ECE 6382

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Notes 20

Bessel Functions

Notes are from D. R. Wilton, Dept. of ECE

Cylindrical Wave Functions

Helmholtz equation:

$$\nabla^2 \psi + k^2 \psi = 0$$

In cylindrical coordinates:

$$\left(\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + k^2 \psi = 0$$

Separation of variables:

$$\psi\left(\rho, \phi, z\right) = R\left(\rho\right) \Phi(\phi) Z(z)$$

Substitute into previous equation and divide by ψ.

Cylindrical Wave Functions (cont.)

$$\left(\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}\right)+k^2\psi=0$$

$$\psi(\rho,\phi,z)=R(\rho)\Phi(\phi)Z(z)$$

$$\left(\frac{d^2R}{d\rho^2}\Phi Z+\frac{1}{\rho}\frac{dR}{d\rho}\Phi Z+\frac{1}{\rho^2}\frac{d^2\Phi}{d\phi^2}RZ+\frac{d^2Z}{dz^2}R\Phi\right)+k^2R\Phi Z=0$$

Divide by ψ

$$\left(\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}+\frac{Z''}{Z}\right)+k^2=0$$

Cylindrical Wave Functions (cont.)

$$\left(\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}+\frac{Z''}{Z}\right)+k^2=0 \qquad (1)$$

Therefore

$$\underbrace{\frac{Z''}{Z}}_{f(z)}=\underbrace{-k^2-\frac{R''}{R}-\frac{1}{\rho}\frac{R'}{R}-\frac{1}{\rho^2}\frac{\Phi''}{\Phi}}_{g(\rho,\phi)}$$

Hence,

$$f(z)=\text{constant}\equiv -k_z^2$$

Cylindrical Wave Functions (cont.)

Hence

$$\frac{Z''}{Z} = -k_z^2$$

$$Z(z) = h(k_z z) = \left\{ e^{\pm i k_z z}, \sin(k_z z), \cos(k_z z) \right\}$$

Next, to isolate the ϕ -dependent term, multiply Eq. (1) by ρ^2 :

$$\rho^2 \left(\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} - k_z^2 \right) + k^2 \rho^2 = 0$$

Cylindrical Wave Functions (cont.)

Hence

$$\frac{\Phi''}{\Phi} = \rho^2 \left(-k^2 + k_z^2 - \frac{1}{\rho}\frac{R'}{R} - \frac{R''}{R} \right) \qquad (2)$$

$$\underbrace{\frac{\Phi''}{\Phi}}_{f(\phi)} = \underbrace{\rho^2 \left(-k^2 + k_z^2 - \frac{1}{\rho}\frac{R'}{R} - \frac{R''}{R} \right)}_{g(\rho)}$$

Hence

$$\frac{\Phi''}{\Phi} = \text{constant} \equiv -\upsilon^2$$

Note:
If ϕ is allowed to change by 2π, then υ must be an integer.

so

$$\Phi = h(\upsilon\phi) = \left\{ e^{\pm i\upsilon\phi}, \sin(\upsilon\phi), \cos(\upsilon\phi) \right\}$$

Cylindrical Wave Functions (cont.)

From Eq. (2) we then have

$$-\upsilon^2 = \rho^2 \left(-k^2 + k_z^2 - \frac{1}{\rho}\frac{R'}{R} - \frac{R''}{R} \right)$$

The next goal is to solve this equation for $R(\rho)$.

First, multiply by R and collect terms:

$$\rho^2 R'' + \rho R' + \rho^2 \left(k^2 - k_z^2 \right) R - \upsilon^2 R = 0$$

Cylindrical Wave Functions (cont.)

Define $$k_\rho^{\,2} \equiv k^2 - k_z^{\,2}$$

Then, $$\rho^2 R'' + \rho R' + \left[\left(k_\rho \rho \right)^2 - \upsilon^2 \right] R = 0$$

Next, define $$\begin{cases} x = k_\rho \rho \\ R(\rho) = y(x) \end{cases}$$

Note that $$R'(\rho) = \frac{dR}{d\rho} = \frac{dy}{dx}\frac{dx}{d\rho} = y'(x) k_\rho$$

and $$R''(p) = y''(x) k_\rho^2$$

Cylindrical Wave Functions (cont.)

Then we have $$x^2 y'' + xy' + \left[x^2 - \upsilon^2 \right] y = 0$$

Bessel equation of order υ

Two independent solutions: $J_\upsilon(x), Y_\upsilon(x)$

Hence

$$y(x) = AJ_\upsilon(x) + BY_\upsilon(x)$$

Therefore

$$R(\rho) = \left\{ J_\upsilon(k_\rho \rho),\ Y_\upsilon(k_\rho \rho) \right\}$$

Cylindrical Wave Functions (cont.)

Summary

$$\nabla^2 \psi + k^2 \psi = 0$$

$$\psi(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z)$$

$$Z(z) = \left\{ e^{\pm i k_z z}, \ \sin(k_z z), \ \cos(k_z z) \right\}$$

$$\Phi = \left\{ e^{\pm i \upsilon \phi}, \ \sin(\upsilon \phi), \ \cos(\upsilon \phi) \right\}$$

