ECE 6382

Fall 2023

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Notes 21

Modified Bessel Functions and Kelvin Functions

Notes are from D. R. Wilton, Dept. of ECE

Modified Bessel Functions

Modified Bessel differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \upsilon^2) y = 0$$

This comes from the usual Bessel differential equation:

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \upsilon^2) y = 0 \qquad \text{(Bessel DE)}$$

Set $z = ix$; $dz = idx$ in the usual Bessel DE:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \upsilon^2) y = 0 \qquad \text{(modified Bessel DE)}$$

The modified Bessel functions are Bessel functions of imaginary argument.

$$\left(J_\upsilon(ix), Y_\upsilon(ix) \right)$$

Modified Bessel Function of the First Kind

Definition:

$$I_\upsilon(x) \equiv (-i)^\upsilon J_\upsilon(ix)$$

(I_υ is a real function of x.)

To see that I_υ is a real function, use the Frobenius solution for J_υ:

$$J_\upsilon(ix) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\upsilon+k)!}\left(\frac{ix}{2}\right)^{\upsilon+2k}$$

Note :

$$(i)^{2k} = (-1)^k$$

$$(i)^\upsilon (-i)^\upsilon = (1)^\upsilon = 1$$

Therefore, the Frobenius series solution for I_υ is:

$$I_\upsilon(x) = \sum_{k=0}^{\infty} \frac{1}{k!(\upsilon+k)!}\left(\frac{x}{2}\right)^{\upsilon+2k}$$

Second Solution of Modified Bessel Equation

For $\upsilon \neq n$, the modified Bessel function of the 2nd kind is defined as:

$$K_\upsilon(x) \equiv \frac{\pi}{2}\frac{I_{-\upsilon}(x) - I_\upsilon(x)}{\sin \upsilon\pi}$$

For $\upsilon = n$ (an integer):

$$K_n(x) \equiv \lim_{\upsilon \to n} \frac{\pi}{2}\frac{I_{-\upsilon}(x) - I_\upsilon(x)}{\sin \upsilon\pi}$$

Relations Between Bessel and Modified Bessel Functions

The modified Bessel functions are related to the regular Bessel functions as

$$I_\upsilon(x) \equiv (-i)^\upsilon J_\upsilon(ix)$$

$$K_\upsilon(x) \equiv \frac{\pi}{2}\frac{I_{-\upsilon}(x) - I_\upsilon(x)}{\sin \upsilon\pi} = \frac{\pi}{2} i^{\upsilon+1} H_\upsilon^{(1)}(ix) \quad \text{(proof omitted)}$$

Note: The added factors in front ensure that the functions are real.

For $\upsilon = n,$ an integer :

$$I_{-n}(x) = I_n(x)$$

$$K_{-n}(x) = K_n(x)$$

Small Argument Approximations

For small arguments we have:

$$I_{\pm\upsilon}(x) \xrightarrow[x\to 0]{} \frac{1}{(\pm\upsilon)!}\left(\frac{x}{2}\right)^{\pm\upsilon}$$

$$K_0(x) \xrightarrow[x\to 0]{} -\ln\left(\frac{x}{2}\right) - \gamma$$

$$K_\upsilon(x) \xrightarrow[x\to 0]{} \frac{(\upsilon-1)!}{2}\left(\frac{x}{2}\right)^{-\upsilon}$$

Large Argument Approximations

For large arguments we have:

