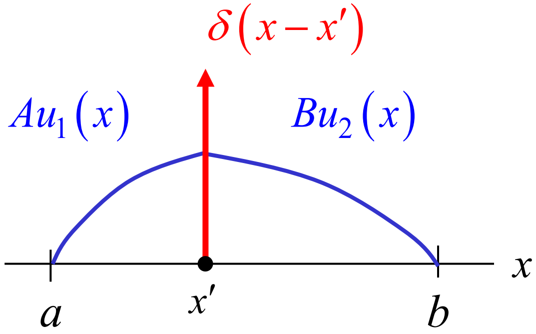
$$\Rightarrow \quad A = \left(-\frac{w(x')}{P(x')}\right)\left(\frac{-u_2(x')}{\Delta}\right), \quad B = \left(-\frac{w(x')}{P(x')}\right)\left(\frac{-u_1(x')}{\Delta}\right)$$

where $\Delta = \text{determinant} = u_1'(x')u_2(x') - u_1(x')u_2'(x')$

Also, $W[u_1, u_2] = W(x') = u_1(x')u_2'(x') - u_1'(x')u_2(x') = -\Delta$

Green's Functions (cont.)

Method 1 (cont.)



We then have:

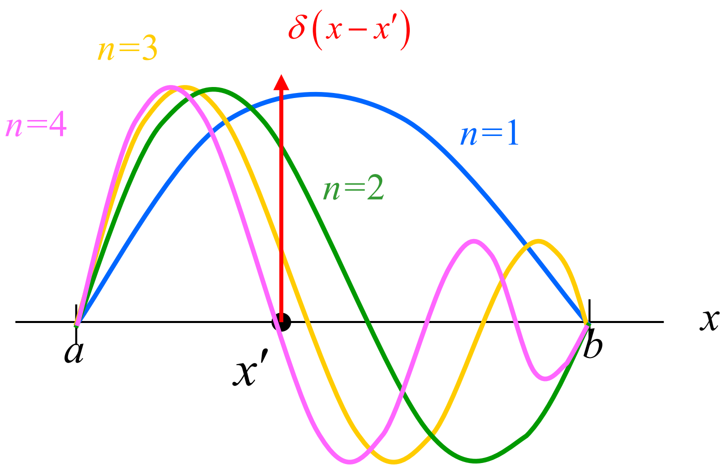
$$G(x,x') = \begin{cases} \left(-\dfrac{w(x')}{P(x')}\right)\dfrac{u_2(x')u_1(x)}{W(x')}, & x < x' \\ \left(-\dfrac{w(x')}{P(x')}\right)\dfrac{u_1(x')u_2(x)}{W(x')}, & x > x' \end{cases}$$

Green's Functions (cont.)

Method 2

The Green's function is expanded as a series of eigenfunctions.

$$G(x,x') = \sum_n a_n \psi_n(x)$$



where

$$\mathcal{L}\psi_n = \lambda_n \psi_n$$

$\psi_n(a) = 0,\ \psi_n(b) = 0$ (Dirichlet BCs)

or

$\psi_n'(a) = 0,\ \psi_n'(b) = 0$ (Neuman BCs)

The eigenfunctions corresponding to distinct eigenvalues are orthogonal (from Sturm-Liouville theory).

Green's Functions (cont.)

Method 2 (cont.)

$$\mathcal{L}G(x,x') = \delta(x-x')$$

⇩

$$\mathcal{L}\sum_n a_n \psi_n(x) = \delta(x-x')$$

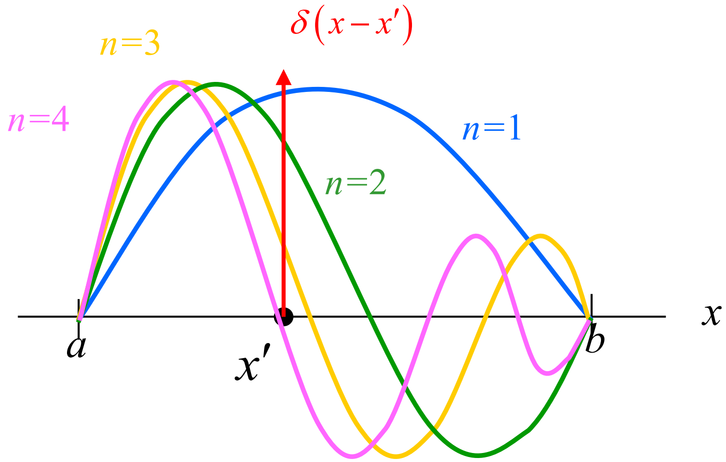
⇩

$$\sum_n a_n \mathcal{L}\psi_n(x) = \delta(x-x')$$

⇩

$$\sum_n a_n \lambda_n \psi_n(x) = \delta(x-x')$$

Multiply both sides by $\psi_m{}^*(x)\, w(x)$ and then integrate from a to b.



$$\sum_n a_n \lambda_n \left\langle \psi_n(x), \psi_m(x) \right\rangle = \left\langle \delta(x-x'), \psi_m(x) \right\rangle$$

⇩ Delta-function property

$$\sum_n a_n \lambda_n \left\langle \psi_n(x), \psi_m(x) \right\rangle = w(x')\psi_m^*(x')$$

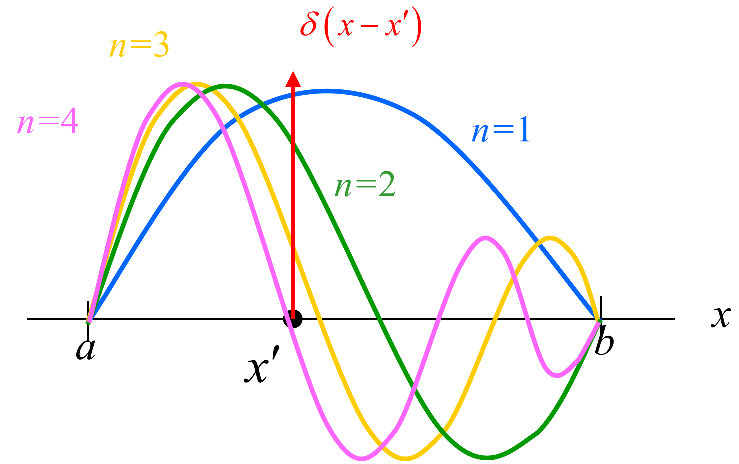
⇩ Orthogonality

$$a_m \lambda_m \left\langle \psi_m(x), \psi_m(x) \right\rangle = w(x')\psi_m^*(x')$$

Recall: $\langle f, g \rangle \equiv \int_a^b f(x)\, g^*(x)\, w(x)\, dx$

Green's Functions (cont.)

Method 2 (cont.)



Hence

$$a_m = \frac{w(x')\psi_m^*(x')}{\lambda_m \langle \psi_m(x), \psi_m(x) \rangle}$$

Therefore, we have (relabeling $m \to n$):

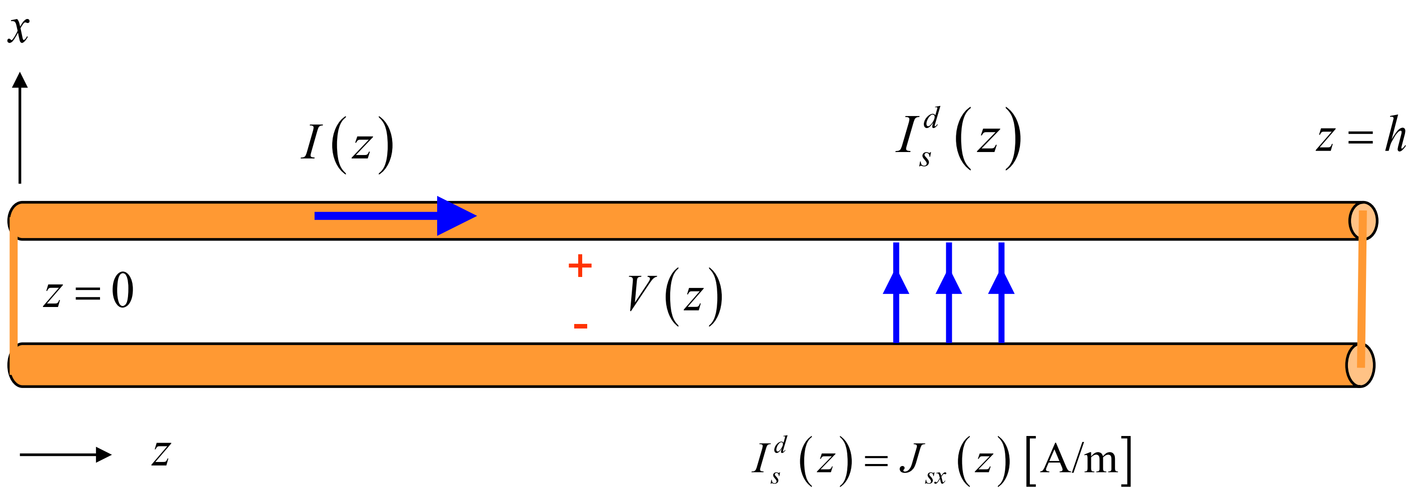
$$G(x,x') = \sum_n \left(\frac{w(x')\psi_n^*(x')}{\lambda_n \langle \psi_n(x), \psi_n(x) \rangle} \right) \psi_n(x)$$

Note: $\langle \psi_n(x), \psi_n(x) \rangle \equiv \int_a^b |\psi_n(x)|^2 \, w(x)\, dx$

Application: Transmission Line

A short-circuited transmission line resonator with a distributed current source.