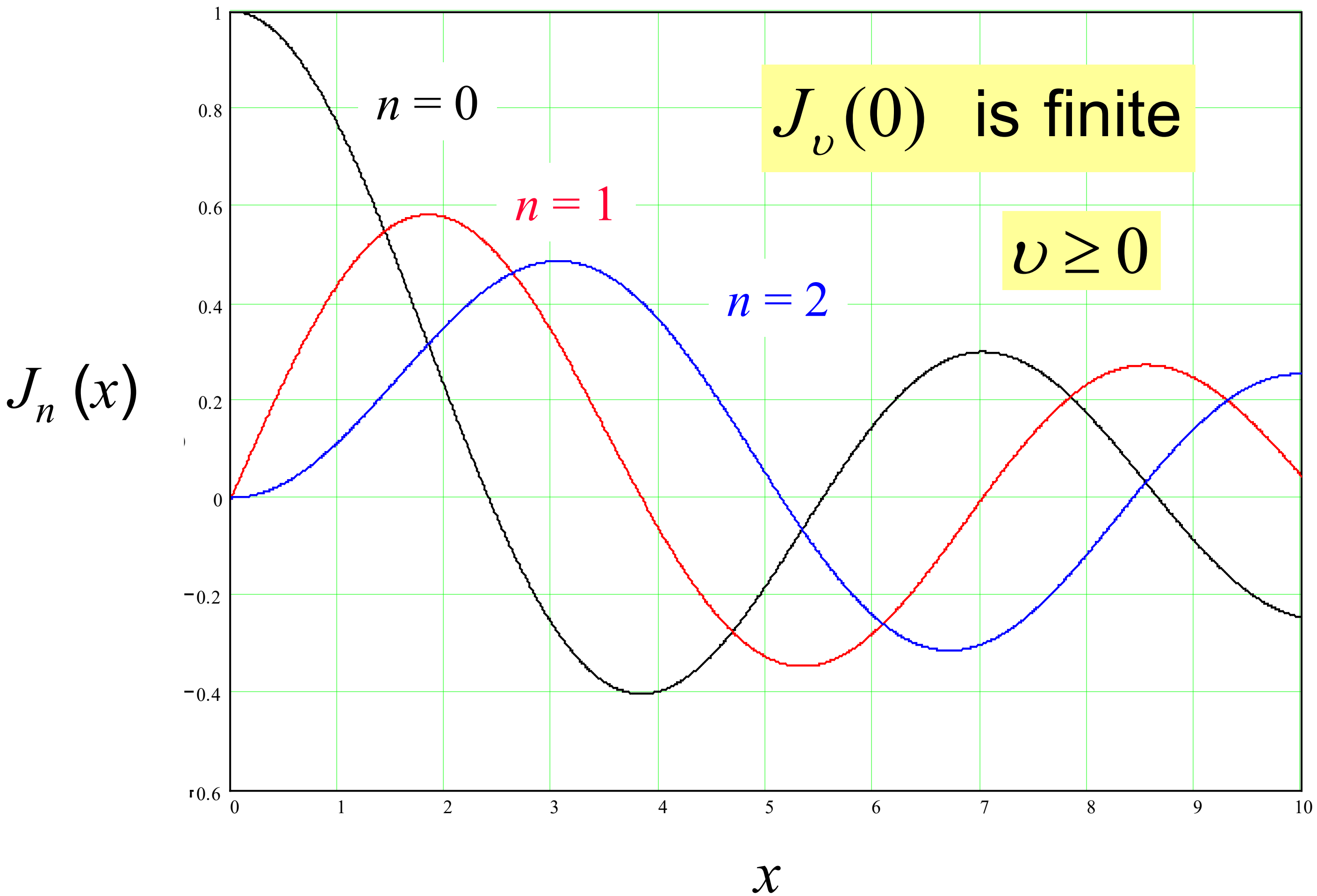
$$R(\rho) = \left\{ J_\upsilon(k_\rho \rho), \ Y_\upsilon(k_\rho \rho) \right\}$$

$$k_\rho^2 + k_z^2 = k^2$$

References for Bessel Functions

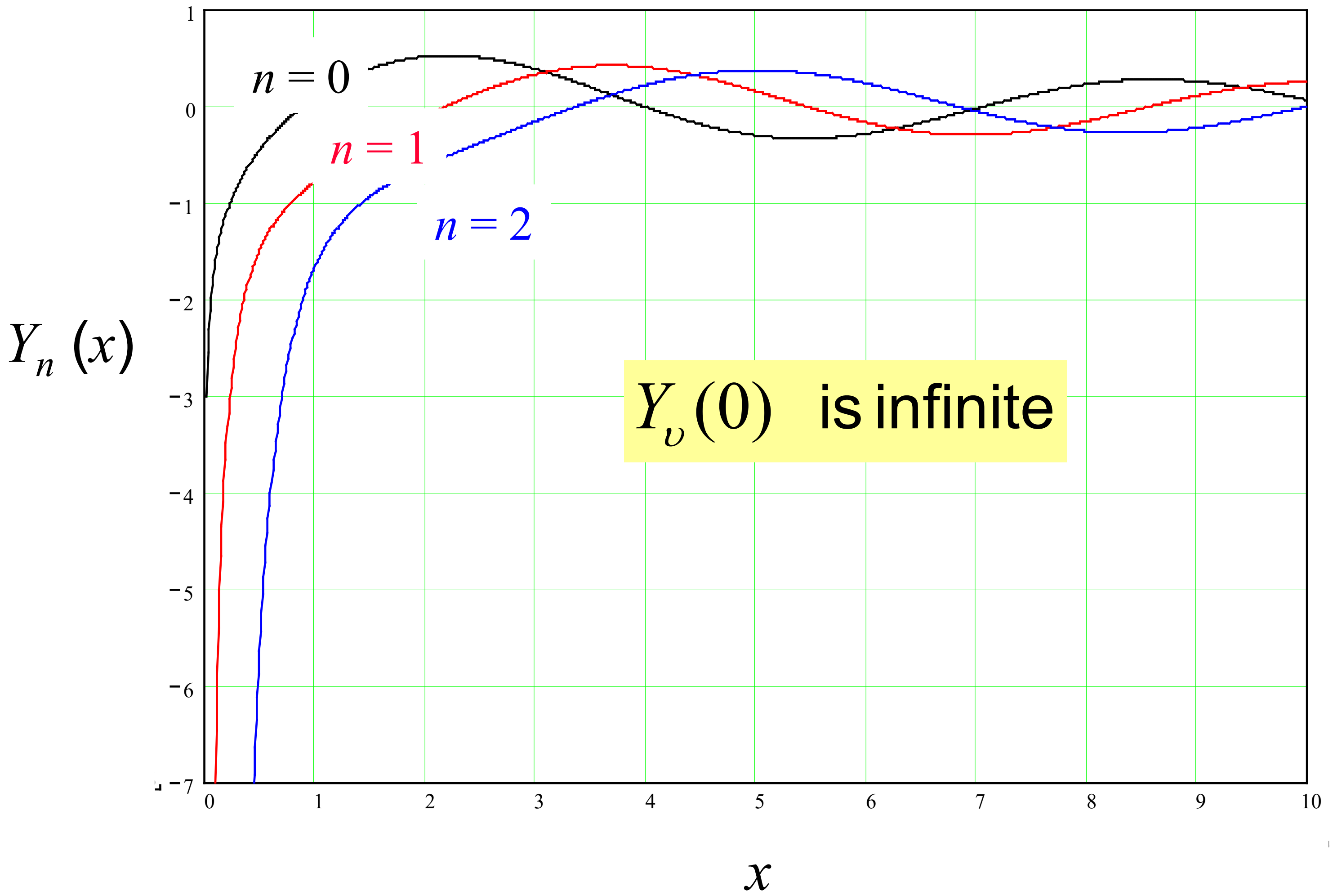
- M. R. Spiegel, *Schaum's Outline Mathematical Handbook*, McGraw-Hill, 4th Edition, 2012.
- M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Government Printing Office, Tenth Printing, 1972.
- NIST online Digital Library of Mathematical Functions (http://dlmf.nist.gov).
- N. N. Lebedev, *Special Functions & Their Applications*, Dover Publications, New York, 1972.
- G. N. Watson, "A Treatise on the Theory of Bessel Functions" (2nd Ed.), Cambridge University Press, 1995.