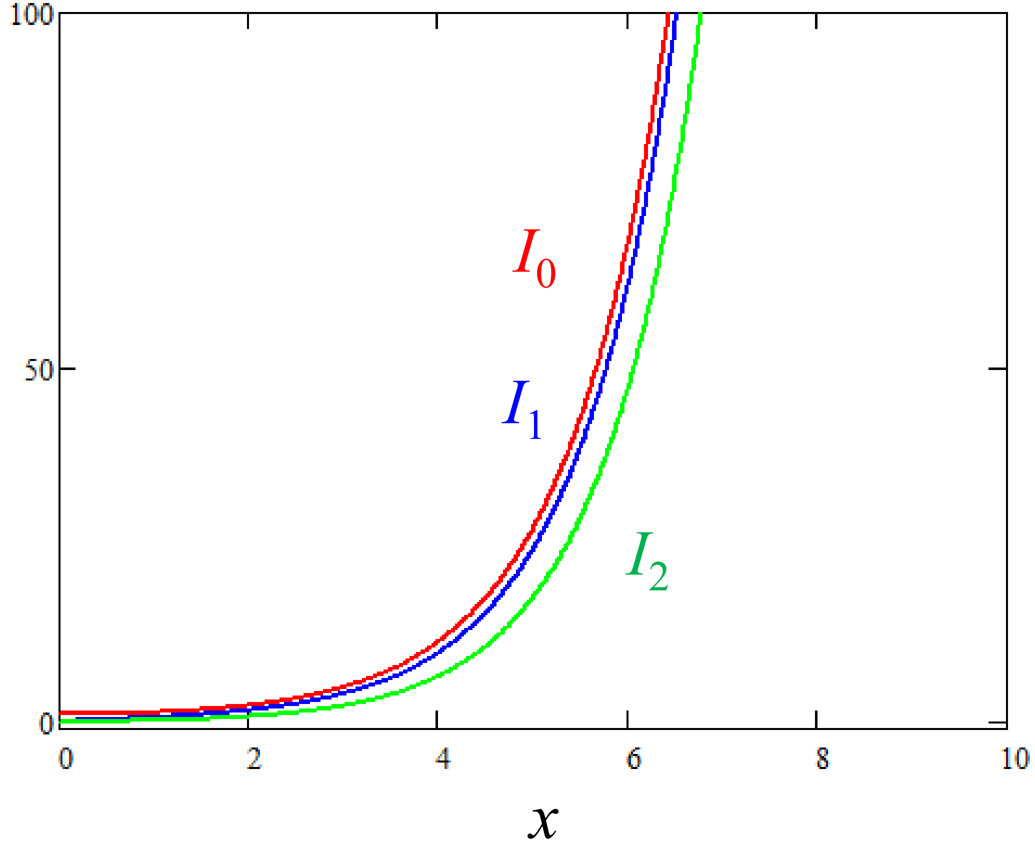
$$I_\upsilon(x) \xrightarrow[x\to\infty]{} \frac{e^x}{\sqrt{2\pi x}}$$

exponentially grows!

$$K_\upsilon(x) \xrightarrow[x\to\infty]{} \sqrt{\frac{\pi}{2x}}\, e^{-x}$$

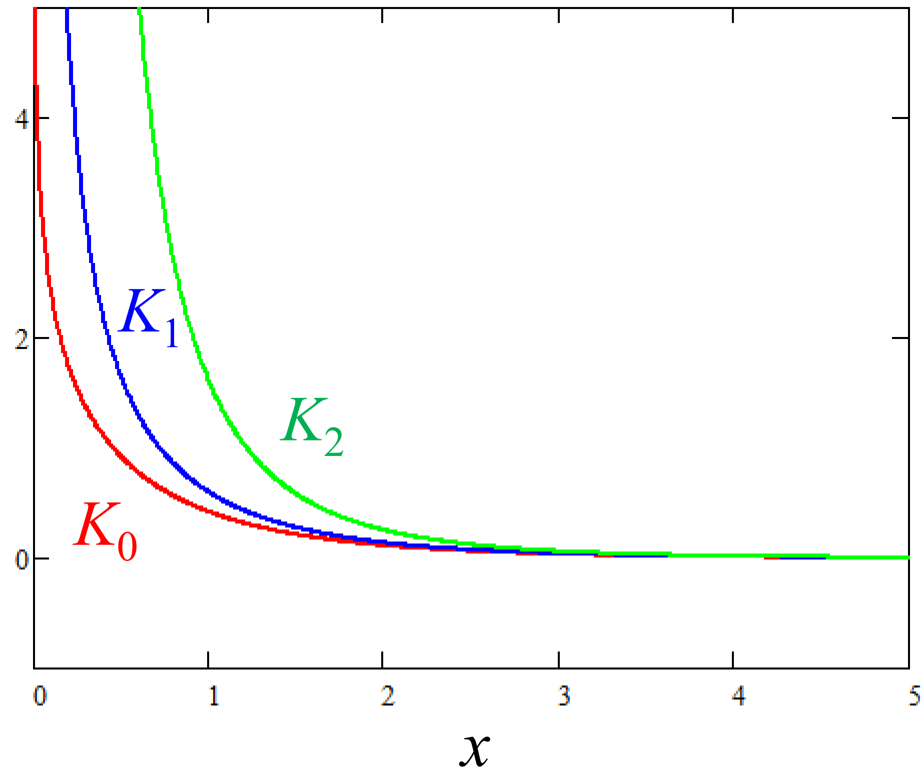
exponentially decays!