The distributed current source is a surface current.



$$I_s^d(z) = J_{sx}(z)\ [\text{A/m}]$$

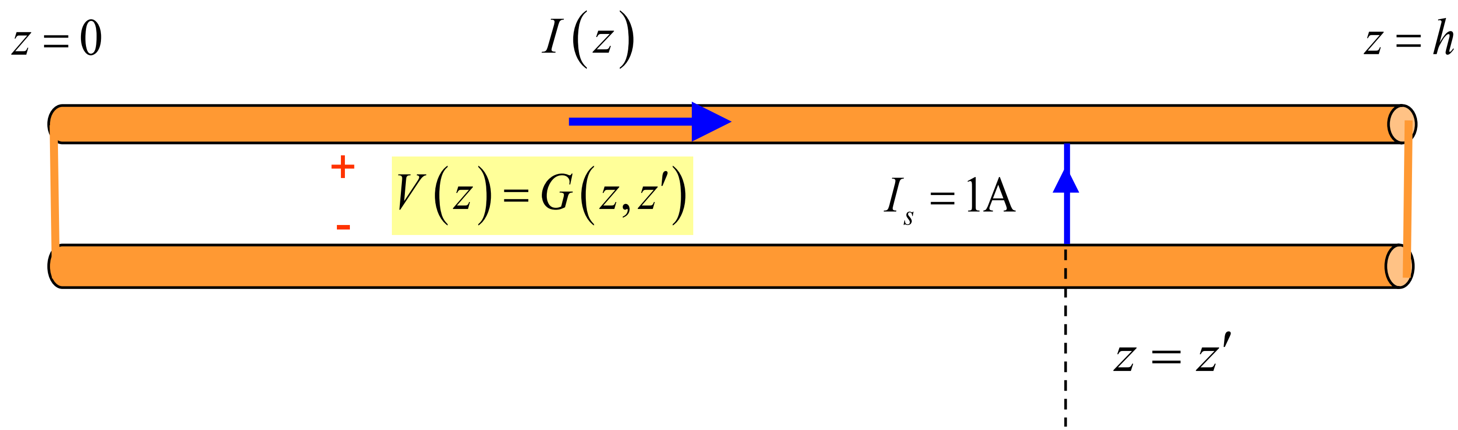
$$I_{\text{total}} = \int_0^h J_{sx}(z)\, dz$$

Green's function problem:

$$I_s^d(z) = \delta(z - z') \Rightarrow \text{Lumped current source}: I_s = 1\text{A} \ @\ z = z'$$

Application: Transmission Line (cont.)

An illustration of the Green's function (for voltage):



The total voltage due to the distributed current source (surface current) is then:

$$V(z)=\int_0^h J_{sx}(z')\,G(z,z')\,dz'$$

Application: Transmission Line (cont.)

Telegrapher's equations for a distributed current source:

$$\frac{dV}{dz} = -j\omega L I$$

$$\frac{dI}{dz} = I_s^d - j\omega C V$$

L = inductance/meter
C = capacitance/meter

(Please see Appendix A.)

Take the derivative of the first and substitute from the second:

$$\frac{d^2V}{dz^2} = -j\omega L\left(I_s^d - j\omega C V\right)$$

Note: j is used instead of i here.

Application: Transmission Line (cont.)

Hence

$$\frac{d^2V}{dz^2} + k_z^2 V = (-j\omega L) J_{sx}(z)$$

or

$$\left(\frac{1}{-j\omega L}\right)\left(\frac{d^2V(z)}{dz^2} + k_z^2 V(z)\right) = J_{sx}(z)$$

Therefore:

$$\left(\frac{1}{-j\omega L}\right)\left(\frac{d^2G(z,z')}{dz^2} + k_z^2 G(z,z')\right) = \delta(z-z')$$

where $k_z = \omega\sqrt{LC}$

Application: Transmission Line (cont.)

$$\left(\frac{1}{-j\omega L}\right)\left(\frac{d^2V(z)}{dz^2}+k_z^2V(z)\right)=J_{sx}(z)$$

Compare with:

$$-\frac{1}{w(z)}\left(\frac{d}{dz}\left(P(z)\frac{d}{dz}\right)u(z)\right)+Q(z)u(z)=f(z)$$

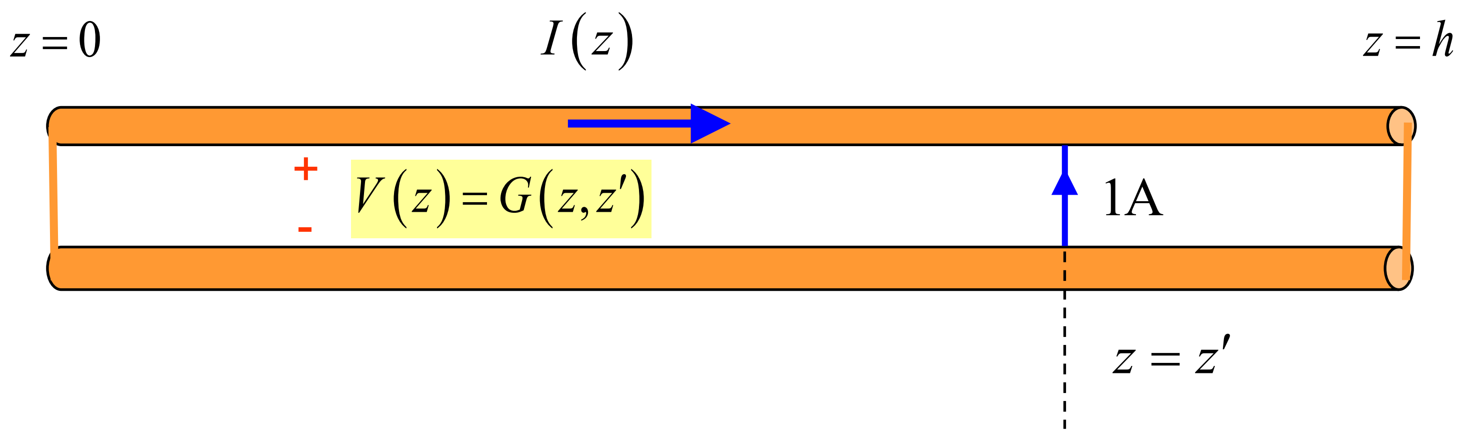
Therefore:

$$w(z)=j\omega L,\quad P(z)=1,\quad Q(z)=k_z^2,\quad f(z)=J_{sx}(z)$$

Note: The self-adjoint property required the operator to be real, which is not the case here since $w(z)$ is not real. However, we can multiply both sides of the equation by j to make the operator real, and then divide the final answer by j at the end. This process does not affect the final result, so it is not done here.

Application: Transmission Line (cont.)

Method 1



The general solution of the homogeneous equation is:

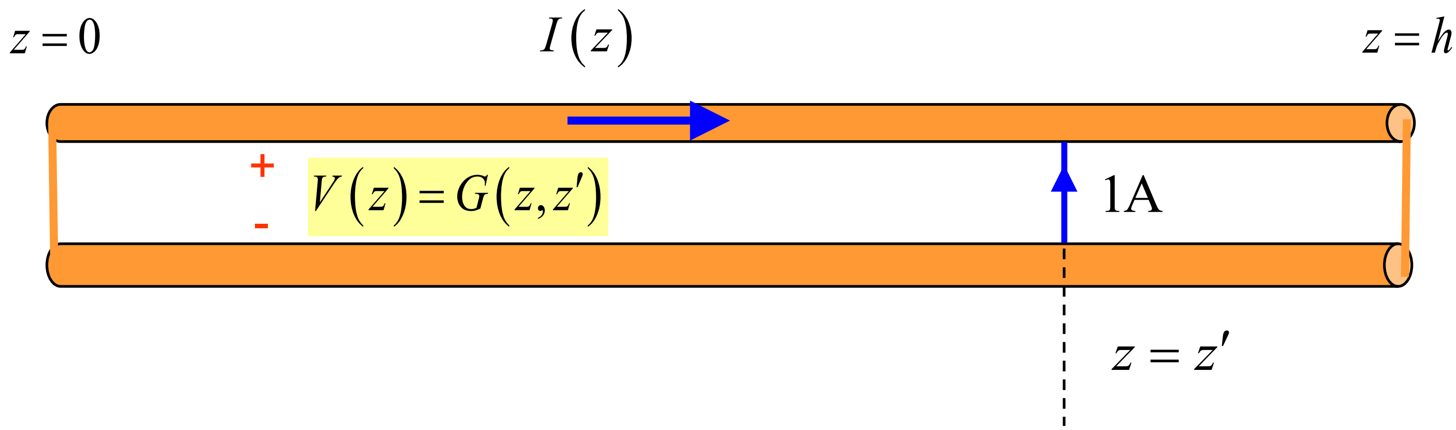
$$V_1(z)=Au_1(z)=A\sin(k_z z)$$

$$V_2(z)=Bu_2(z)=B\sin(k_z(z-h))$$

Homogeneous equation:

$$\frac{d^2V}{dz^2}+k_z^2V=0$$

Application: Transmission Line (cont.)



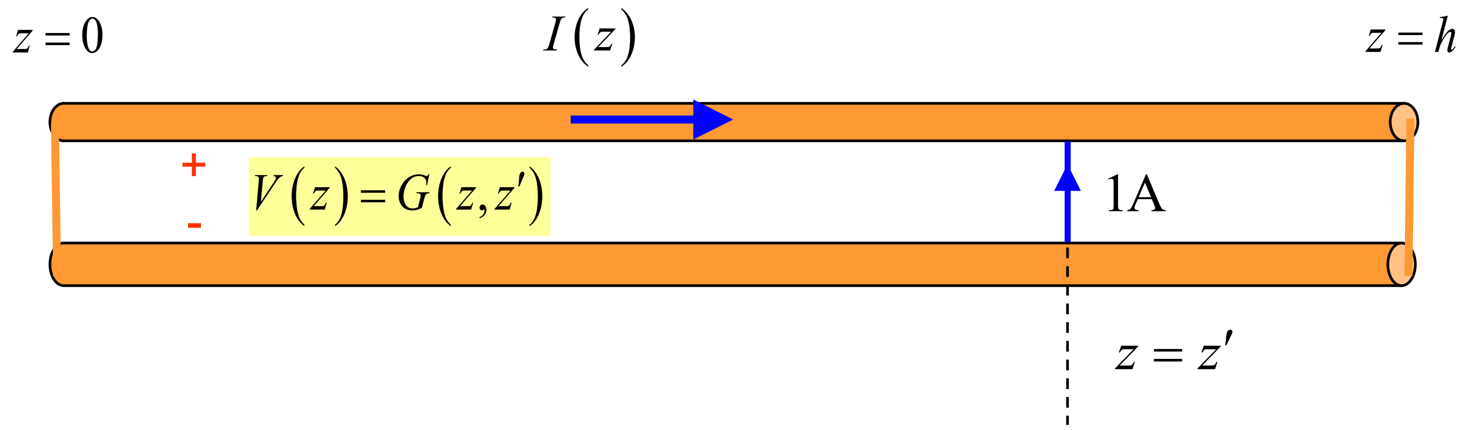
The Green's function is:

$$G(z,z') = \begin{cases} \left(-\dfrac{w(z')}{P(z')}\right)\dfrac{u_2(z')u_1(z)}{W(z')}, & z < z' \\ \left(-\dfrac{w(z')}{P(z')}\right)\dfrac{u_1(z')u_2(z)}{W(z')}, & z > z' \end{cases}$$

$$u_1(z) = \sin(k_z z)$$

$$u_2(z) = \sin(k_z (z - h))$$

Application: Transmission Line (cont.)