Plot of Bessel Functions



Note : $J_0(0) \neq 0.$

Plot of Bessel Functions (cont.)



Note : $Y_0(0)$ tends to infinity slower than the rest.

Frobenius Solution

Frobenius solution*:

$$J_\upsilon(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\upsilon+k)!}\left(\frac{x}{2}\right)^{\upsilon+2k}$$

$$z! = \Gamma(z+1)$$

This series always converges.

Note : $J_\upsilon(z)$ is analytic (has a Taylor series) for $\upsilon = n$ (integer).

* The Frobenius series is a generalized Taylor series that has a non-integer set of powers.

Non-Integer Order

Non-integer order: $\upsilon \neq n$

$$y(x) = \left\{ J_\upsilon(x), J_{-\upsilon}(x) \right\}$$ These are two valid solutions.

Note:
- Bessel equation is unchanged by $\upsilon \to -\upsilon$
- $J_{-\upsilon}(x)$ is always a valid solution

These are linearly independent when υ is not an integer:

$$J_\upsilon(x) \approx A_1 x^\upsilon, \quad J_{-\upsilon}(x) \approx A_2 x^{-\upsilon} \quad \text{as} \quad x \to 0$$

Non-Integer Order (cont.)

Bessel function of second kind:

$$Y_\upsilon(x) \equiv \frac{J_\upsilon(x)\cos(\upsilon\pi) - J_{-\upsilon}(x)}{\sin(\upsilon\pi)}$$

$$\upsilon \neq \ldots -2, -1, 0, 1, 2 \ldots$$

This definition gives the $Y_\upsilon(x)$ function a nice (simple) asymptotic form for large x.

Integer Order

Integer order: $\upsilon = n$

$$Y_n(x) \equiv \lim_{\upsilon \to n} Y_\upsilon(x)$$

$$Y_\upsilon(x) \equiv \frac{J_\upsilon(x)\cos(\upsilon\pi) - J_{-\upsilon}(x)}{\sin(\upsilon\pi)}$$

Integer Order (cont.)

From the Frobenius solution we have:

Notice the branch point at $z = 0$.

$$Y_n(x) = \frac{2}{\pi} J_n(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right] - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\Phi(k)+\Phi(n+k)\right]\frac{1}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k+n}$$

(Schaum's Outline Mathematical Handbook, Eq. (24.9))

where

$$\gamma \doteq 0.577216$$

(Euler's constant)

$$\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p} \quad (p > 0), \quad \Phi(0) = 0$$

Integer Order (cont.)

Symmetry property:

$$J_{-n}(x) = (-1)^n J_n(x)$$

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

The functions J_υ and $J_{-\upsilon}$ are no longer linearly independent when υ is an integer.

Properties for General Orders

Properties for General Orders

- In the following discussion, the order can be an integer or a non-integer value, as indicated.

$$n = \text{integer}$$

$$\upsilon = \text{arbitrary (with restrictions as indicated)}$$

Small-Argument Formulas

Small-Argument Properties ($x \to 0$):

$$J_n(x) \approx Ax^n, \quad n = 0,1,2\ldots \qquad \left(J_{-n}(x) = (-1)^n J_n(x)\right)$$

$$J_\upsilon(x) \approx Ax^\upsilon, \quad \upsilon \neq -1,-2,-3,\ldots \qquad \left(\upsilon = -n:\; J_{-n}(x) = (-1)^n J_n(x)\right)$$

$$Y_0(x) \approx C\ln(x)$$

$$Y_n(x) \approx Dx^{-n}, \quad n = 1,2,3\ldots$$

$$Y_\upsilon(x) \approx Bx^{-|\upsilon|}, \quad \upsilon \neq 0$$

Examples:

$$J_0(x) \approx 1$$

$$J_1(x) \approx x/2$$

$$J_{-1}(x) \approx -x/2$$

$$J_{1/2}(x) \approx \sqrt{x}\left(\frac{1/\sqrt{2}}{\left(\frac{1}{2}\right)!}\right)$$

$$J_{-1/2}(x) \approx \frac{1}{\sqrt{x}}\left(\frac{\sqrt{2}}{\left(-\frac{1}{2}\right)!}\right)$$

For <u>order zero</u>, the Bessel function of the second kind behaves <u>logarithmically</u> rather than algebraically.

Asymptotic Formulas

Asymptotic Formulas:

$$x \to \infty$$

$$J_\upsilon(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\upsilon\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_\upsilon(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\upsilon\pi}{2} - \frac{\pi}{4}\right)$$

Hankel Functions

$$H_\upsilon^{(1)}(x) \equiv J_\upsilon(x) + iY_\upsilon(x)$$
$$H_\upsilon^{(2)}(x) \equiv J_\upsilon(x) - iY_\upsilon(x)$$

As $x \to \infty$

$$H_\upsilon^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{+i\left(x - \upsilon\frac{\pi}{2} - \frac{\pi}{4}\right)}$$ Incoming wave*

$$H_\upsilon^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x - \upsilon\frac{\pi}{2} - \frac{\pi}{4}\right)}$$ Outgoing wave*

(*assuming $\exp(+i\omega t)$ time convention)

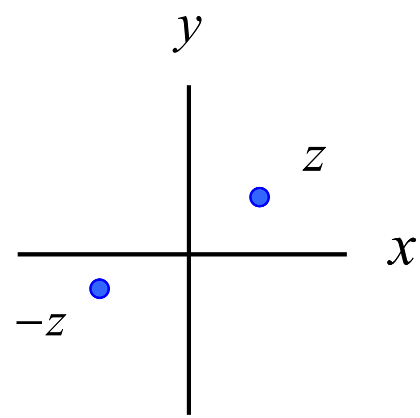
These are valid for arbitrary order υ.

