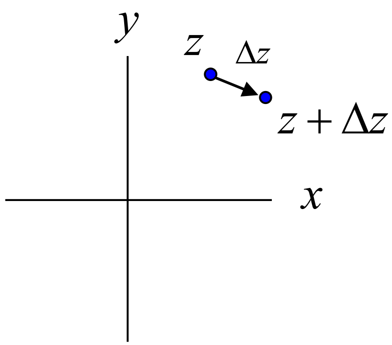
ECE 6382

Fall 2023

David R. Jackson



Notes 2

Differentiation of Functions of a Complex Variable

Notes are adapted from D. R. Wilton, Dept. of ECE

Functions of a Complex Variable

- **Function of a complex variable :** $w = f(z)$

$$z = x + iy, \quad w = u + iv$$

$$w = f(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$$

(e.g., $f(z) = z^2$**,** $u(x,y) = x^2 - y^2,\ v(x,y) = 2xy$ **)**

- **Examples of functions :**

$$w = a + bz + cz^2, \qquad w = A\sinh(\sqrt{z})$$

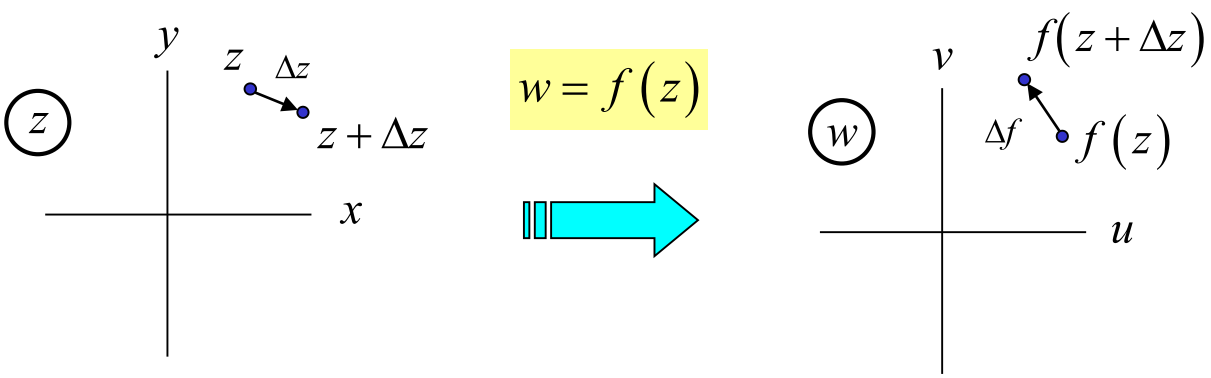
$$w = \frac{a + bz}{c + dz + ez^2}, \qquad w = \sum_{n=0}^{\infty} z^n$$

Differentiation of Functions of a Complex Variable

- **Derivative of a function of a complex variable :**

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

- **To define a unique derivative at a point z, the limit**
  - **must exist at z**
  - **must be independent of the direction of $\Delta z = \arg(\Delta z)$ at z**



The Cauchy – Riemann Conditions

Denote $\Delta z = \Delta x + i\Delta y$ $\quad w = f(z) = u(x,y) + iv(x,y)$

Augustin-Louis Cauchy

- **First, let** $\Delta z = \Delta x$ **:**

$$\frac{df}{dz} = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(z+\Delta x)-f(z)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + i\lim_{\Delta x\to 0}\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$$

$$\Rightarrow \boxed{\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

- **Next let** $\Delta z = i\Delta y$ **:**

$$\frac{df}{dz} = \lim_{\Delta y\to 0}\frac{\Delta f}{i\Delta y} = \lim_{\Delta y\to 0}\frac{f(z+i\Delta y)-f(z)}{i\Delta y}$$

$$= \lim_{\Delta y\to 0}\frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y} + \cancel{i}\lim_{\Delta y\to 0}\frac{v(x,y+\Delta y)-v(x,y)}{\cancel{i}\Delta y}$$

$$\Rightarrow \boxed{\frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

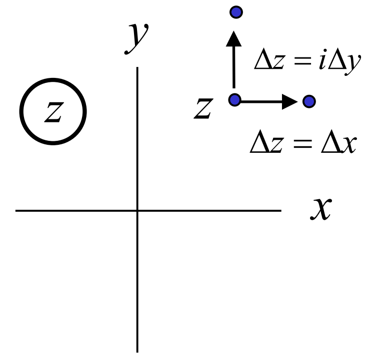
Question: Is $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ **?**

Bernhard Riemann

The Cauchy – Riemann Conditions (cont.)