The final form of the Green's function is:

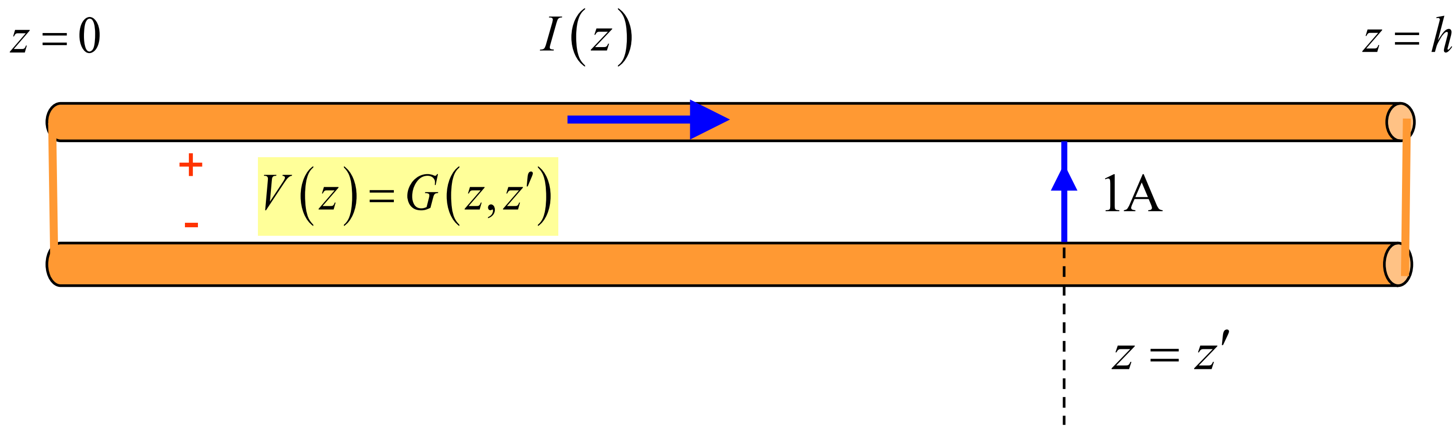
$$G(z,z') = \begin{cases} (-j\omega L)\dfrac{\sin\left(k_z(z'-h)\right)\sin\left(k_z z\right)}{W(z')}, & z<z' \\ (-j\omega L)\dfrac{\sin\left(k_z z'\right)\sin\left(k_z(z-h)\right)}{W(z')}, & z>z' \end{cases}$$

where

$$W(z') = W[u_1(z'),u_2(z')] = k_z\left[\sin(k_z z')\cos\left(k_z(z'-h)\right) - \cos(k_z z')\sin\left(k_z(z'-h)\right)\right] = k_z\sin(k_z h)$$

Application: Transmission Line (cont.)

Method 2



The eigenvalue problem is

$$-\frac{1}{j\omega L}\left(\frac{d^2\psi}{dz^2}+k_z^2\psi\right)=\lambda\psi$$

Note:
A minus sign in introduced in the eigenvalue problem for convenience.

This may be written as

$$\frac{d^2\psi}{dz^2}=-\lambda'^2\psi$$

where $\lambda'^2 \equiv (j\omega L)\lambda + k_z^2$

Application: Transmission Line (cont.)

We then have:

$$\frac{d^2\psi}{dz^2} = -\lambda'^2\psi, \quad \psi(0) = \psi(h) = 0$$

Note: We do not need to consider $n = 0$ (trivial eigenfunction).

The solution is:

$$\psi_n(z) = \sin(\lambda_n' z)$$

$$\lambda_n' = \frac{n\pi}{h}$$

$$\Rightarrow$$

$$\psi_n(z) = \sin\left(\frac{n\pi z}{h}\right)$$

$$\lambda_n = \frac{1}{j\omega L}\left(\left(\frac{n\pi}{h}\right)^2 - k_z^2\right)$$

$$n = 1, 2, 3 \ldots$$

Recall: $\lambda_n = \frac{1}{j\omega L}\left(\lambda_n'^2 - k_z^2\right)$

Note: $\langle \psi_n(z), \psi_n(z) \rangle = \int_a^b \psi_n(z)\psi_n^*(z)w(z)\,dz = \int_0^h \sin^2\left(\frac{n\pi z}{h}\right)(j\omega L)\,dz = \frac{h}{2}(j\omega L)$

$$(n \neq 0)$$

Application: Transmission Line (cont.)

We then have:

$$G(z,z') = \sum_n \left(\frac{w(z')\psi_n^*(z')}{\lambda_n \langle \psi_n(z), \psi_n(z) \rangle} \right) \psi_n(z)$$

where

$$\lambda_n = \frac{1}{j\omega L}\left(\lambda_n'^2 - k_z^2\right) = \frac{1}{j\omega L}\left(\left(\frac{n\pi}{h}\right)^2 - k_z^2\right)$$

$$\psi_n(z) = \sin\left(\frac{n\pi z}{h}\right) \qquad w(z') = j\omega L$$

$$\langle \psi_n(z), \psi_n(z) \rangle = \frac{h}{2}(j\omega L)$$

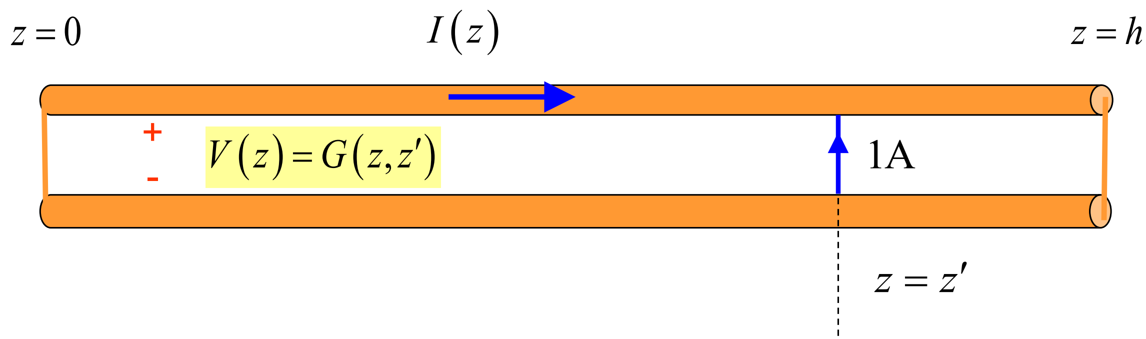
Application: Transmission Line (cont.)

The final solution is then:

$$G(z,z') = -(j\omega L)\left(\frac{2}{h}\right)\sum_{n=1}^{\infty}\left(\frac{\sin\left(\dfrac{n\pi z'}{h}\right)}{\left(k_z^2 - \left(\dfrac{n\pi}{h}\right)^2\right)}\right)\sin\left(\frac{n\pi z}{h}\right)$$

Application: Transmission Line (cont.)

Summary



$k_z = \omega\sqrt{LC}$

$$G(z,z') = \begin{cases} (-j\omega L)\dfrac{\sin\left(k_z(z'-h)\right)\sin(k_z z)}{k_z\sin(k_z h)}, & z<z' \\ (-j\omega L)\dfrac{\sin(k_z z')\sin\left(k_z(z-h)\right)}{k_z\sin(k_z h)}, & z>z' \end{cases}$$

Method 1

$$G(z,z') = -(j\omega L)\left(\frac{2}{h}\right)\sum_{n=1}^{\infty}\left(\frac{\sin\left(\dfrac{n\pi z'}{h}\right)}{\left(k_z^2-\left(\dfrac{n\pi}{h}\right)^2\right)}\right)\sin\left(\frac{n\pi z}{h}\right)$$

Method 2

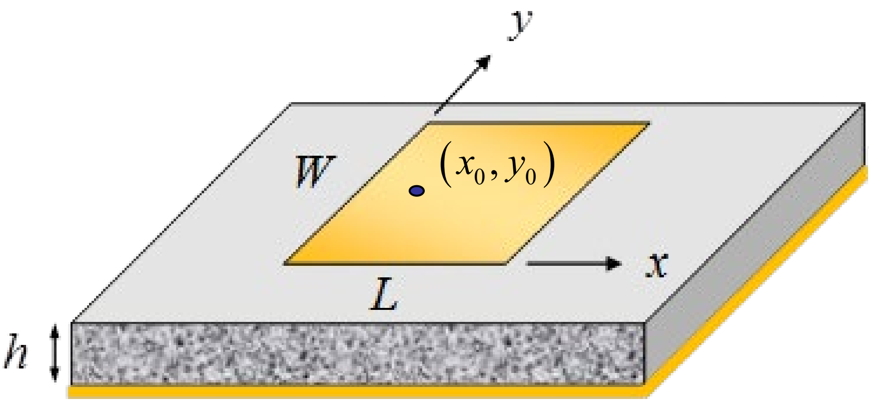
Application: Transmission Line (cont.)