Hankel Functions (cont.)

Useful identity:

$$H_n^{(2)}(-z) = (-1)^{n+1} H_n^{(1)}(+z)$$

$$\operatorname{Im}(z) > 0$$



This is a symmetry property of the Hankel function.

N. N. Lebedev, *Special Functions & Their Applications*, Dover Publications, New York, 1972.

Generating Function

The integer order Bessel function of the first kind can also be defined through a <u>generating function</u> $g(x,t)$:

$$g(x,t) \equiv e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

(derivation omitted)

The generating function definition leads to various useful identities and representations:

- $e^{-ikx} = \sum_{n=-\infty}^{\infty} (-i)^n J_n(k\rho) e^{in\phi}$ (plane wave expansion, a.k.a. Jacobi - Anger expansion)
- $J_m(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\phi - m\phi)\, d\phi$ (integral representation of Bessel function)

(Please see next slides.)

Generating Function (cont.)

Plane-wave expansion:

- $$e^{-ikx} = e^{-ik(\rho\cos\phi)} = e^{-ik\rho(\cos\phi)} = e^{\frac{k\rho}{2}\left(-ie^{i\phi}-ie^{-i\phi}\right)} = e^{\frac{\alpha}{2}\left(t-\frac{1}{t}\right)}\Bigg|_{\substack{\alpha=k\rho,\\ t=-ie^{i\phi}}}$$

Hence, using the generating function identity, we have:

$$e^{-ikx} = \sum_{n=-\infty}^{\infty} J_n(\alpha)\,t^n\Bigg|_{\substack{\alpha=k\rho,\\ t=-ie^{i\phi}}}$$

so

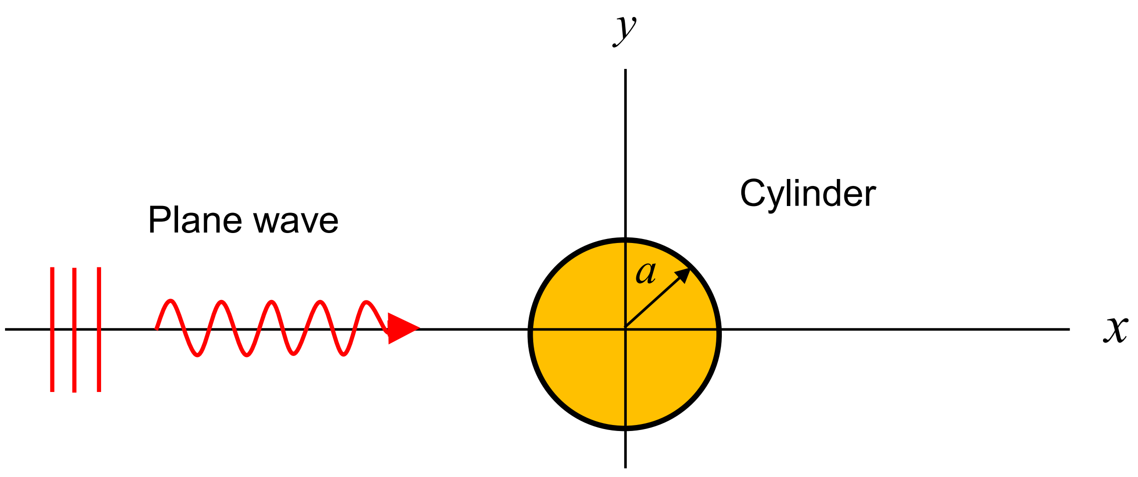
$$e^{-ikx} = \sum_{n=-\infty}^{\infty} (-i)^n J_n(k\rho)\,e^{in\phi}$$

Generating Function (cont.)

The plane wave expansion is very useful for solving scattering problems in cylindrical coordinates:

$$e^{-jkx} = \sum_{n=-\infty}^{\infty} (-j)^n J_n(k\rho) e^{jn\phi}$$

(using j instead of i)



Top view of scattering of a cylinder by a plane wave

Example

Example: Acoustic plane-wave scattering by a cylinder.

$\psi(x, y, z) =$ acoustic pressure function

$$\psi^{\text{inc}}(x,y,z) = e^{-jkx} = \sum_{n=-\infty}^{\infty} (-j)^n J_n(k\rho) e^{jn\phi}$$

(using j instead of i)

Assume: $$\psi^{\text{sca}}(x,y,z) = \sum_{n=-\infty}^{\infty} (-j)^n a_n H_n^{(2)}(k_\rho \rho) e^{jn\phi}$$

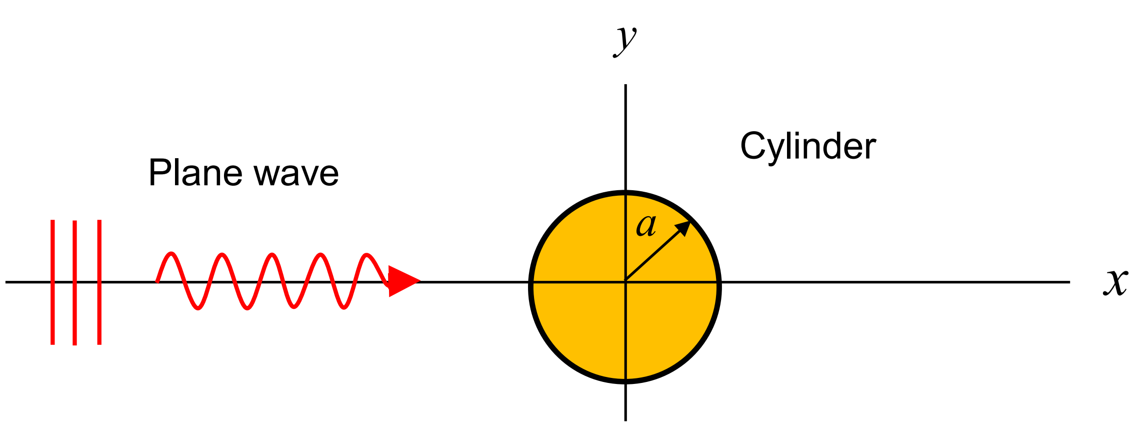
$$\psi(x,y,z) = \psi^{\text{inc}}(x,y,z) + \psi^{\text{sca}}(x,y,z)$$