- **We found**

$$\frac{df}{dz} = \underbrace{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}_{\Delta z = \Delta x} \qquad \frac{df}{dz} = \underbrace{\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}_{\Delta z = i\Delta y}$$



- **For a unique derivative, these expressions must be equal. That is, a *necessary* condition for the existence of a derivative of function of a complex variable is that**

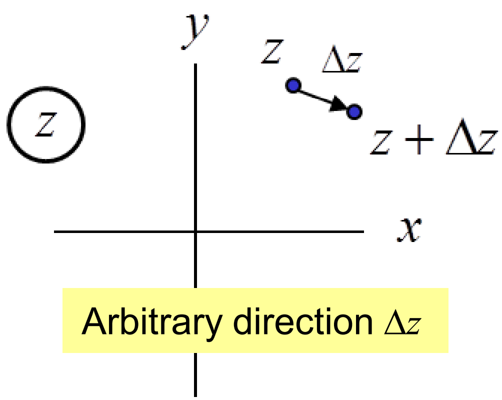
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann equations

- **We've proved that if** $\frac{df}{dz}$ **exists** $\Rightarrow$ **Cauchy - Riemann conditions.**

The Cauchy – Riemann Conditions (cont.)

- **Next, we prove that Cauchy - Riemann conditions $\Rightarrow \dfrac{df}{dz}$ exists (sufficiency):**



$$\frac{\Delta f}{\Delta z}=\frac{\Delta u+i\Delta v}{\Delta z}\approx\frac{\left(\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y\right)+i\left(\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y\right)}{\Delta x+i\Delta y}$$

$$=\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y}{\Delta x+i\Delta y}$$

Use C.R. conditions

$$=\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\Delta y}{\Delta x+i\Delta y}$$

$$=\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(i\Delta y)}{\Delta x+i\Delta y}$$

$$=\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\cancel{(\Delta x+i\Delta y)}}{\cancel{\Delta x+i\Delta y}}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

, independent of $\arg(\Delta z)=\tan^{-1}\dfrac{\Delta y}{\Delta x}$

Total differentials :

$$\Delta u(x,y)\approx\frac{\partial u(x,y)}{\partial x}\Delta x+\frac{\partial u(x,y)}{\partial y}\Delta y$$

$$\Delta v(x,y)\approx\frac{\partial v(x,y)}{\partial x}\Delta x+\frac{\partial v(x,y)}{\partial y}\Delta y$$

The Cauchy – Riemann Conditions (cont.)

Hence, we have the following equivalent statements:

- $\dfrac{df}{dz}$ **exists** $\Leftrightarrow$ **Cauchy - Riemann conditions.**

or

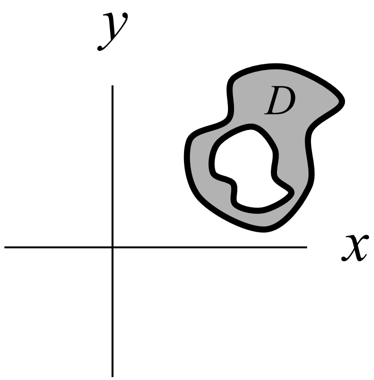
- $\dfrac{df}{dz}$ **exists *if and only if* (iff) the Cauchy - Riemann conditions hold.**

or

- **The Cauchy - Riemann conditions are a *necessary and sufficient* condition for the existence of the derivative** $\dfrac{df}{dz}$ **of a complex variable** f**.**

The Cauchy – Riemann Conditions (cont.)

- We say that a function is "analytic" at a point if the derivative exists there (and at all points in some neighborhood of the point).



- $f(z)$ is said to be "analytic" in a domain D if the derivative exsits at each point in D.

- The theory of complex variables largely exploits the remarkable properties of analytic functions.

- The terms "holomorphic", "regular", and "differentiable" are also used instead of "analytic."

Applying the Cauchy – Riemann Conditions

◊ **Example 1:**

$$f(z) = z = (x+iy) = \overbrace{x}^{u(x,y)} + i\,\overbrace{y}^{v(x,y)}$$

$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y} \quad \checkmark$$

$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} \quad \checkmark$$

$\Rightarrow$ **C.R. conditions hold everywhere (for z finite)**

$\Rightarrow$ z **is analytic everywhere**

◊ **Example 2:**

$$f(z) = z^* = (x+iy)^* = \overbrace{x}^{u(x,y)} + i\,\overbrace{(-y)}^{v(x,y)}$$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \quad ✗$$

$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} \quad \checkmark$$

$\Rightarrow$ **C.R. conditions hold *nowhere***

$\Rightarrow$ z^* **is analytic nowhere**

Applying the Cauchy – Riemann Conditions (cont.)