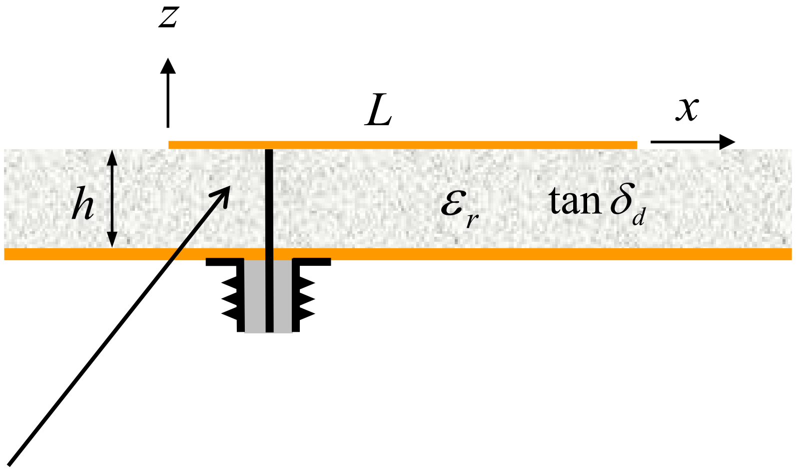
Other possible Green's functions for the transmission line:

- We solve for the Green's function that gives us the current $I(z)$ due to the 1A parallel current source.
- We solve for the Green's function giving the voltage due a 1V series voltage source instead of a 1A parallel current source.
- We solve for the Green's function giving the current to due a 1V series voltage source instead of a 1A parallel current source.

Cavity Model for Patch Antenna

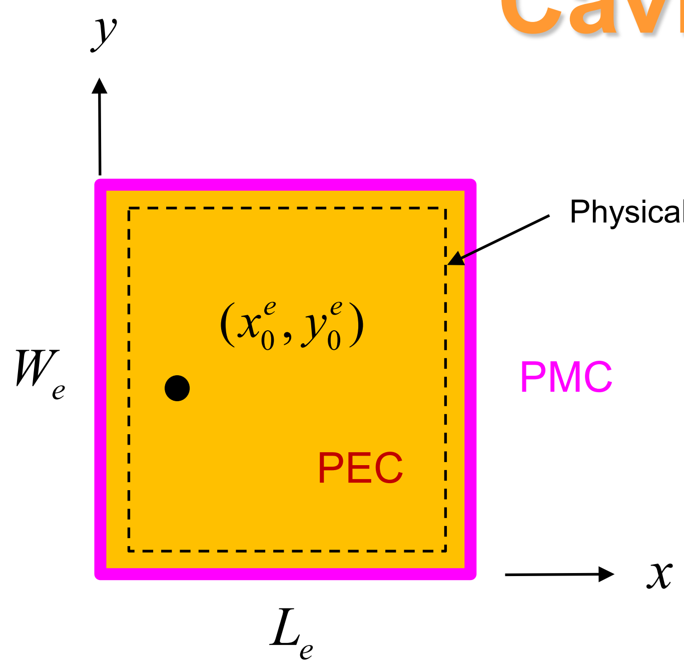
Next we use the cavity model and the method of eigenfunction expansion to solve for the input impedance of the rectangular microstrip patch antenna.





I_0 current feed at (x_0, y_0)

Cavity Model



Accounting for fringing:

$$L_e = L + 2\Delta L$$

$$W_e = W + 2\Delta W$$

$$x_0^e = x_0 + \Delta L$$

$$y_0^e = y_0 + \Delta W$$

$$\Delta L / h = 0.412 \left[\frac{\left(\varepsilon_r^{eff} + 0.3\right)\left(\dfrac{W}{h} + 0.264\right)}{\left(\varepsilon_r^{eff} - 0.258\right)\left(\dfrac{W}{h} + 0.8\right)} \right]$$

(Hammerstad formula)

Note: The coordinates (x_0, y_0) are measured from the corner of the <u>physical</u> patch.

$$\varepsilon_r^{eff} = \frac{\varepsilon_r + 1}{2} + \left(\frac{\varepsilon_r - 1}{2}\right)\left[1 + 12\left(\frac{h}{W}\right)\right]^{-1/2}$$

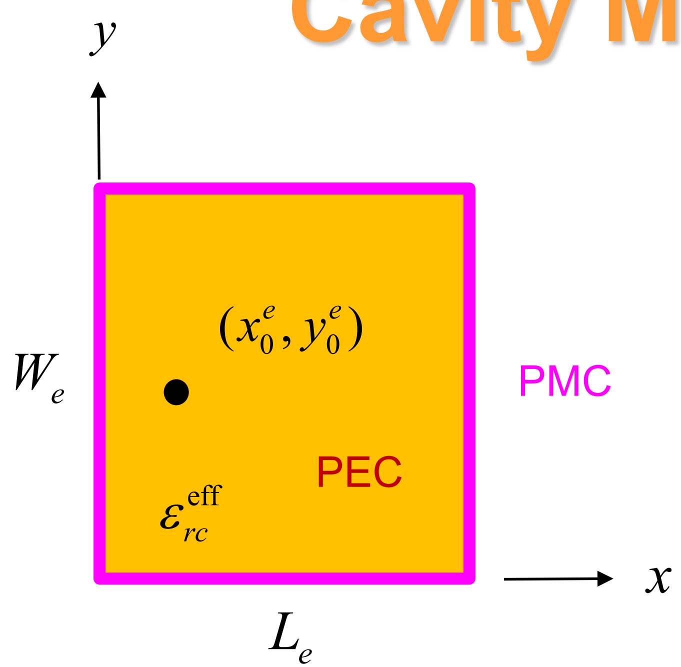
$$\Delta W / h = \frac{\ln 4}{\pi}$$

(Wheeler formula)

Note:

ΔL is often chosen from Hammerstad's formula.
ΔW is often chosen from Wheeler's formula.

Cavity Model (cont.)



Accounting for loss and radiation:

$$k_e = k_0 \sqrt{\varepsilon_{rc}^{\text{eff}}}$$

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r \left(1 - j l_{\text{eff}}\right)$$

$$l_{\text{eff}} = \tan \delta_{\text{eff}} = \frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{sp}} + \frac{1}{Q_{sw}}$$

Note : $\tan \delta_d = \dfrac{1}{Q_d}$

Assume no z variation (the probe current is constant in the z direction.)

Cavity Model (cont.)

CAD Formulas for Q Factors*

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r \left(1 - j l_{\text{eff}}\right) \qquad l_{\text{eff}} = \tan\delta_{\text{eff}} = \frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{sp}} + \frac{1}{Q_{sw}}$$

where

$$Q_d = \frac{1}{\tan\delta_d}$$

$$Q_c = \left(\frac{\eta_0}{2}\right)\left[\frac{(k_0 h)}{R_s^{\text{ave}}}\right] \qquad R_s^{\text{ave}} = \left(R_s^{\text{patch}} + R_s^{\text{ground}}\right)/2 \qquad R_s = \frac{1}{\sigma\delta}$$

$$Q_{sp} \approx \frac{3}{16}\left(\frac{\varepsilon_r}{p c_1}\right)\left(\frac{L_e}{W_e}\right)\left(\frac{1}{h/\lambda_0}\right) \qquad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$Q_{sw} = Q_{sp}\left(\frac{e_r^{\text{hed}}}{1 - e_r^{\text{hed}}}\right)$$

*Derived in ECE 6345.

Cavity Model (cont.)

CAD Formulas for Q Factors (cont.)

$$e_r^{\text{hed}} = \frac{1}{1 + \frac{3}{4}\pi\left(k_0 h\right)\left(\frac{1}{c_1}\right)\left(1 - \frac{1}{\varepsilon_r}\right)^3}$$

$$c_2 = -0.0914153$$

$$a_2 = -0.16605$$

$$a_4 = 0.00761$$

$$c_1 = 1 - \frac{1}{\varepsilon_r} + \frac{2/5}{\varepsilon_r^2}$$

$$p = 1 + \frac{a_2}{10}\left(k_0 W\right)^2 + \left(a_2^2 + 2a_4\right)\left(\frac{3}{560}\right)\left(k_0 W\right)^4 + c_2\left(\frac{1}{5}\right)\left(k_0 L\right)^2$$
$$+ a_2 c_2 \left(\frac{1}{70}\right)\left(k_0 W\right)^2 \left(k_0 L\right)^2$$

Helmholtz Equation for E_z

We first derive the Helmholtz equation for E_z.

$$\nabla \times \underline{H} = \underline{J}^i + j\omega\varepsilon_c^{\text{eff}}\,\underline{E}$$
$$\nabla \times \underline{E} = -j\omega\mu\underline{H}$$

Substituting Faraday's law for $\underline{H}$ into Ampere's law, we have

$$-\frac{1}{j\omega\mu}\nabla \times \left(\nabla \times \underline{E}\right) = \underline{J}^i + j\omega\varepsilon_c^{\text{eff}}\,\underline{E}$$
$$\Rightarrow \nabla \times \left(\nabla \times \underline{E}\right) = -j\omega\mu\underline{J}^i + k_e^2\underline{E}$$
$$\Rightarrow \nabla\left(\nabla \cdot \underline{E}\right) - \nabla^2\underline{E} = -j\omega\mu\underline{J}^i + k_e^2\underline{E}$$
$$\Rightarrow \nabla^2\underline{E} + k_e^2\underline{E} = j\omega\mu\underline{J}^i$$

Helmholtz Equation for E_z (cont.)

Hence

$$\nabla^2 E_z + k_e^2 E_z = j\omega\mu J_z^i$$

$$J_z^i = J_z^i\left(x, y\right) = I_0\,\delta\left(x - x_0^e\right)\delta\left(y - y_0^e\right)$$

Feed (impressed) current

We take it here to be a filamentary source (zero radius).

Denote

$$\psi(x, y) = E_z(x, y)$$

$$f(x, y) = j\omega\mu J_z^i(x, y)$$

Then

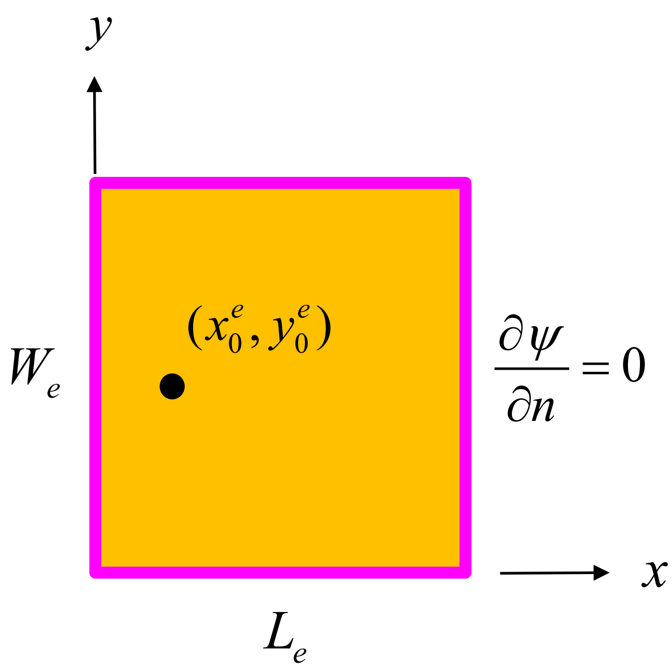
$$\nabla^2 \psi + k_e^2 \psi = f(x, y)$$

Mathematical Problem

$$\nabla^2 \psi + k_e^2 \psi = f(x, y)$$

$$\psi(x, y) = E_z(x, y) \qquad k_e = k_0 \sqrt{\varepsilon_{rc}^{\text{eff}}} \qquad f(x, y) = (j\omega\mu I_0)\,\delta\left(x - x_0^e\right)\delta\left(y - y_0^e\right)$$

The function ψ is really a 2-D Green's function, if the feed current is filamentary.



