Differentiation of Functions of a Complex Variable

Notes are adapted from D. R. Wilton, Dept. of ECE
Function of a complex variable: \( w = f(z) \)

\[ z = x + iy, \quad w = u + iv \]

\[ w = f(z) = u(x, y) + iv(x, y) \]

(e.g., \( f(z) = z^2 \), \( u(x, y) = x^2 - y^2 \), \( v(x, y) = 2xy \))

Examples:

\[ w = a + bz + cz^2, \quad w = A \sinh(\sqrt{z}) \]

\[ w = \frac{a + bz}{c + dz + ez^2}, \quad w = \sum_{n=-\infty}^{n=\infty} a_n z^n \]
Differentiation of Functions of a Complex Variable

- **Derivative of a function of a complex variable:**

\[ f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \]

- To define a unique derivative at a point \( z \), the limit
  - must exist at \( z \)
  - must be independent of the direction of \( \Delta z = \arg(\Delta z) \) at \( z \)

In the diagram, \( z \) and \( z + \Delta z \) are points in the complex plane, and the derivative is illustrated by the change in the function \( f \) as \( \Delta z \) approaches zero.
The Cauchy – Riemann Conditions

- **Define** $\Delta z = \Delta x + i\Delta y$
- **First, let** $\Delta z = \Delta x$:
  \[
  \frac{df}{dz} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x}
  = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}
  \]

  \[
  \Rightarrow \quad \frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
  \]

- **Next let** $\Delta z = i\Delta y$:
  \[
  \frac{df}{dz} = \lim_{\Delta y \to 0} \frac{\Delta f}{i\Delta y} = \lim_{\Delta y \to 0} \frac{f(z + i\Delta y) - f(z)}{i\Delta y}
  = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\lambda \Delta y}
  \]

  \[
  \Rightarrow \quad \frac{df}{dz} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}
  \]

**Question:** Is $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$?
The Cauchy – Riemann Conditions (cont.)

- We found
  \[
  \frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \Delta z = \Delta x
  \]
  \[
  \frac{df}{dz} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} = \Delta z = i \Delta y
  \]

- For a unique derivative, these expressions must be equal. That is, a necessary condition for the existence of a derivative of function of a complex variable is that
  \[
  \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}
  \]
  Cauchy-Riemann equations

- We've proved that if \( \frac{df}{dz} \) exists \( \Rightarrow \) Cauchy - Riemann conditions.
Next, we prove that Cauchy-Riemann conditions exist (sufficiency):

$$\frac{df}{dz} = \frac{\Delta f}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta z} = \frac{\left( \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \right) + i \left( \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y \right)}{\Delta x + i\Delta y}$$

Use C.R. conditions:

$$\Delta u(x, y) = \frac{\partial u(x, y)}{\partial x} \Delta x + \frac{\partial u(x, y)}{\partial y} \Delta y$$
$$\Delta v(x, y) = \frac{\partial v(x, y)}{\partial x} \Delta x + \frac{\partial v(x, y)}{\partial y} \Delta y$$

Total differentials:

$$\Delta u(x, y) = \frac{\partial u(x, y)}{\partial x} \Delta x + \frac{\partial u(x, y)}{\partial y} \Delta y$$
$$\Delta v(x, y) = \frac{\partial v(x, y)}{\partial x} \Delta x + \frac{\partial v(x, y)}{\partial y} \Delta y$$

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \text{ independent of } \arg(\Delta z) = \tan^{-1} \frac{\Delta y}{\Delta x}$$
The Cauchy – Riemann Conditions (cont.)

Hence we have the following equivalent statements:

- \( \frac{df}{dz} \) exists \( \iff \) Cauchy - Riemann conditions.

  or

- \( \frac{df}{dz} \) exists if and only if (iff) the Cauchy - Riemann conditions hold.

  or

- The Cauchy - Riemann conditions are a necessary and sufficient condition for the existence of the derivative \( \frac{df}{dz} \) of a complex variable \( f \).
We say that a function is "analytic" at a point if the derivative exists there (and at all points in some neighborhood of the point).

\[ f(z) \] is said to be "analytic" in a domain \( D \) if the derivative exists at each point in \( D \).

The theory of complex variables largely exploits the remarkable properties of analytic functions.

The terms "holomorphic", "regular", and "differentiable" are also used instead of "analytic."
Applying the Cauchy – Riemann Conditions

◊ Example 1:

\[ f(z) = z = (x + iy) = \frac{u(x,y)}{x} + i \frac{v(x,y)}{y} \]

\[ \frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y} \quad \checkmark \]

\[ \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} \quad \checkmark \]

⇒ C.R. conditions hold everywhere (for \( z \) finite)

⇒ \( z \) is analytic everywhere

◊ Example 2:

\[ f(z) = z^* = (x + iy)^* = \frac{u(x,y)}{x} + i \frac{v(x,y)}{(-y)} \]

\[ \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \quad \times \]

⇒ C.R. conditions hold nowhere

⇒ \( z^* \) is analytic nowhere
Applying the Cauchy – Riemann Conditions (cont.)