Plots of Modified Bessel Functions for Real Arguments



The I_n functions <u>increase</u> exponentially. They are finite at $x = 0$.

Plots of Modified Bessel Functions for Real Arguments



The K_n functions <u>decrease</u> exponentially. The are infinite at $x = 0$.

Recurrence Relations

Some recurrence relations are:

$$I_{\upsilon-1}(x) - I_{\upsilon+1}(x) = \frac{2\upsilon}{x} I_{\upsilon}(x)$$

$$I_{\upsilon-1}(x) + I_{\upsilon+1}(x) = 2I'_{\upsilon}(x)$$

$$K_{\upsilon-1}(x) - K_{\upsilon+1}(x) = -\frac{2\upsilon}{x} K_{\upsilon}(x)$$

$$K_{\upsilon-1}(x) + K_{\upsilon+1}(x) = -2K'_{\upsilon}(x)$$

Wronskian

A Wronskian identity is:

$$W\left[I_\upsilon, K_\upsilon\right] = I_\upsilon(x)K'_\upsilon(x) - I'_\upsilon(x)K_\upsilon(x) = -\frac{1}{x}$$

Kelvin Functions

The Kelvin functions are defined as

$$\mathrm{Ber}_{\upsilon}(x) \equiv \mathrm{Re}\left(J_{\upsilon}\left(xe^{i3\pi/4}\right)\right)$$

$$\mathrm{Bei}_{\upsilon}(x) \equiv \mathrm{Im}\left(J_{\upsilon}\left(xe^{i3\pi/4}\right)\right)$$

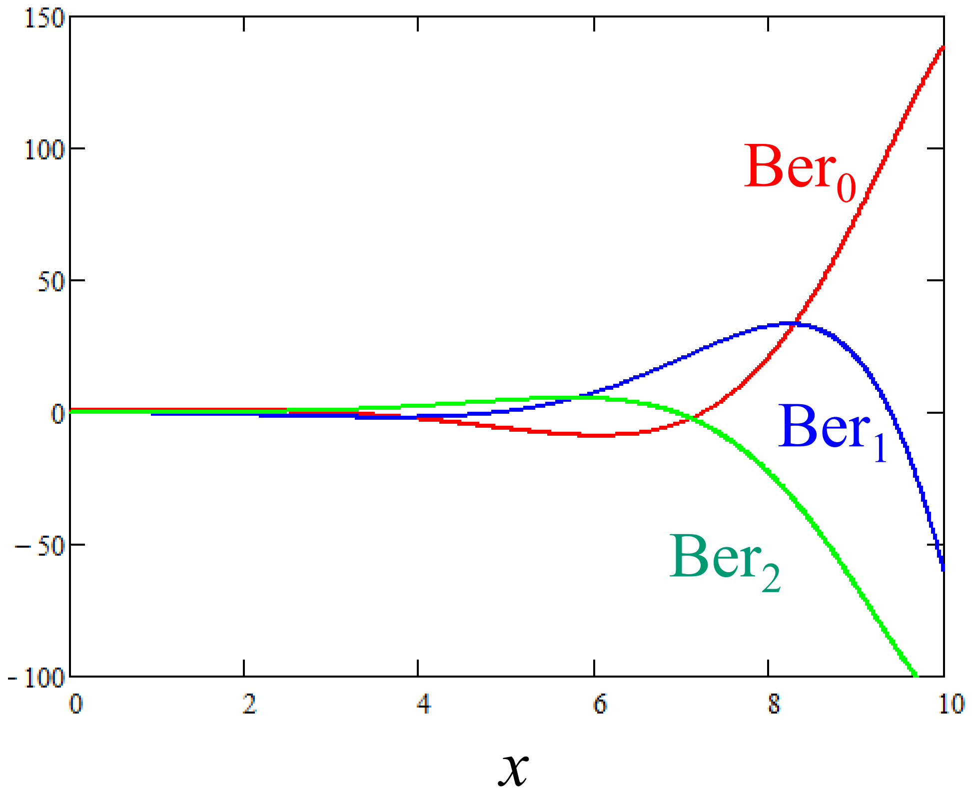
$$\mathrm{Ker}_{\upsilon}(x) \equiv \mathrm{Re}\left(K_{\upsilon}\left(xe^{i\pi/4}\right)\right)$$