◊ **Example 3 :**

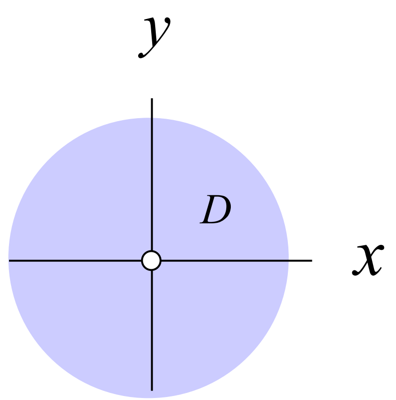
$$f(z)=\frac{1}{z}=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$$

$$=\underbrace{\frac{x}{x^2+y^2}}_{u(x,y)}+i\underbrace{\left(\frac{-y}{x^2+y^2}\right)}_{v(x,y)}$$

$$\frac{\partial u}{\partial x}=\frac{x^2+y^2-2x^2}{\left(x^2+y^2\right)^2}\overset{?}{=}\frac{\partial v}{\partial y}=\frac{-x^2-y^2+2y^2}{\left(x^2+y^2\right)^2} \quad ✓$$

$$\frac{\partial u}{\partial y}=\frac{-2xy}{\left(x^2+y^2\right)^2}=-\frac{\partial v}{\partial x} \quad ✓$$

$\Rightarrow$ **C.R. conditions hold everywhere except** $x=y=0$ $(z=0)$.



$\frac{1}{z}$ is analytic except at $z=0$

$\Rightarrow \; D: |z|>0$

$f(z)$ **is analytic everywhere except at** $z=0$. **The point** $z=0$ **is called a "singularity."**

A singularity is a point where the function is <u>not</u> analytic.

Applying the Cauchy – Riemann Conditions (cont.)

◊ **Example 4 :**

$$f(z) = \sin(z) = \sin(x+iy) = \sin x \cos(iy) + \sin(iy)\,\cos x$$

but $$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2} = \cosh y,$$

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = -i\,\frac{e^{-y} - e^{y}}{2} = i\sinh y$$

so $$\sin(z) = \sin(x+iy) = \underbrace{\sin x \cosh y}_{u(x,y)} + i\,\underbrace{\sinh y\,\cos x}_{v(x,y)}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y}, \checkmark \quad \frac{\partial u}{\partial y} = \sin x \sinh y = -\frac{\partial v}{\partial x} \checkmark$$

$\Rightarrow$ **C.R. conditions hold for all finite** z

Now use : $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cos x \cosh y - i \sin x \sinh y$$

$$= \cos x \cos(iy) - \sin x \sin(iy)$$

$$= \cos(x+iy)$$

$$= \cos z$$

$$\Rightarrow \boxed{f'(z) = \frac{d}{dz}\sin(z) = \cos z}$$

Differentiation Rules (cont.)

◊ Example

$$\frac{d}{dz}\left(z^2\right) = \lim_{\Delta z \to 0} \frac{\left(z+\Delta z\right)^2 - \left(z\right)^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\cancel{\left(z\right)^2} + 2z\Delta z + \left(\Delta z\right)^2 - \cancel{\left(z\right)^2}}{\Delta z}$$

$$= 2z + \lim_{\Delta z \to 0} \Delta z$$

$$= 2z$$

Note: The above "brute-force" derivation, directly using the definition of the derivative, is exactly what is done in usual calculus, with x being used there instead of z.

Differentiation Rules

- **It is relatively simple to prove on a case - by - case basis that practically all formulas for differentiating functions of real variables also apply to the corresponding function of a complex variable :**

$$\frac{d\,z^n}{dz} = nz^{n-1},\ \frac{d\,e^{az}}{dz} = ae^{az},\quad \frac{d\sin z}{dz} = \cos z,\quad \frac{d\cos z}{dz} = -\sin z,\ \text{etc.}$$

$\frac{d}{dz}\left(z^n\right) = nz^{n-1}$ $\Rightarrow$ **every polynomial of degree** $N,\ P_N(z)$**,**
in z **is analytic (differentiable).**

$\Rightarrow$ **every rational function** $\frac{P(z)}{Q(z)}$ **in** z **is analytic** ***except*** **where** $Q(z)$ **vanishes.**

Differentiation Rules

- **Replacing x by z in the usual derivations for functions of a real variable, we find practically all differentiation rules for functions of a complex variable turn out to be identical to those for real variables :**

$$\frac{d\left(f(z)\pm g(z)\right)}{dz}=f'(z)\pm g'(z)$$

$$\frac{d\left(f(z)g(z)\right)}{dz}=f'(z)g(z)+f(z)g'(z)$$

$$\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right)=\frac{g\,f'-f\,g'}{(g)^2}$$

A Theorem Related to z^*

If $f = f(z, z^*)$ is analytic, then

$$\frac{\partial f}{\partial z^*} = 0$$

(An analytic function cannot vary with z^*, and therefore cannot be a function of z^*, except in a trivial way.)