Eigenvalue Problem

$$\nabla^2 \psi + k_e^2 \psi = f(x, y)$$

Eigenvalue problem:

$$\nabla^2 \psi + k_e^2 \psi = \lambda \psi$$

$$\Rightarrow \quad \nabla^2 \psi = \left(\lambda - k_e^2\right) \psi$$

$$\Rightarrow \quad \nabla^2 \psi = -\lambda'^2 \psi \qquad \left(-\lambda'^2 = \lambda - k_e^2\right)$$

The original eigenvalue problem is thus reduced to this simpler "reduced" eigenvalue problem.

New notation: $\lambda' = \lambda'_{mn}$

Eigenvalue Problem (cont.)

Introduce eigenfunctions of the 2-D Laplace operator:

$$\psi_{mn}(x,y)$$

$$\nabla^2 \psi_{mn}(x,y) = -\lambda'^2_{mn}\psi_{mn}(x,y)$$

$$\frac{\partial \psi_{mn}}{\partial n} = 0\Big|_C \qquad -\lambda'^2_{mn} = \text{eigenvalue}$$

For a rectangular patch we have, from separation of variables (or guessing),

$$\psi_{mn}(x,y) = \cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$\lambda'^2_{mn} = \left[\left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2\right]$$

Note:
The eigenvalues are real and the eigenvalues are orthogonal.

Eigenfunction Expansion

Assume an "eigenfunction expansion":

$$\psi(x,y) = \sum_{m,n} A_{mn}\psi_{mn}(x,y) \qquad (m,n) = 0,1,2\ldots$$

This must satisfy $\nabla^2\psi + k_e^2\psi = f(x,y)$

Hence

$$\sum_{m,n} A_{mn}\nabla^2\psi_{mn} + k_e^2\sum_{m,n} A_{mn}\psi_{mn} = f(x,y)$$

Using the properties of the eigenfunctions, we have

$$\sum_{m,n} A_{mn}\left(k_e^2 - \lambda_{mn}'^2\right)\psi_{mn}(x,y) = f(x,y)$$

Eigenfunction Expansion (cont.)

Multiply the previous equation by $\psi^{*}_{m'n'}(x,y)$ and integrate.

Note that the eigenfunctions are orthogonal, so that

$$\int_S \psi_{mn}(x,y)\,\psi^{*}_{m'n'}(x,y)\,dS = 0 \qquad (m,n) \neq (m',n')$$

Note: The eigenfunctions are real, so we can drop the conjugate here if we want.

Define

$$< u, v > \equiv \int_S u(x,y)\,v^{*}(x,y)\,dS$$

$$\Rightarrow \quad < \psi_{mn}, \psi_{m'n'} > = \int_S \psi_{mn}(x,y)\,\psi^{*}_{m'n'}(x,y)\,dS = 0, \quad (m,n) \neq (m',n')$$

We then have

$$A_{m'n'}\left(k_e^2 - \lambda'^2_{m'n'}\right) < \psi_{m'n'}, \psi_{m'n'} > = < f, \psi_{m'n'} >$$

Eigenfunction Expansion (cont.)

Hence, we have (removing the primes in the notation)

$$A_{mn} = \frac{< f, \psi_{mn} >}{< \psi_{mn}, \psi_{mn} >} \left(\frac{1}{k_e^2 - \lambda_{mn}'^2} \right)$$

Recall: $f(x, y) = j\omega\mu J_z^i(x, y)$

so

$$A_{mn} = j\omega\mu \left(\frac{< J_z^i, \psi_{mn} >}{< \psi_{mn}, \psi_{mn} >} \right) \left(\frac{1}{k_e^2 - \lambda_{mn}'^2} \right)$$

The field inside the patch cavity is then given by

$$E_z(x, y) = \psi(x, y) = \sum_{m,n} A_{mn} \psi_{mn}(x, y)$$

Eigenfunction Expansion (cont.)

For the rectangular patch:

$$\psi_{mn} = \cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$\lambda_{mn}'^2 = \left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2$$

$$k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}}$$

where

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r\left(1 - j l_{\text{eff}}\right)$$

We need:

$$\left\langle \psi_{mn}, \psi_{mn} \right\rangle = \int_0^{L_e} \cos^2\left(\frac{m\pi x}{L_e}\right) dx \int_0^{W_e} \cos^2\left(\frac{n\pi y}{W_e}\right) dy$$

Eigenfunction Expansion (cont.)

so

$$\langle \psi_{mn}, \psi_{mn} \rangle = \left(\frac{W_e}{2} \right) \left(\frac{L_e}{2} \right) (1 + \delta_{m0})(1 + \delta_{n0})$$

$$\delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

Eigenfunction Expansion (cont.)

For a filamentary feed current we have:

$$\left\langle J_z^i, \psi_{mn} \right\rangle = \int_{-W_e/2}^{W_e/2} \int_{-L_e/2}^{L_e/2} I_0 \delta\left(x - x_0^e\right) \delta\left(y - y_0^e\right) \psi_{mn}^*\left(x, y\right) dx dy$$

$$= I_0 \psi_{mn}^*\left(x_0^e, y_0^e\right)$$

Hence, we have

$$\left\langle J_z^i, \psi_{mn} \right\rangle = I_0 \cos\left(\frac{m\pi x_0^e}{L_e}\right) \cos\left(\frac{n\pi y_0^e}{W_e}\right)$$

Eigenfunction Expansion (cont.)

The final form for the field inside the patch cavity is then given by:

$$E_z(x,y) = \psi(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\psi_{mn}(x,y)$$

$$\psi_{mn} = \cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$\lambda_{mn}'^2 = \left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2$$

$$k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}}$$

$$A_{mn} = j\omega\mu\left(\frac{< J_z^i, \psi_{mn} >}{< \psi_{mn}, \psi_{mn} >}\right)\left(\frac{1}{k_e^2 - \lambda_{mn}'^2}\right)$$

$$\left\langle J_z^i, \psi_{mn}\right\rangle = I_0\cos\left(\frac{m\pi x_0^e}{L_e}\right)\cos\left(\frac{n\pi y_0^e}{W_e}\right)$$

$$\left\langle \psi_{mn}, \psi_{mn}\right\rangle = \left(\frac{W_e}{2}\right)\left(\frac{L_e}{2}\right)\left(1+\delta_{m0}\right)\left(1+\delta_{n0}\right)$$

$$J_z^i = J_z^i(x,y) = I_0\,\delta\left(x - x_0^e\right)\delta\left(y - y_0^e\right)$$

Final Field Inside Cavity

Substituting in for all of the terms, we have:

$$E_z(x,y) = j\omega\mu I_0\left(\frac{4}{W_e L_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos\left(\frac{m\pi x_0^e}{L_e}\right)\cos\left(\frac{n\pi y_0^e}{W_e}\right)}{k_e^2-\left(\frac{m\pi}{L_e}\right)^2-\left(\frac{n\pi}{W_e}\right)^2}\right)\cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}}$$

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r\left(1 - j l_{\text{eff}}\right)$$

$$l_{\text{eff}} = \tan\delta_{\text{eff}} = \frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{sp}} + \frac{1}{Q_{sw}}$$

Note:
It is usually the (1,0) mode that is resonant.

Note: It is not obvious, but the field goes to infinity when $(x,y) \to (x_0^e, y_0^e)$

Green's Function

Using the Green's function notation, we have (setting $I_0 = 1$):

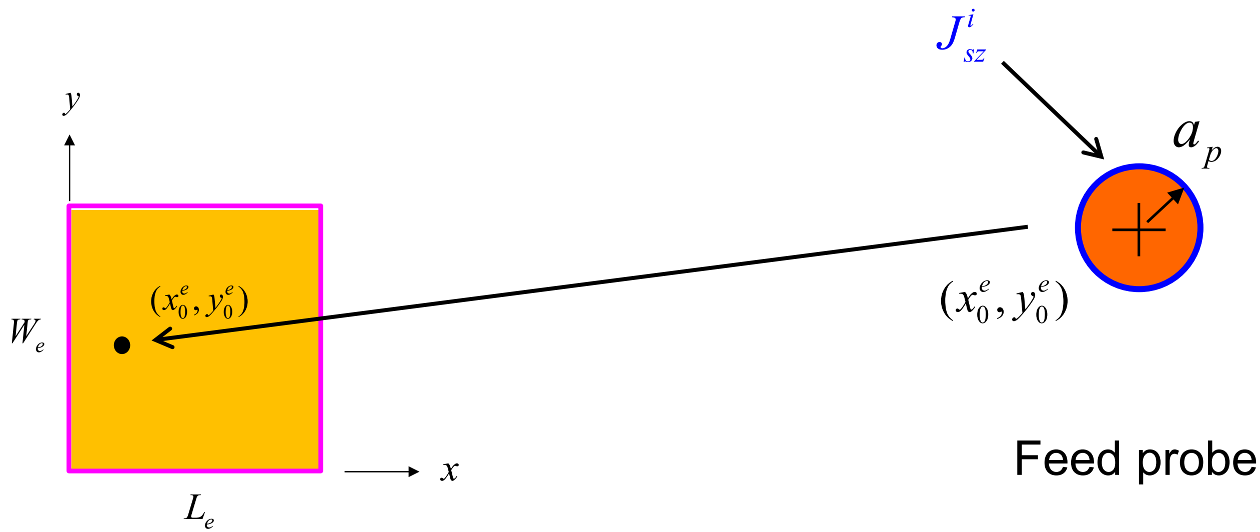
$$G(x,y;x',y') = j\omega\mu\left(\frac{4}{W_e L_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos\left(\frac{m\pi x'}{L_e}\right)\cos\left(\frac{n\pi y'}{W_e}\right)}{k_e^2-\left(\frac{m\pi}{L_e}\right)^2-\left(\frac{n\pi}{W_e}\right)^2}\right)\cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