Example 3:

\[ f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} \]

\[ = \left( \frac{x}{x^2 + y^2} \right) + i \left( \frac{-y}{x^2 + y^2} \right) \]

\[ \frac{\partial u}{\partial x} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} \quad \Rightarrow \quad \frac{\partial v}{\partial y} = \frac{-x^2 - y^2 + 2y^2}{(x^2 + y^2)^2} \]

\[ \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \text{C.R. conditions hold everywhere except } x = y = 0 \ (z = 0). \]

\[ f(z) \] is analytic everywhere except at \( z = 0 \). The point \( z = 0 \) is called a "singularity."

A singularity is a point where the function is not analytic.
Applying the Cauchy – Riemann Conditions (cont.)

◊ Example 4:

\[ f(z) = \sin(z) = \sin(x + iy) = \sin x \cos(iy) + \sin(iy) \cos x \]

but \[ \cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2} = \cosh y, \]

\[ \sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = -i \frac{e^{-y} - e^{y}}{2} = i \sinh y \]

so \[ \sin(z) = \sin(x + iy) = \underbrace{\sin x \cosh y}_{u(x,y)} + i \underbrace{\sinh y \cos x}_{v(x,y)} \]

\[ \Rightarrow \frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y}, \quad \checkmark \]
\[ \frac{\partial u}{\partial y} = \sin x \sinh y = -\frac{\partial v}{\partial x} \quad \checkmark \]

\[ \Rightarrow \text{C.R. conditions hold for all finite } z \]

Now use: \[ f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \cos x \cosh y - i \sin x \sinh y \]

\[ = \cos x \cos(iy) - \sin x \sin(iy) \]
\[ = \cos(x + iy) \]
\[ = \cos z \]

\[ \Rightarrow f'(z) = \frac{d}{dz} \sin(z) = \cos z \]
Replacing $x$ by $z$ in the usual derivations for functions of a real variable, we find practically all differentiation rules for functions of a complex variable turn out to be identical to those for real variables:

\[
\frac{d}{dz}\left(f(z) \pm g(z)\right) = f'(z) \pm g'(z)
\]

\[
\frac{d}{dz}\left(f(z)g(z)\right) = f'(z)g(z) + f(z)g'(z)
\]

\[
\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{gf' - fg'}{(g)^2}
\]
differentiation rules (cont.)

- It is relatively simple to prove on a case-by-case basis that practically all formulas for differentiating functions of real variables also apply to the corresponding function of a complex variable:

\[
\frac{d}{dz} z^n = nz^{n-1}, \quad \frac{d}{dz} e^{az} = ae^{az}, \quad \frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z, \quad \text{etc.}
\]

\[
\frac{d}{dz} (z^n) = nz^{n-1} \quad \Rightarrow \quad \text{every polynomial of degree } N, \quad P_N(z), \quad \text{in } z \text{ is analytic (differentiable)}.
\]

\[
\Rightarrow \quad \text{every rational function } \frac{P(z)}{Q(z)} \text{ in } z \text{ is analytic except where } Q(z) \text{ vanishes.}
\]
Diamond Example

\[ \frac{d}{dz} \left( z^2 \right) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - (z)^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\left( z \right)^2 + 2z\Delta z + (\Delta z)^2 - \left( z \right)^2}{\Delta z} \]

\[ = 2z + \lim_{\Delta z \to 0} \Delta z \]

\[ = 2z \]
A Theorem Related to $z^*$

- **C.R. conditions:**

  \[
  \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}
  \]

  If $f = f(z,z^*)$ is analytic, then

  \[
  \frac{\partial f}{\partial z^�} = 0
  \]

  (The function cannot really vary with $z^*$.)

  **Note that**

  \[z = x + iy, \quad z^* = x - iy \quad \Rightarrow \quad x = \frac{z + z^*}{2}, \quad y = \frac{z - z^*}{2i}\]

  Make the above substitution for $x$ and $y$ and treat $z$ and $z^*$ as independent variables:

  \[
  \frac{\partial u(z,z^*)}{\partial x} = \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial u}{\partial z^*} \frac{\partial z^*}{\partial x} = \frac{\partial v(z,z^*)}{\partial y} = \frac{\partial v}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial v}{\partial z^*} \frac{\partial z^*}{\partial y}
  \]

  \[
  = i \left( \frac{\partial v}{\partial z} - \frac{\partial v}{\partial z^*} \right)
  \]

  \[
  \Rightarrow \quad \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z^*} = i \left( \frac{\partial v}{\partial z} - \frac{\partial v}{\partial z^*} \right)
  \]

  \[
  (1)
  \]
Similarly,
\[
\frac{\partial v(z, z^*)}{\partial x} = \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial v}{\partial z^*} \frac{\partial z^*}{\partial x} = \frac{1}{1} + \frac{1}{1} \quad \text{C.R. consds.}
\]
\[
\frac{\partial u(z, z^*)}{\partial y} = -\frac{\partial u}{\partial z} \frac{\partial z}{\partial y} - \frac{\partial u}{\partial z^*} \frac{\partial z^*}{\partial y} = i - i
\]
\[
\Rightarrow \frac{\partial v}{\partial z} + \frac{\partial v}{\partial z^*} = -i \left( \frac{\partial u}{\partial z} - \frac{\partial u}{\partial z^*} \right) \tag{2}
\]