$$\mathrm{Kei}_{\upsilon}(x) \equiv \mathrm{Im}\left(K_{\upsilon}\left(xe^{i\pi/4}\right)\right)$$

Note: These are important for studying the fields inside of a conducting wire.

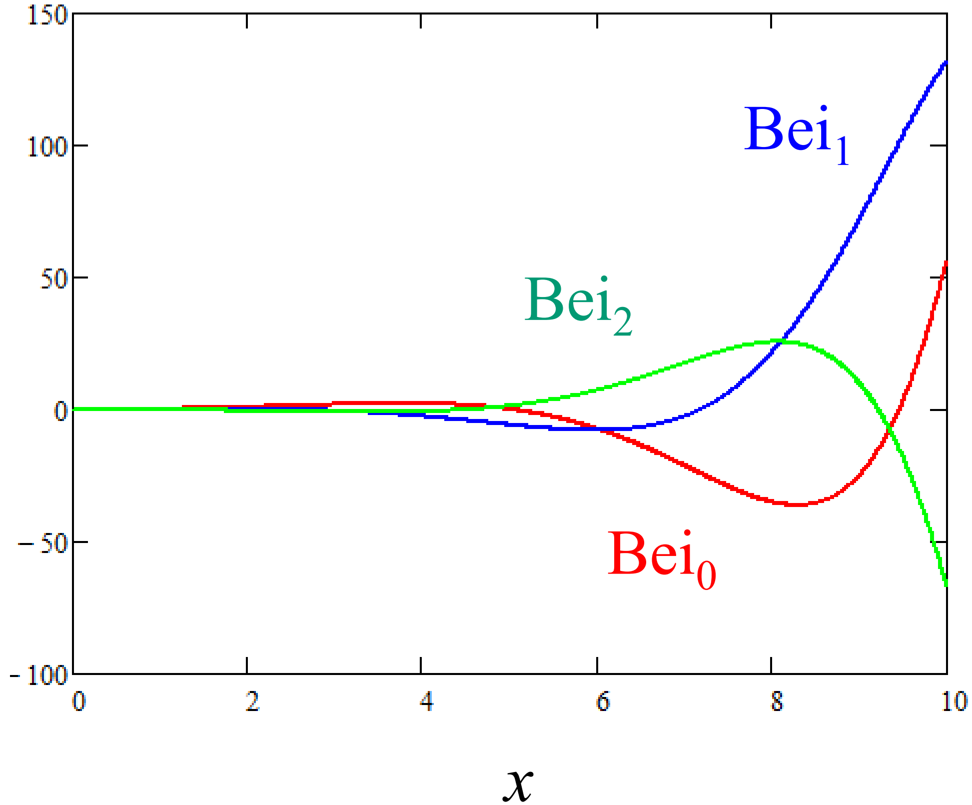
Kelvin Functions (cont.)

The Ber functions <u>increase</u> exponentially.
They are finite at $x = 0$.



Kelvin Functions (cont.)