For an arbitrary impressed current excitation inside the cavity, we then have:

$$E_z(x,y) = \int_0^{W_e}\int_0^{L_e} J_z^i(x',y')\,G(x,y;x',y')\,dx'dy'$$

Input Impedance

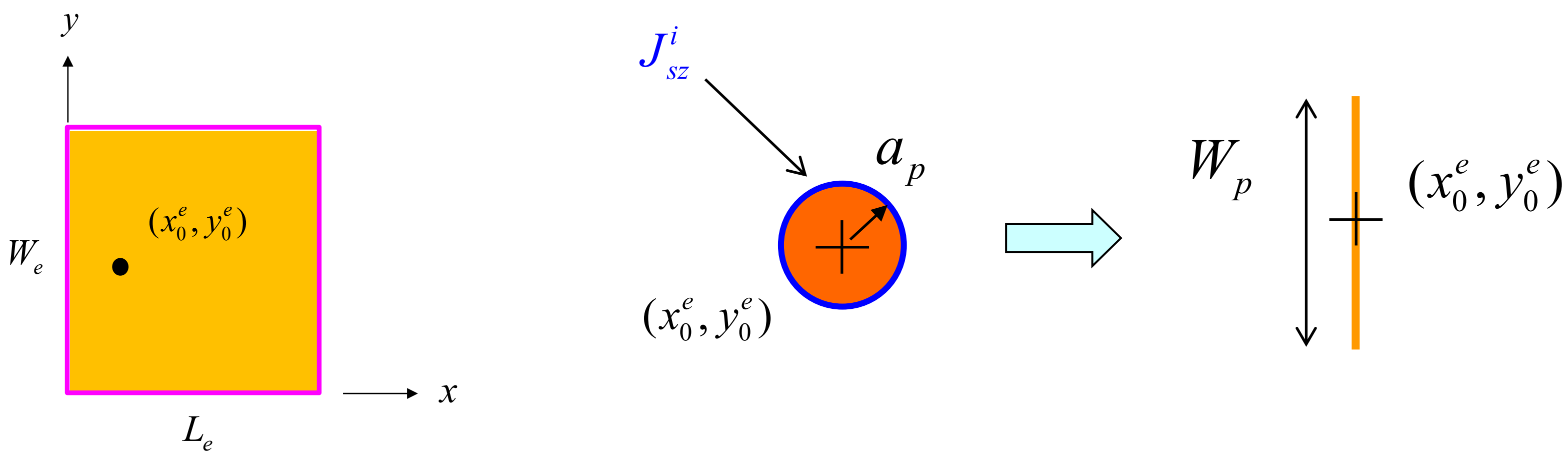
To calculate the input impedance, we need to consider a nonzero radius of the feed probe.



Note: Because the probe is made of PEC, there is a surface current on it.

Input Impedance (cont.)

- We first calculate the electric field E_z inside the patch cavity due to the probe.
- It is convenient to use a strip model of the probe.

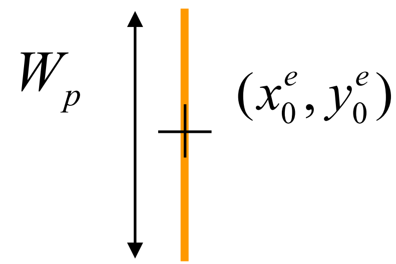


Input Impedance (cont.)

For a “Maxwell” strip current assumption, we have:

$$J_{sz}^{i}\left(y'\right)=\frac{I_0}{\pi\sqrt{\left(\frac{W_p}{2}\right)^2-\left(y'-y_0^e\right)^2}},\qquad y'\in\left(y_0^e-\frac{W_p}{2},\,y_0^e+\frac{W_p}{2}\right)$$

$$W_p=4a_p$$



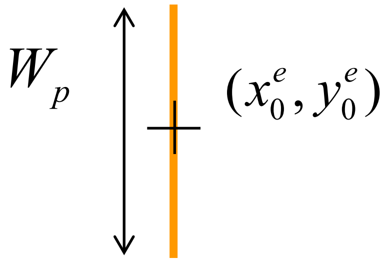
Note: The total probe current is I_0 amps.

Input Impedance (cont.)

For a uniform strip current assumption, we have:

$$J_{sz}^{i}\left(y'\right)=\frac{I_0}{W_p}, \qquad y'\in\left(y_0^e-\frac{W_p}{2},\, y_0^e+\frac{W_p}{2}\right)$$

$$W_p = a_p\, e^{\frac{3}{2}} \doteq 4.482\, a_p$$



Note: The total probe current is I_0 amps.

(We will use this model.)

Input Impedance (cont.)

Field inside cavity due to probe:

$$E_z(x,y)=\int_0^{W_e}\int_0^{L_e} J_z^i(x',y')\,G(x,y;x',y')\,dx'dy'$$ (arbitrary impressed volumetric current in cavity)

$$E_z(x,y)=\int_C J_{sz}^i(x',y')\,dS'$$ (arbitrary impressed surface current on a contour)

$$E_z(x,y)=\int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}} J_{sz}^i(y')\,G\left(x,y;x_0^e,y'\right)dy'$$ (strip current)

$$J_{sz}^i(y')=\frac{I_0}{W_p}$$ (uniform strip current model)

$$G(x,y;x_0^e,y')=j\omega\mu\left(\frac{4}{W_eL_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos\left(\frac{m\pi x_0^e}{L_e}\right)\cos\left(\frac{n\pi y'}{W_e}\right)}{k_e^2-\left(\frac{m\pi}{L_e}\right)^2-\left(\frac{n\pi}{W_e}\right)^2}\right)\cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

Terms that need to be integrated

Input Impedance (cont.)

Integration over the strip current:

$$\int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}} J_{sz}^i(y')\cos\left(\frac{n\pi y'}{W_e}\right)dy' = \int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}} \frac{I_0}{W_p}\cos\left(\frac{n\pi y'}{W_e}\right)dy$$

$$= \frac{I_0}{W_p}\int_{-\frac{W_p}{2}}^{+\frac{W_p}{2}} \cos\left(\frac{n\pi}{W_e}\left[y_0^e + y'\right]\right)dy'$$

Integrates to zero (odd)

$$= \frac{I_0}{W_p}\int_{-\frac{W_p}{2}}^{+\frac{W_p}{2}} \cos\left(\frac{n\pi y_0^e}{W_e}\right)\cos\left(\frac{n\pi y'}{W_e}\right) - \sin\left(\frac{n\pi y_0^e}{W_e}\right)\sin\left(\frac{n\pi y'}{W_e}\right)dy'$$

$$= \frac{I_0}{W_p}\left[\cos\left(\frac{n\pi y_0^e}{W_e}\right)W_p\,\mathrm{sinc}\left(\frac{n\pi W_p}{2W_e}\right)\right]$$

$$\mathrm{sinc}(x) \equiv \frac{\sin x}{x}$$

Input Impedance (cont.)

The field inside the cavity due to the strip probe current is then:

$$E_z(x,y) = j\omega\mu\left(\frac{4}{W_e L_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos\left(\frac{m\pi x_0^e}{L_e}\right)}{k_e^2-\left(\frac{m\pi}{L_e}\right)^2-\left(\frac{n\pi}{W_e}\right)^2}\right)\left[\frac{I_0}{W_p}\cos\left(\frac{n\pi y_0^e}{W_e}\right)W_p\,\text{sinc}\left(\frac{n\pi W_p}{2W_e}\right)\right]\cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

We next use the field inside the cavity to find the input impedance. We first calculate the complex power going into the patch, which is the complex power radiated by the probe current.

$$P_{\text{in}} = -\frac{1}{2}\int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}} E_z\left(x_0^e, y\right) J_{sz}^{i*}\left(y\right)\,dy$$

$$J_{sz}^{i}\left(y\right) = \frac{I_0}{W_p}$$

$$P_{\text{in}} = \frac{1}{2}Z_{\text{in}}\left|I_0\right|^2$$

$$\Rightarrow \quad Z_{\text{in}} = 2\frac{P_{\text{in}}}{\left|I_0\right|^2}$$

Input Impedance (cont.)

$$P_{\text{in}} = -\frac{1}{2} h \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} E_z\left(x_0^e, y\right) J_{sz}^{i*}(y)\, dy \qquad J_{sz}^{i*}(y) = \frac{I_0^*}{W_p}$$

$$E_z(x,y) = j\omega\mu \left(\frac{4}{W_e L_e}\right) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(1+\delta_{m0})(1+\delta_{n0})} \left(\frac{\cos\left(\frac{m\pi x_0^e}{L_e}\right)}{k_e^2 - \left(\frac{m\pi}{L_e}\right)^2 - \left(\frac{n\pi}{W_e}\right)^2} \right) \left[\frac{I_0}{W_p} \cos\left(\frac{n\pi y_0^e}{W_e}\right) W_p \operatorname{sinc}\left(\frac{n\pi W_p}{2W_e}\right)\right] \cos\left(\frac{m\pi x}{L_e}\right) \cos\left(\frac{n\pi y}{W_e}\right)$$

We need this integral:

$$\int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \cos\left(\frac{n\pi y}{W_e}\right)\left(\frac{I_0^*}{W_p}\right) dy = \frac{I_0^*}{W_p} \cos\left(\frac{n\pi y_0^e}{W_e}\right) W_p \operatorname{sinc}\left(\frac{n\pi W_p}{2W_e}\right)$$

Input Impedance (cont.)