Next, consider
\[
\frac{\partial f}{\partial z^*} = \frac{\partial u}{\partial z^*} + i \frac{\partial v}{\partial z^*} = -i \left( \frac{\partial v}{\partial z} + \frac{\partial v}{\partial z^*} \right) + \frac{\partial u}{\partial z} + i \frac{\partial v}{\partial z} - \left( \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z^*} \right)
\]
\[
= - \left( \frac{\partial u}{\partial z^*} + i \frac{\partial v}{\partial z^*} \right) = -\frac{\partial f}{\partial z^*} \quad \Rightarrow \frac{\partial f}{\partial z^*} = 0
\]
\[
\Rightarrow f \text{ is independent of } z^*
\]
Examples:

\[ f(z) = z^* \quad \text{is analytic nowhere since } \quad \frac{\partial f}{\partial z^*} = 1 \neq 0 \quad \text{(not independent of } z^*) \]

\[ f(z) = \sin z^* \quad \text{is not analytic since } \quad \frac{\partial f}{\partial z^*} = \cos z^* \neq 0 \quad \text{(unless } z = (2n + 1)\pi / 2) \]

\[ f(z) = |z|^2 = zz^* \quad \text{is not analytic since } \quad \frac{\partial f}{\partial z^*} = z \neq 0 \quad \text{(unless } z = 0) \]
- **Typical functions that are analytic everywhere (in the finite complex plane):**
  
  \[ 1, z, z^2, z^3, z^4, z^5, \ldots, z^n, \ldots \]

  \[ e^z, \sin z, \cos z, \sinh z, \cosh z \]

  A function that is analytic everywhere is called “entire”.

- **Typical functions analytic almost everywhere:**
  
  \[ \frac{1}{z^2}, \frac{1}{z^2 - 1}, z^{1/2}, \tan z, \cot z, \tanh z, \coth z \]

- **Finite linear combinations** of analytic functions are analytic:
  
  If \( f(z), g(z), h(z) \) are analytic
  
  \[ \Rightarrow \, a f(z) + b g(z) + c h(z) \text{ analytic} \]

- **Infinite linear combinations (series)** of analytic functions may be:
  
  - Analytic everywhere
  - Analytic nowhere
  - Analytic almost everywhere
Important theorem (proven later)

The derivative of an analytic function is also analytic.

\[ f(z) \text{ is analytic} \]
\[ \downarrow \]
\[ f'(z) \text{ is analytic} \]
\[ \downarrow \]
\[ f''(z) \text{ is analytic} \]
\[ \downarrow \]
\[ f'''(z) \text{ is analytic} \]

Hence, all derivatives of an analytic function are also analytic.
Analytic Functions

Examples

Composite functions of analytic functions are also analytic.

\[ f(z) = z^2 \]
\[ g(z) = \sin z \]
\[ h(z) = g(f(z)) = \sin(z^2) \]

Derivatives of analytic functions are also analytic.

\[ f'(z) = \sin z \]
\[ f''(z) = \cos z \]
Real and Imaginary Parts of Analytic Functions Are Harmonic Functions

Assume an analytic function: \( f(z) = u + iv \)

\[ \nabla^2 u(x, y) = \nabla^2 v(x, y) = 0 \]

The functions \( u \) and \( v \) are harmonic (i.e., they satisfy Laplace’s equation)

Notation:
\[
\begin{align*}
  u &= u(z) = u(x, y) \\
  v &= v(z) = v(x, y)
\end{align*}
\]

This result is extensively used in conformal mapping to solve electrostatics and other problems involving the 2D Laplace equation (discussed later).
Real and Imaginary Parts of Analytic Functions Are Harmonic Functions

Proof

\[ f \text{ is analytic} \Rightarrow df / dz \text{ is also analytic (see note on slide 19)} \]

Analytic \[ \Rightarrow \frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \equiv U(x, y) + iV(x, y) \]

where \( f'(z) \equiv U + iV \)

We have : \( U(x, y) \equiv \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad iV(x, y) \equiv i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} \)

Apply the C.R. conditions to \( f'(z) \):

\[ \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad \nabla^2 u = 0 \]

\[ \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y} \Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2} \quad \Rightarrow \quad \nabla^2 v = 0 \]
Real and Imaginary Parts of Analytic Functions Are Harmonic Functions

Example: \[ w = f(z) = z^2 \]

\[ w = u + iv = (x + iy)^2 \]

\[ \begin{cases} 
    u = x^2 - y^2 \\
    v = 2xy 
\end{cases} \]

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \]

\[ \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 + 0 = 0 \]