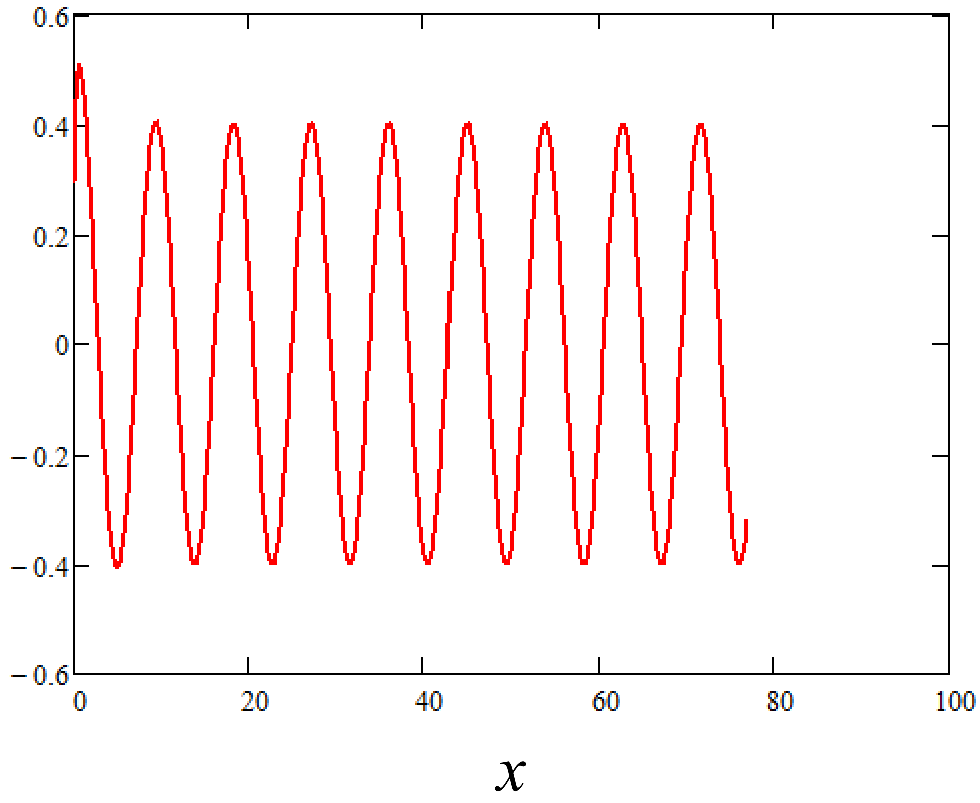
The Bei functions <u>increase</u> exponentially.
They are finite at $x = 0$.



Kelvin Functions (cont.)

Normalizing makes it more obvious that the Ber and Bei functions increase exponentially and also oscillate as x increases.

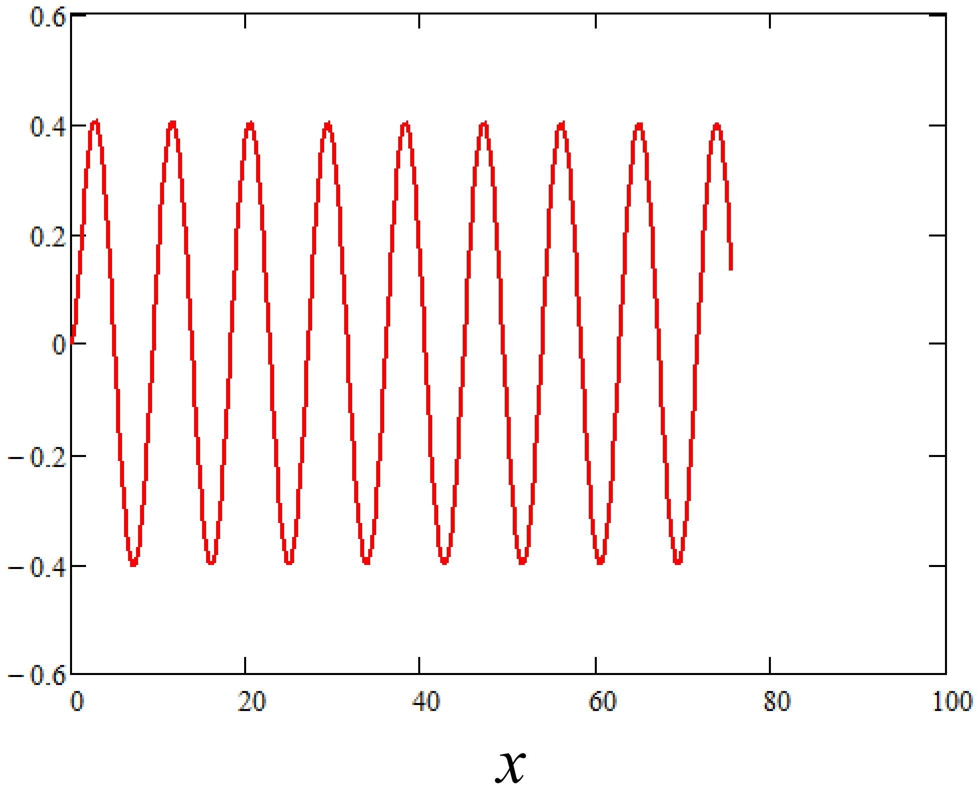
$$\frac{\mathrm{Ber}_0(x)}{\dfrac{1}{\sqrt{x}}e^{x/\sqrt{2}}}$$