The final result is:

$$Z_{\text{in}} = -j\omega\mu h\left(\frac{4}{W_e L_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos^2\left(\dfrac{m\pi x_0^e}{L_e}\right)\cos^2\left(\dfrac{n\pi y_0^e}{W_e}\right)\operatorname{sinc}^2\left(\dfrac{n\pi W_p}{2W_e}\right)}{k_e^2-\left(\dfrac{m\pi}{L_e}\right)^2-\left(\dfrac{n\pi}{W_e}\right)^2}\right)$$

$$W_p = a_p\, e^{\frac{3}{2}} \doteq 4.482\, a_p$$

$$k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}}$$

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r\left(1-jl_{\text{eff}}\right)$$

$$l_{\text{eff}} = 1/Q$$

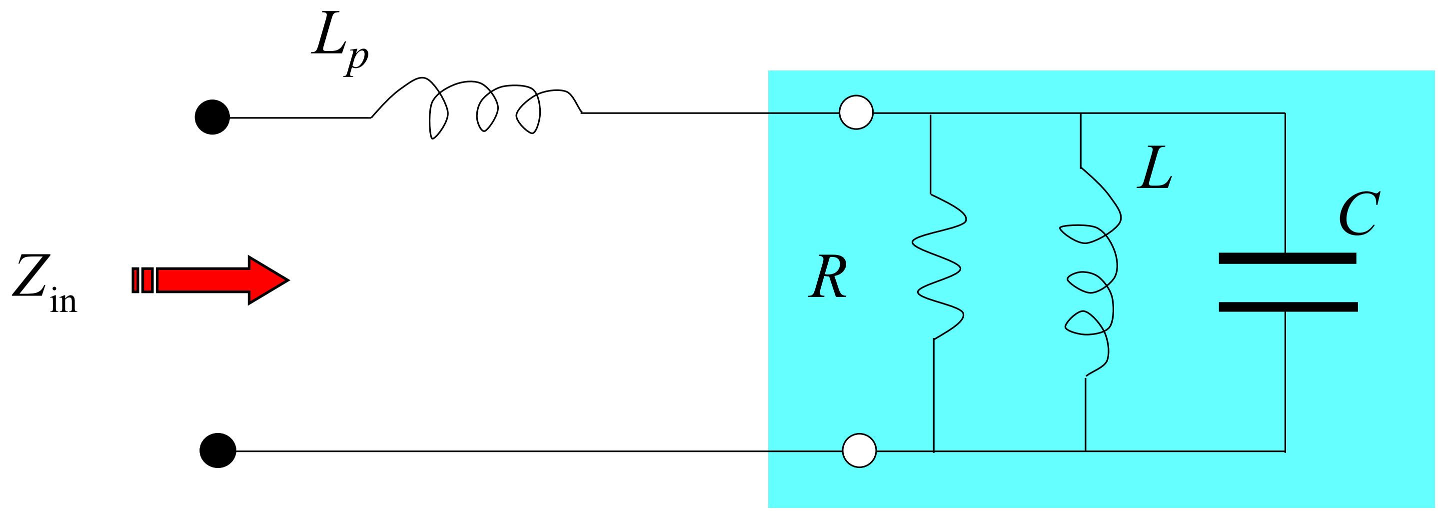
Note:
We cannot assume a probe of zero radius, or else the series will not converge – the input reactance will be infinite.

Circuit Model

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 R}{L}$$

Probe (feed) inductance
(accounts for all modes $(m,n) \neq (1,0)$)



(1,0 mode resonator)

$$Z_{\text{in}} \approx j\omega L_p + \frac{R}{1 + jQ\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)}$$

This formula, which describes the above circuit model, comes from approximating the input impedance formula. (Please see Appendix B.)

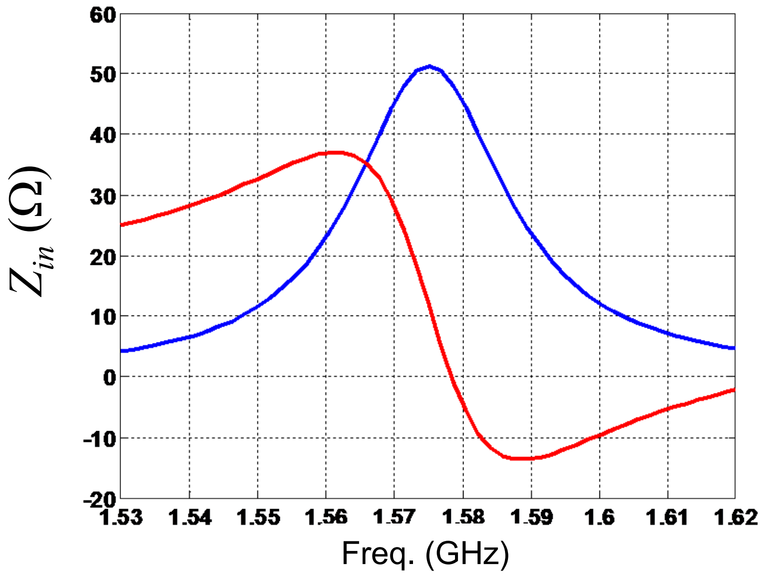
Resonance frequency:

$$\text{Re}\left(k_e^2\right) = \left(\frac{\pi}{L_e}\right)^2$$

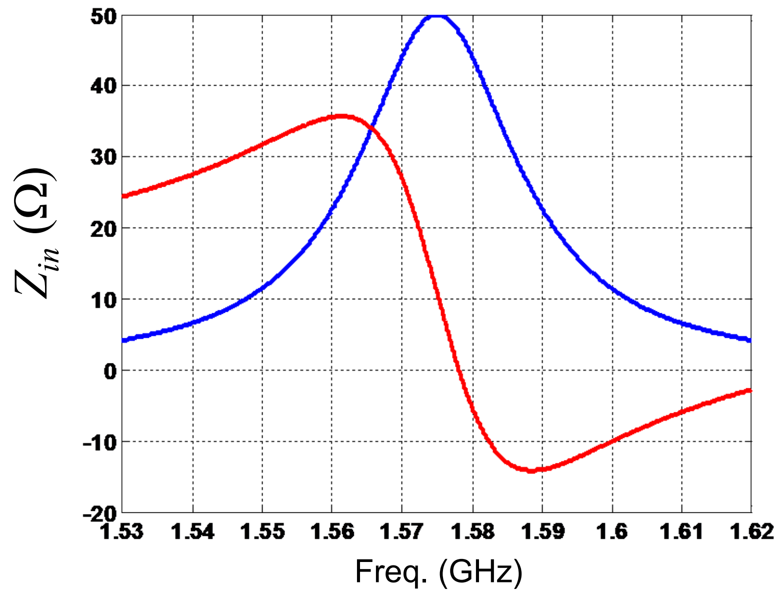
$$\Rightarrow \quad k_0\sqrt{\varepsilon_r} \approx \frac{\pi}{L_e}$$

Results

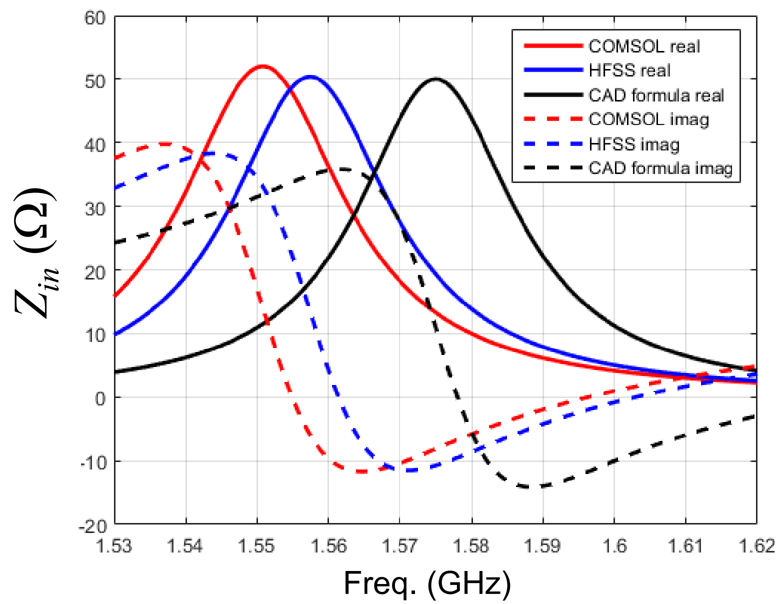
Cavity model (eigenfunction expansion) of patch