Acoustic "hard" cylinder:

$$\frac{\partial \psi}{\partial n} = 0 \quad (\rho = a)$$

$$k = \frac{2\pi}{\lambda_0} = \frac{\omega}{c_0}$$

$c_0 =$ speed of acoustic wave



Example (cont.)

$$\psi^{\text{inc}}(x,y,z) = \sum_{n=-\infty}^{\infty} (-j)^n J_n(k\rho) e^{jn\phi}$$

$$\psi^{\text{sca}}(x,y,z) = \sum_{n=-\infty}^{\infty} (-j)^n a_n H_n^{(2)}(k_\rho \rho) e^{jn\phi}$$

$$\frac{\partial \psi}{\partial n} = \frac{\partial \psi}{\partial \rho} = 0 \quad (\rho = a)$$

$$\sum_{n=-\infty}^{\infty} (-j)^n k J_n'(ka) e^{jn\phi} = -\sum_{n=-\infty}^{\infty} (-j)^n a_n k H_n'^{(2)}(k_\rho a) e^{jn\phi}$$

$$\Rightarrow \quad a_n = -\frac{J_n'(ka)}{H_n'^{(2)}(k_\rho a)}$$

Generating Function (cont.)

Integral representation of Bessel function:

Start with: $$e^{-ik\rho\cos\phi} = \sum_{n=-\infty}^{\infty} (-i)^n J_n(k\rho) e^{in\phi}$$

- Set $k = 1$ in above result, multiply both sides by $e^{-im\phi}$
- Integrate over $(0, 2\pi)$, use orthogonality of $e^{in\phi}$

Orthogonality:

$$\int_0^{2\pi} e^{-i(m-n)\phi} d\phi = \begin{cases} 2\pi, & m = n \\ 0, & m \neq n \end{cases}$$

$$\int_0^{2\pi} e^{-i\rho\cos\phi} e^{-im\phi} d\phi = 2\pi (-i)^m J_m(\rho)$$

$$\Rightarrow \quad J_m(\rho) = \frac{i^m}{2\pi} \int_0^{2\pi} e^{-i\rho\cos\phi} e^{-im\phi} d\phi$$

Generating Function (cont.)

Integral representation of Bessel function (cont.):

$$J_m(\rho) = \frac{i^m}{2\pi} \int_0^{2\pi} e^{-i\rho\cos\phi} e^{-im\phi} d\phi$$

- Use $\phi = \phi' + \pi/2$, symmetry, $\phi' \to \phi$:

$$J_m(\rho) = \frac{i^m}{2\pi} \int_{-\pi/2}^{2\pi-\pi/2} e^{-i\rho\cos(\phi'+\pi/2)} e^{-im(\phi'+\pi/2)} d\phi'$$

$$= \frac{i^m}{2\pi} \int_{-\pi/2}^{2\pi-\pi/2} e^{+i\rho\sin(\phi')} e^{-im\phi'} (-i)^m \, d\phi'$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{+i\rho\sin(\phi')} e^{-im\phi'} d\phi'$$

Periodic integrand

Generating Function (cont.)

Integral representation of Bessel function (cont.):

$$J_m(\rho) = \frac{1}{2\pi} \int_0^{2\pi} e^{+i\rho\sin(\phi')} e^{-im\phi'} d\phi'$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{+i\rho\sin(\phi')} e^{-im\phi'} d\phi'$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\rho\sin\phi' - m\phi') + i\sin(\rho\sin\phi' - m\phi')\, d\phi'$$

odd function

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\rho\sin\phi' - m\phi')\, d\phi'$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos(\rho\sin\phi' - m\phi')\, d\phi$$

$$\Rightarrow \quad J_m(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\phi - m\phi)\, d\phi$$

(ρ is relabeled as x)

Recurrence Relations

Many recurrence relations can be derived from the generating function.

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

$$\Rightarrow \frac{\partial}{\partial t} e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

$$\Rightarrow \frac{x}{2}\left(1+\frac{1}{t^2}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

$$\Rightarrow \frac{x}{2}\left(1+\frac{1}{t^2}\right) \sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

$$\Rightarrow \frac{x}{2} \sum_{n=-\infty}^{\infty} J_n(x)\left(t^n + t^{n-2}\right) = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

On LHS use: $n \to n-1 \;\; \& \;\; n \to n+1$

$$\frac{x}{2} \sum_{n=-\infty}^{\infty} \left[J_{n-1}(x) + J_{n+1}(x)\right] t^{n-1} = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

Recurrence Relations (cont.)

$$\frac{x}{2}\sum_{n=-\infty}^{\infty}\left[J_{n-1}(x)+J_{n+1}(x)\right]t^{n-1}=\sum_{n=-\infty}^{\infty} n\,J_n(x)\,t^{n-1}$$

Equating like powers of t yields:

$$J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x)$$

Recurrence Relations (cont.)

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

This can then be used to generate other useful recurrence relations:

$$n \to n-1: \quad J_n(x) = \frac{2(n-1)}{x} J_{n-1}(x) - J_{n-2}(x)$$ ("upward recursion")

$$n \to n+1: \quad J_n(x) = \frac{2(n+1)}{x} J_{n+1}(x) - J_{n+2}(x)$$ ("downward recursion")

Recurrence Relations (cont.)