All functions that contain z^* are therefore not analytic, except for some trivial cases (where the function does not really vary with z^*).

A Theorem Related to z^* (cont.)

Examples :

$f(z) = z^*$ **is analytic nowhere, since** $\frac{\partial f}{\partial z^*} = 1 \neq 0$ **(not independent of z^*)**

$f(z) = \sin z^*$ **is not analytic, since** $\frac{\partial f}{\partial z^*} = \cos z^* \neq 0$ (unless $z = (2n+1)\pi/2$)

$f(z) = |z|^2 = zz^*$ **is not analytic, since** $\frac{\partial f}{\partial z^*} = z \neq 0$ (unless $z = 0$)

$f(z) = \sin(z) + \cos(z^*)$ **is not analytic, since** $\frac{\partial f}{\partial z^*} = -\sin(z^*) \neq 0$ (unless $z = n\pi$)

$f(z) = \frac{z^*}{z^*} = 1$ **is analytic everywhere, since** $\frac{\partial f}{\partial z^*} = \frac{\partial}{\partial z^*}(1) = 0$

Proof of z^* Theorem

If $f = f(z, z^*)$ is analytic, then

$$\frac{\partial f}{\partial z^*} = 0$$

(An analytic function cannot vary with z^*, and therefore cannot really be a function of z^*, except in a trivial way.)

- **C.R. conditions :**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Note that

$$z = x + iy, \quad z^* = x - iy$$

Treating z and z^* as independent variables :

$$\frac{\partial u(z, z^*)}{\partial x} = \frac{\partial u}{\partial z}\underbrace{\frac{\partial z}{\partial x}}_{=1} + \frac{\partial u}{\partial z^*}\underbrace{\frac{\partial z^*}{\partial x}}_{=1} \overset{\text{C.R. conds.}}{=} \frac{\partial v(z, z^*)}{\partial y} = \frac{\partial v}{\partial z}\underbrace{\frac{\partial z}{\partial y}}_{=i} + \frac{\partial v}{\partial z^*}\underbrace{\frac{\partial z^*}{\partial y}}_{=-i}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial z} + \frac{\partial u}{\partial z^*} = i\left(\frac{\partial v}{\partial z} - \frac{\partial v}{\partial z^*}\right)} \quad \textbf{(1)}$$

A Theorem Related to z^* (cont.)

Similarly,

$$\frac{\partial v(z,z^*)}{\partial x} = \frac{\partial v}{\partial z}\underbrace{\frac{\partial z}{\partial x}}_{=1} + \frac{\partial v}{\partial z^*}\underbrace{\frac{\partial z^*}{\partial x}}_{=1} \overset{\text{C.R. conds.}}{=} -\frac{\partial u(z,z^*)}{\partial y} = -\frac{\partial u}{\partial z}\underbrace{\frac{\partial z}{\partial y}}_{=i} - \frac{\partial u}{\partial z^*}\underbrace{\frac{\partial z^*}{\partial y}}_{=-i}$$

$$\Rightarrow \quad \boxed{\frac{\partial v}{\partial z} + \frac{\partial v}{\partial z^*} = -i\left(\frac{\partial u}{\partial z} - \frac{\partial u}{\partial z^*}\right)} \qquad \textbf{(2)}$$

Next, consider

$$\frac{\partial f}{\partial z^*} = \frac{\partial u}{\partial z^*} + i\frac{\partial v}{\partial z^*} = \overbrace{-i\left(\cancel{\frac{\partial v}{\partial z}} + \frac{\partial v}{\partial z^*}\right) + \cancel{\frac{\partial u}{\partial z}}}^{\text{from (2)}} + \overbrace{i\cancel{\frac{\partial v}{\partial z}} - \left(\cancel{\frac{\partial u}{\partial z}} + \frac{\partial u}{\partial z^*}\right)}^{\text{from (1)}}$$

$$= -\left(\frac{\partial u}{\partial z^*} + i\frac{\partial v}{\partial z^*}\right) = -\frac{\partial f}{\partial z^*} \quad \Rightarrow \frac{\partial f}{\partial z^*} = 0$$

$\Rightarrow f$ is independent of z^*

Entire Functions

A function that is analytic everywhere in the finite* complex plane is called "entire".