CAD Circuit model of patch

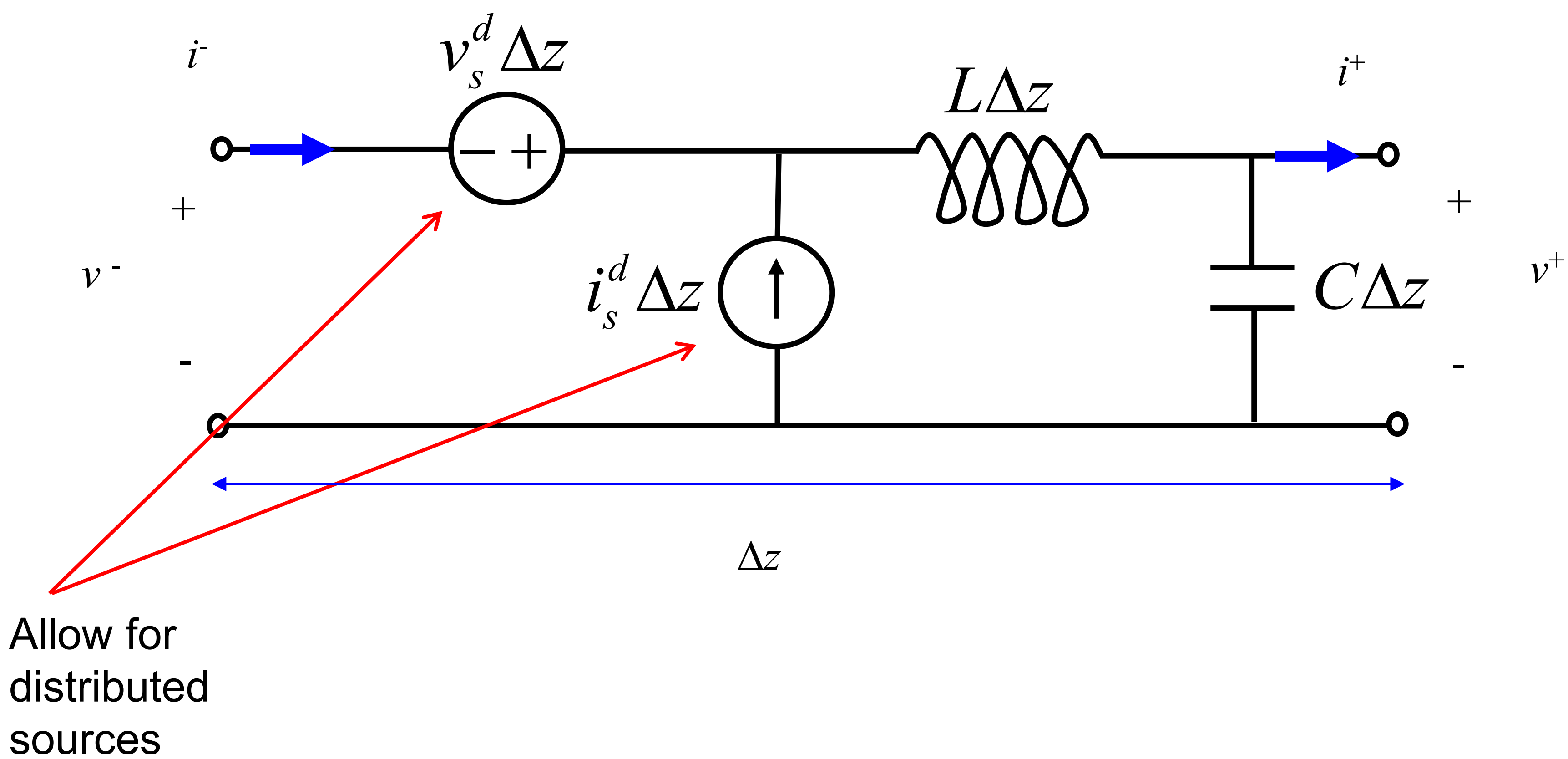


Patch	Feed
$\varepsilon_r = 2.2$	$x_0 = 1.85$ cm
$\tan\delta = 0.001$	$y_0 = W/2$
$h = 1.524$ mm	$a = 0.635$ mm
$L = 6.255$ cm	
$W/L = 1.5$	
$\sigma = 3.0 \times 10^7$ S/m	



Appendix A

This appendix presents a derivation of the telegrapher's equations with distributed sources.

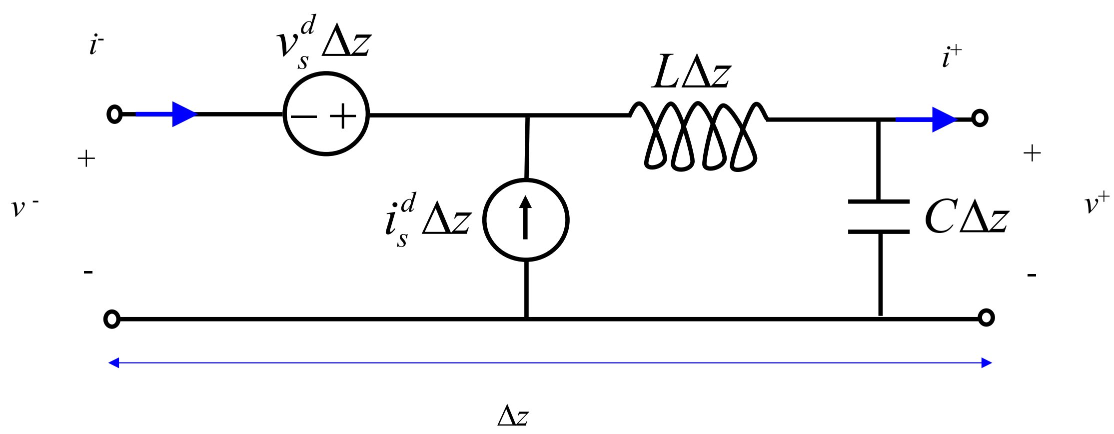


Appendix A (cont.)

$$v^+ - v^- = v_s^d \Delta z - (L\Delta z)\frac{\partial}{\partial t}\left(i^- + i_s^d \Delta z\right)$$

$$\approx \Delta z\left(v_s^d - L\frac{\partial i}{\partial t}\right) \quad \left(\text{Neglect } (\Delta z)^2,\ i^- \approx i\right)$$

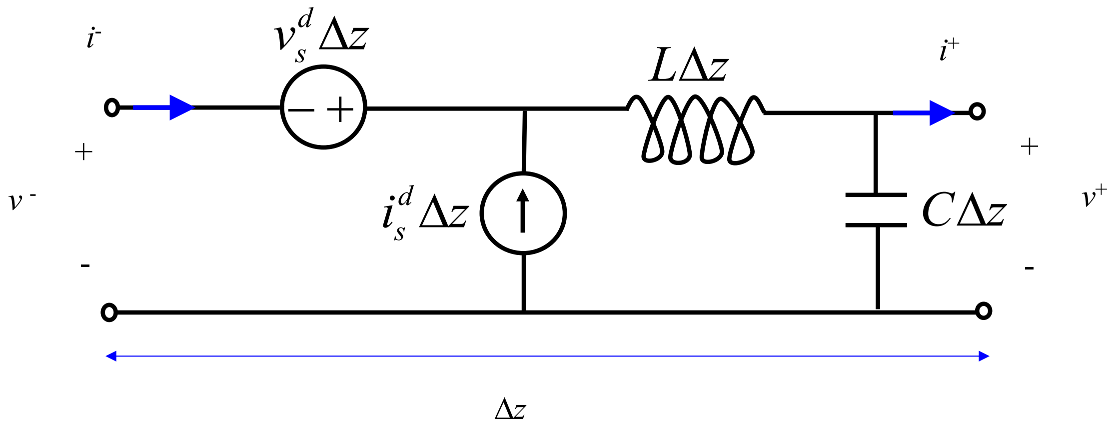
$$\Delta z \to 0: \quad \frac{\partial v}{\partial z} = v_s^d - L\frac{\partial i}{\partial t}$$



Appendix A (cont.)

$$i^{+} - i^{-} = i_s^d \Delta z - \left(C\Delta z\right)\frac{\partial v^{+}}{\partial t}$$

$$\Delta z \to 0: \quad \frac{\partial i}{\partial z} = i_s^d - C\frac{\partial v}{\partial t}$$



Appendix A (cont.)

In the phasor domain:

$$\frac{\partial v}{\partial z} = v_s^d - L\frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial z} = i_s^d - C\frac{\partial v}{\partial t}$$

$$\Downarrow \qquad \frac{\partial}{\partial t} \to j\omega$$

$$\frac{dV}{dz} = V_s^d - j\omega L I$$

$$\frac{dI}{dz} = I_s^d - j\omega C V$$

Appendix B

We can write each (m,n) term of the series for Z_{in} as:

$$Z_{\text{in}}^{m,n} = -j\omega\left(\frac{P_{mn}}{k_e^2 - k_{mn}^2}\right)$$

$$= -j\omega\left(\frac{P_{mn}}{k^2\left(1 - jl_{\text{eff}}\right) - k_{mn}^2}\right)$$

$$= -j\omega\left(\frac{P_{mn}}{\left(k^2 - k_{mn}^2\right) - jk^2 l_{\text{eff}}}\right)$$

$$= \omega\frac{P_{mn}}{k^2 l_{\text{eff}} + j\left(k^2 - k_{mn}^2\right)}$$

where

$$P_{mn} = -j\omega\mu h\left(\frac{4}{W_e L_e}\right)\frac{\cos^2\left(\dfrac{m\pi x_0^e}{L_e}\right)\cos^2\left(\dfrac{n\pi y_0^e}{W_e}\right)\text{sinc}^2\left(\dfrac{n\pi W_p}{2W_e}\right)}{\left(1+\delta_{m0}\right)\left(1+\delta_{n0}\right)}$$

$$k_e^2 = k^2\left(1 - jl_{\text{eff}}\right)$$

$$k^2 = k_0^2 \varepsilon_r$$

$$k_{mn}^2 = \left(\frac{m\pi}{L_e}\right)^2 - \left(\frac{n\pi}{W_e}\right)^2$$

Appendix B (cont.)

or

$$Z_{\text{in}}^{m,n} = \left(\frac{P_{mn}}{k_{mn}^2 l_{\text{eff}}}\right)\left(\frac{\omega}{\dfrac{k^2}{k_{mn}^2} + j\left(\dfrac{1}{l_{\text{eff}}}\right)\left(\dfrac{k^2}{k_{mn}^2} - 1\right)}\right)$$

Next, use:

$$f_{rmn} \equiv \frac{f}{f_{mn}}$$

$$\frac{1}{l_{\text{eff}}} = Q$$

$$\omega = \omega_{mn}\left(\frac{\omega}{\omega_{mn}}\right) = \omega_{mn} f_{rmn}$$

$$\frac{k^2}{k_{mn}^2} = \frac{\omega^2}{\omega_{mn}^2} = f_{rmn}^2$$

Also, define

$$R_{mn} \equiv \left(\frac{P_{mn}}{k_{mn}^2 l_{\text{eff}}}\right)\omega_{mn}$$

$$\Rightarrow \left(\frac{P_{mn}}{k_{mn}^2 l_{\text{eff}}}\right)\omega = \left(\frac{P_{mn}}{k_{mn}^2 l_{\text{eff}}}\right)\omega_{mn}\left(\frac{\omega}{\omega_{mn}}\right) = R_{mn} f_{rmn}$$

Appendix B (cont.)

Then

$$Z_{\text{in}}^{m,n} = R_{mn}\left(\frac{f_{rmn}}{f_{rmn}^2 + jQ\left(f_{rmn}^2 - 1\right)}\right)$$

or

$$Z_{\text{in}}^{m,n} = \left(\frac{R_{mn}}{f_{rmn} + jQ\left(f_{rmn} - \frac{1}{f_{rmn}}\right)}\right)$$

Appendix B (cont.)

For $f_{rmn}^2 \approx 1$, we have:

$$Z_{\text{in}}^{m,n} \approx \frac{R_{mn}}{1 + jQ\left(f_{rmn} - \dfrac{1}{f_{rmn}}\right)}$$

This is the formula for the input impedance of a parallel RLC circuit.

(This justifies the RLC model near resonance.)

All of the non-resonant terms in the infinite series for Z_{in} $(m, n \neq 1,0)$ are slowing varying near the resonance frequency of the $(1,0)$ mode, and are nearly imaginary, and can thus get lumped together into a "probe reactance" term.