Another recurrence relation for the <u>derivative</u> of the Bessel function:

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\,t^n$$

For the LHS:

$$\frac{\partial}{\partial x} g(x,t) = \frac{\partial e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{\partial x} = \frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{n=-\infty}^{\infty} J_n(x)\,t^n$$

$$= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} J_n(x)\,t^{n+1} - \sum_{n=-\infty}^{\infty} J_n(x)\,t^{n-1}\right)$$

$$= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} J_{n-1}(x)\,t^{n} - \sum_{n=-\infty}^{\infty} J_{n+1}(x)\,t^{n}\right) = \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]t^n$$

Also, we have, for the RHS:

$$\frac{\partial}{\partial x} g(x,t) = \sum_{n=-\infty}^{\infty} J_n'(x)\,t^n$$

Equating like powers of t yields:

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Recurrence Relations (cont.)

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Then use the previous identity:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

This can be used to replace J_{n+1} or J_{n-1}.

This yields:

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x} J_n(x)$$

Recurrence Relations (cont.)

The same recurrence formulas actually apply to <u>all</u> Bessel functions of all orders.

If $Z_\upsilon(x)$ denotes any Bessel, Neumann, or Hankel function of order υ, then we have:

$$Z_{\upsilon-1}(x)+Z_{\upsilon+1}(x)=\frac{2\upsilon}{x}Z_\upsilon(x)$$

$$Z_\upsilon(x)=\frac{2(\upsilon-1)}{x}J_{\upsilon-1}(x)-J_{\upsilon-2}(x)$$

$$Z_\upsilon(x)=\frac{2(\upsilon+1)}{x}J_{\upsilon+1}(x)-J_{\upsilon+2}(x)$$

$$Z_\upsilon'=Z_{\upsilon-1}-\frac{\upsilon}{x}Z_\upsilon \qquad Z_\upsilon'=-Z_{\upsilon+1}+\frac{\upsilon}{x}Z_\upsilon$$

Recurrence Relations (cont.)

Integral identities also follow from the recurrence identities.

Example of integral identity:

$$Z'_\upsilon = Z_{\upsilon-1} - \frac{\upsilon}{x} Z_\upsilon \quad \Rightarrow \quad xZ'_\upsilon + \upsilon Z_\upsilon = xZ_{\upsilon-1}$$ (multiplying both sides by x)

Multiply by $x^{\upsilon-1}$ $\quad \Rightarrow \quad x^\upsilon Z'_\upsilon + \upsilon x^{\upsilon-1} Z_\upsilon = x^\upsilon Z_{\upsilon-1}$

Hence,

$$x^\upsilon Z'_\upsilon + \upsilon x^{\upsilon-1} Z_\upsilon = \frac{d}{dx}\left[x^\upsilon Z_\upsilon(x)\right] = x^\upsilon Z_{\upsilon-1}(x)$$

$$\Rightarrow \quad \int x^\upsilon Z_{\upsilon-1}(x)\,dx = x^\upsilon Z_\upsilon(x)$$

Similarly, we have

$$\int x^{-\upsilon} Z_{\upsilon+1}(x)\,dx = -x^{-\upsilon} Z_\upsilon(x)$$

Recurrence Relations (cont.)

Examples:

$$\int J_0(x)\,x\,dx = x\,J_1(x) \qquad \text{(First one, } \upsilon = 1\text{)}$$

$$\int J_1(x)\,dx = -J_0(x) \qquad \text{(Second one, } \upsilon = 0\text{)}$$

Wronskians

$$W(x) = W[J_\upsilon, J_{-\upsilon}] \equiv J_\upsilon(x) J'_{-\upsilon}(x) - J'_\upsilon(x) J_{-\upsilon}(x) = -\frac{2\sin\upsilon\pi}{\pi x}$$

Please see the note below.

$$W[J_\upsilon, Y_\upsilon] = J_\upsilon(x) Y'_\upsilon(x) - J'_\upsilon(x) Y_\upsilon(x) = \frac{2}{\pi x}$$

$$W\left[J_\upsilon, H_\upsilon^{(1,2)}\right] = J_\upsilon(x) H_\upsilon^{(1,2)\prime}(x) - J'_\upsilon(x) H_\upsilon^{(1,2)}(x) = \pm\frac{2i}{\pi x}$$

$+$ for $H_\upsilon^{(1)}$

$-$ for $H_\upsilon^{(2)}$

Note: For $\upsilon \neq n$, the Wronskian is $[J_\upsilon, J_{-\upsilon}]$ is not identically zero (in fact, it is not zero anywhere), and hence the two functions are linearly independent.

(Please see Appendix A for a derivation.)

Fourier-Bessel Series

Fourier-Bessel series:

$$f(\rho) = \sum_{n=1}^{\infty} c_{\upsilon n} J_{\upsilon}\left(p_{\upsilon n} \frac{\rho}{a} \right), \quad 0 \le \rho \le a$$

Note:
The order υ and the length a are arbitrary here.

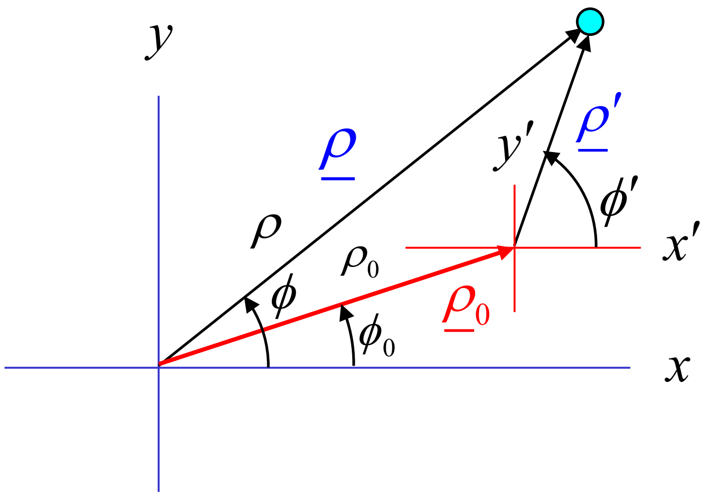
where $p_{\upsilon n}$ is the n^{th} zero of $J_{\upsilon}(x)$: $J_{\upsilon}(p_{\upsilon n}) = 0$.