- **Typical functions that are entire (analytic everywhere in the finite complex plane) :**

$$1,\ z,\ z^2,\ z^3,\ z^4, z^5, \cdots, z^n, \cdots$$

$$e^z, \sin z,\ \cos z, \sinh z,\ \cosh z$$

- **Typical functions analytic *almost* everywhere :**

$$\frac{1}{z^2},\ \frac{1}{z^2-1},\ \ z^{1/2},\ \tan z,\ \cot z,\ \tanh z,\ \coth z$$

* A function is said to be analytic everywhere in the finite complex plane if it is analytic everywhere except possibly at infinity.

Analytic at infinity: Let $w = 1/z$ Is the function analytic at $w = 0$?

Combinations of Analytic Functions

Combinations of functions:

- ***Finite linear combinations* of analytic functions are analytic :**

  If $f(z), g(z), h(z)$ **are analytic**

  $\Rightarrow \; a f(z) + b\, g(z) + c\, h(z)$ **is analytic**

- ***Composite combinations* of analytic functions are analytic :**

  If $f(z), g(z)$ **are analytic**

  $\Rightarrow \; f(g(z))$ **is analytic**

Combinations of Analytic Functions (cont.)

Infinite series:

- ***Infinite series* may be :**
  - **- Analytic everywhere**
  - **- Analytic somwhere**

The “somewhere” might depend on the form used to represent the function.

Example:

$$f(z) = \frac{1}{1-z}$$

$$f(z) = 1 + z + z^2 + z^3 + \ldots, \quad |z| < 1$$

The first form is analytic everywhere except $z = 1$.

The second form is analytic for $|z| < 1$ (the series does not converge on or outside the unit circle).

Combination of Analytic Functions (cont.)

Examples

Composite functions of analytic functions are also analytic.

$$f(z) = z^2$$

$$g(z) = \sin z$$

$$h(z) = g(f(z)) = \sin(z^2) \quad \text{analytic}$$

Derivatives of analytic functions are also analytic (proof given later).

$$f(z) = \sin z$$

$$f'(z) = \cos z \quad \text{analytic}$$

Derivatives of Analytic Function

Important theorem (proven later)

The derivative of an analytic function is also analytic.

$f(z)$ is analytic

$\Downarrow$

$f'(z)$ is analytic

$\Downarrow$

$f''(z)$ is analytic

$\Downarrow$

$\vdots$

Hence, all derivatives of an analytic function are also analytic.

Real and Imaginary Parts of Analytic Functions Are Harmonic Functions

Assume an analytic function: $f(z) = u(x,y) + iv(x,y)$

$$\Rightarrow \quad \nabla^2 u(x,y) = \nabla^2 v(x,y) = 0$$

The functions u and v are harmonic (i.e., they satisfy Laplace's equation)

Notation: $\begin{cases} u = u(z) = u(x,y) \\ v = v(z) = v(x,y) \end{cases}$

This result is extensively used in conformal mapping to solve electrostatics and other problems involving the 2D Laplace equation (discussed later).

Pierre-Simon Laplace

Real and Imaginary Parts of Analytic Functions Are Harmonic Functions (cont.)

Proof

f *is analytic* $\Rightarrow$ df/dz is also analytic (see slide 23)

Analytic $\Rightarrow \dfrac{df}{dz} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y}$ (Notation: $f(z) = u(x,y) + iv(x,y)$)

Denote $f'(z) \equiv U + iV$

We have : $U(x,y) = \mathrm{Re}\left(f'(z)\right) = \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$; $\cancel{i}V(x,y) = \cancel{i}\dfrac{\partial v}{\partial x} = -\cancel{i}\dfrac{\partial u}{\partial y}$

Apply the C.R. conditions to $f'(z)$:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \boxed{\nabla^2 u = 0}$$

$$\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y} \Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \boxed{\nabla^2 v = 0}$$

Real and Imaginary Parts of Analytic Functions Are Harmonic Functions (cont.)

Example: $w = f(z) = z^2$

$$w = u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$$

$$\Rightarrow \begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \qquad \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 + 0 = 0$$

Real and Imaginary Parts of Analytic Functions Are Harmonic Functions (cont.)

Example: $w = f(z) = \sin(z)$

$$w = u + iv = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

$$\Rightarrow \begin{cases} u(x,y) = \sin x \cosh y \\ v(x,y) = \cos x \sinh y \end{cases}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \cosh y + \sin x \cosh y = 0$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\cos x \sinh y + \cos x \sinh y = 0$$