The coefficients are given by

$$c_{\upsilon n} = \frac{2}{a^2 \left[J_{\upsilon+1}\left(p_{\upsilon n} \right) \right]^2} \int_0^a f(\rho)\, J_{\upsilon}\left(p_{\upsilon n} \frac{\rho}{a} \right) \rho \, d\rho$$

(Please see Appendix B for a derivation.)

Addition Theorems



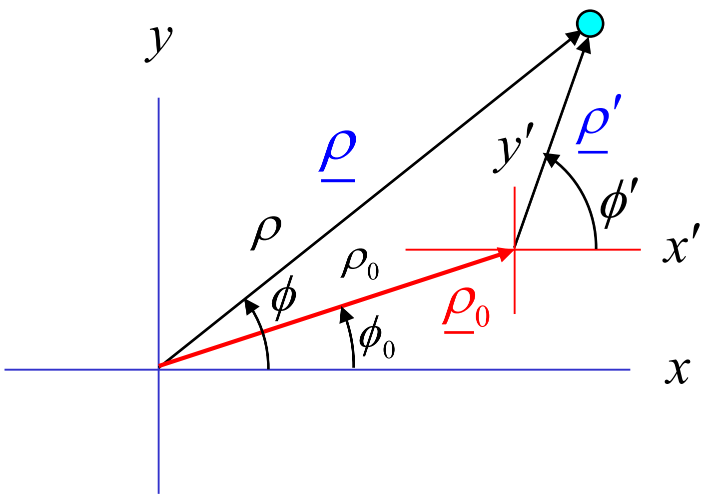
Addition theorems allow cylindrical harmonics in one coordinate system to be expanded in terms of those of a shifted coordinate system.

Shifting from global origin to local origin:

$$J_n(k\rho)\,e^{in\phi} = \sum_{m=-\infty}^{\infty} J_{n-m}(k\rho_0)e^{i(n-m)\phi_0} J_m(k\rho')\,e^{im\phi'}$$

$$H_n^{(2)}(k\rho)\,e^{in\phi} = \begin{cases} \displaystyle\sum_{m=-\infty}^{\infty} H_{n-m}^{(2)}(k\rho_0)e^{i(n-m)\phi_0} J_m(k\rho')\,e^{im\phi'}, & \rho < \rho_0 \\ \displaystyle\sum_{m=-\infty}^{\infty} J_{n-m}(k\rho_0)e^{i(n-m)\phi_0} H_m^{(2)}(k\rho')\,e^{im\phi'}, & \rho > \rho_0 \end{cases}$$

Addition Theorems (cont.)

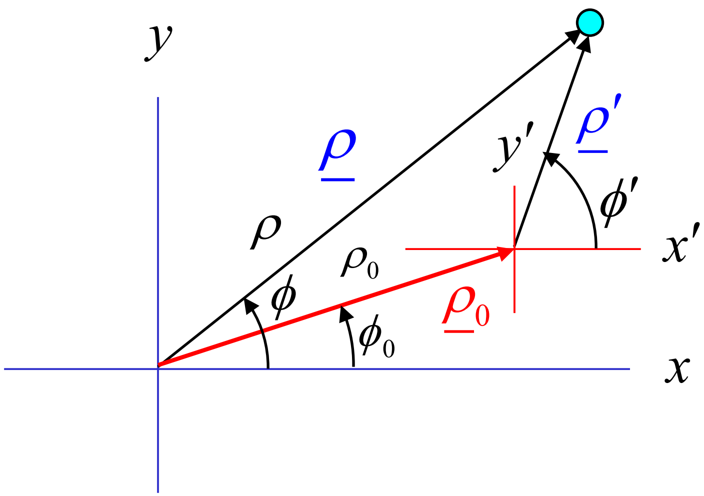


Shifting from local origin to global origin:

$$J_n(k\rho')\,e^{in\phi'} = \sum_{m=-\infty}^{\infty} J_{n-m}(k\rho_0)e^{i(n-m)(\pi+\phi_0)}J_m(k\rho)\,e^{im\phi}$$

$$H_n^{(2)}(k\rho')\,e^{in\phi'} = \begin{cases} \displaystyle\sum_{m=-\infty}^{\infty} H_{n-m}^{(2)}(k\rho_0)e^{i(n-m)(\pi+\phi_0)}J_m(k\rho)\,e^{im\phi}, & \rho < \rho_0 \\ \displaystyle\sum_{m=-\infty}^{\infty} J_{n-m}(k\rho_0)e^{i(n-m)(\pi+\phi_0)}H_m^{(2)}(k\rho)\,e^{im\phi}, & \rho > \rho_0 \end{cases}$$

Addition Theorems (cont.)



Recall :

$$J_{-m}(x) = (-1)^m J_m(x)$$

$$H_{-m}^{(2)}(x) = (-1)^m H_m^{(2)}(x)$$

$$e^{-im\pi} = (-1)^m$$

Shifting from local origin to global origin:

Special case ($n = 0$):

$$J_0(k\rho') = \sum_{m=-\infty}^{\infty} J_m(k\rho_0) J_m(k\rho) e^{im(\phi-\phi_0)}$$

$$H_0^{(2)}(k\rho') = \begin{cases} \sum_{m=-\infty}^{\infty} H_m^{(2)}(k\rho_0) J_m(k\rho) e^{im(\phi-\phi_0)}, & \rho < \rho_0 \\ \sum_{m=-\infty}^{\infty} J_m(k\rho_0) H_m^{(2)}(k\rho) e^{im(\phi-\phi_0)}, & \rho > \rho_0 \end{cases}$$

Appendix A: Wronskians

From the Sturm-Liouville properties, the Wronskians for the Bessel differential equation are found to have the following form:

$$W(x) = W(a)\, e^{-\int_a^x \frac{dx}{x}} = W(a) e^{-\ln\left(\frac{x}{a}\right)} = W(a) e^{+\ln\left(\frac{a}{x}\right)} = \frac{W(a)\,a}{x} = \frac{C}{x}$$

Recall:

$$W(x) = W(a) e^{-\int_a^x p(x')dx'}$$

$$p(x) \equiv \frac{p_1(x)}{p_0(x)}$$

$$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$$

$$p_0(x) = x^2$$

$$p_1(x) = x$$

The constant C can be found using the small-argument approximations for the Bessel functions (keep $k = 0$ term in the Frobenius series). The result is:

$$W(x) = W\left[J_\upsilon, J_{-\upsilon}\right] \equiv J_\upsilon(x) J'_{-\upsilon}(x) - J'_\upsilon(x) J_{-\upsilon}(x) = -\frac{2\sin\upsilon\pi}{\pi x}$$

(Please see next slide.)

Note: For $\upsilon \neq n$, the Wronskian is not identically zero (in fact, it is not zero anywhere), and hence the two functions are linearly independent.

Appendix A (cont.)

$$J_\upsilon(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(\upsilon+k)!}\left(\frac{x}{2}\right)^{\upsilon+2k}$$

$$\Rightarrow \quad J_\upsilon(x)\sim\frac{1}{\upsilon!}\left(\frac{x}{2}\right)^{\upsilon}\ ,\quad J_{-\upsilon}(x)\sim\frac{1}{(-\upsilon)!}\left(\frac{x}{2}\right)^{-\upsilon}$$

$$\Rightarrow \quad J'_\upsilon(x)\sim\frac{1}{2}\frac{\upsilon}{\upsilon!}\left(\frac{x}{2}\right)^{\upsilon-1}\ ,\quad J'_{-\upsilon}(x)\sim\frac{1}{2}\frac{(-\upsilon)}{(-\upsilon)!}\left(\frac{x}{2}\right)^{-\upsilon-1}$$

$$\Rightarrow \quad W(x)=J_\upsilon(x)J'_{-\upsilon}(x)-J'_\upsilon(x)J_{-\upsilon}(x)=-\frac{\upsilon}{\upsilon!(-\upsilon)!x}-\frac{\upsilon}{\upsilon!(-\upsilon)!x}=-\frac{2\upsilon}{\upsilon!(-\upsilon)!x}$$

$$\Rightarrow \quad C=-\frac{2\upsilon}{\upsilon!(-\upsilon)!}=-\frac{2\sin\upsilon\pi}{\pi}$$

$$\Rightarrow \quad W(x)=-\frac{2\sin\upsilon\pi}{\pi x}$$

Recall:

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$$

$$\Rightarrow \Gamma(\upsilon)\Gamma(1-\upsilon)=\frac{\pi}{\sin\pi\upsilon}$$

$$\Rightarrow (\upsilon-1)!(-\upsilon)!=\frac{\pi}{\sin\pi\upsilon}$$

$$\Rightarrow (\upsilon)!(-\upsilon)!=\frac{\pi\upsilon}{\sin\pi\upsilon}$$

Appendix A (cont.)

Similarly, we have

$$W\left[J_\upsilon, Y_\upsilon\right] = J_\upsilon(x)Y_\upsilon'(x) - J_\upsilon'(x)Y_\upsilon(x) = \frac{2}{\pi x}$$

$$W\left[J_\upsilon, H_\upsilon^{(1,2)}\right] = J_\upsilon(x)H_\upsilon^{(1,2)\prime}(x) - J_\upsilon'(x)H_\upsilon^{(1,2)}(x) = \pm\frac{2i}{\pi x}$$

$+$ for $H_\upsilon^{(1)}$

$-$ for $H_\upsilon^{(2)}$

Appendix B: Fourier-Bessel Series

To derive the Fourier-Bessel expansion, start with:

$$f(\rho) = \sum_{n=1}^{\infty} c_{\upsilon n} J_{\upsilon}\left(p_{\upsilon n}\frac{\rho}{a}\right), \quad 0 \le \rho \le a$$

Multiply both sides by $J_{\upsilon}\left(p_{\upsilon m}\frac{\rho}{a}\right)\rho$ and integrate from 0 to a:

$$\int_0^a f(\rho) J_{\upsilon}\left(p_{\upsilon m}\frac{\rho}{a}\right)\rho\, d\rho = \int_0^a \left(\sum_{n=1}^{\infty} c_{\upsilon n} J_{\upsilon}\left(p_{\upsilon n}\frac{\rho}{a}\right) J_{\upsilon}\left(p_{\upsilon m}\frac{\rho}{a}\right)\right)\rho\, d\rho$$

$$= \sum_{n=1}^{\infty}\int_0^a c_{\upsilon n} J_{\upsilon}\left(p_{\upsilon n}\frac{\rho}{a}\right) J_{\upsilon}\left(p_{\upsilon m}\frac{\rho}{a}\right)\rho\, d\rho$$

$$= c_{\upsilon m}\frac{a^2}{2}\left[J_{\upsilon+1}\left(p_{\upsilon m}\right)\right]^2$$

(using orthogonality + result from next slide)

Note:
See Notes 18 for a derivation of the orthogonality when $m \neq n$.

Appendix B (cont.)

Derivation of the orthogonality formula

Start with this integral identity (derivation omitted):

$$\int J_\upsilon^2(px)\,x\,dx = \frac{x^2}{2}\left(J_\upsilon'(px)\right)^2 - \frac{x^2}{2}\left(1-\frac{\upsilon^2}{(px)^2}\right)\left(J_\upsilon(px)\right)^2$$

Choose $p = p_{\upsilon m}/a$:

$$\int_0^a J_\upsilon^2\left(\frac{p_{\upsilon m}}{a}x\right)x\,dx = \frac{a^2}{2}\left(J_\upsilon'\left(\frac{p_{\upsilon m}}{a}a\right)\right)^2 - \frac{a^2}{2}\left(1-\frac{\upsilon^2}{\left(\frac{p_{\upsilon m}}{a}a\right)^2}\right)\left(J_\upsilon\left(\frac{p_{\upsilon m}}{a}a\right)\right)^2$$

$$= \frac{a^2}{2}\left(J_\upsilon'(p_{\upsilon m})\right)^2$$

$$= \frac{a^2}{2}\left(-J_{\upsilon+1}(p_{\upsilon m}) + \frac{\upsilon}{p_{\upsilon m}}J_\upsilon(p_{\upsilon m})\right)^2$$

$$= \frac{a^2}{2}\left(-J_{\upsilon+1}(p_{\upsilon m})\right)^2$$

Recall:

$$J_\upsilon'(x) = -J_{\upsilon+1}(x) + \frac{\upsilon}{x}J_\upsilon(x)$$