Kelvin Functions (cont.)

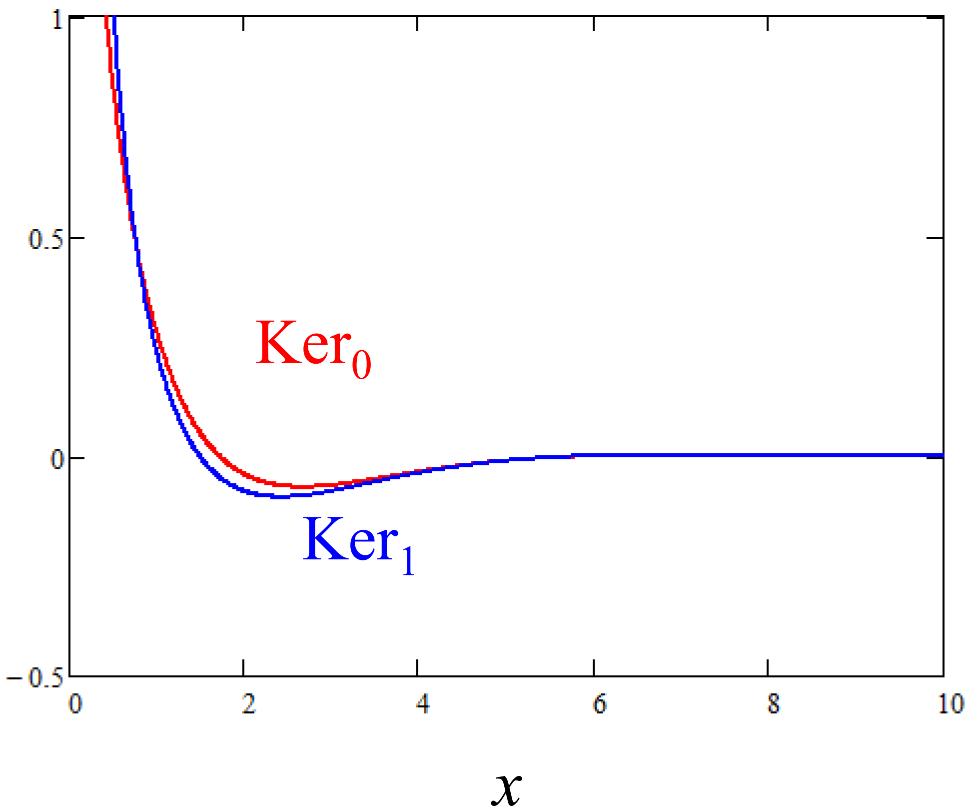
Normalizing makes it more obvious that the Ber and Bei functions increase exponentially and also oscillate as x increases.

$$\frac{\mathrm{Bei}_0(x)}{\dfrac{1}{\sqrt{x}}e^{x/\sqrt{2}}}$$



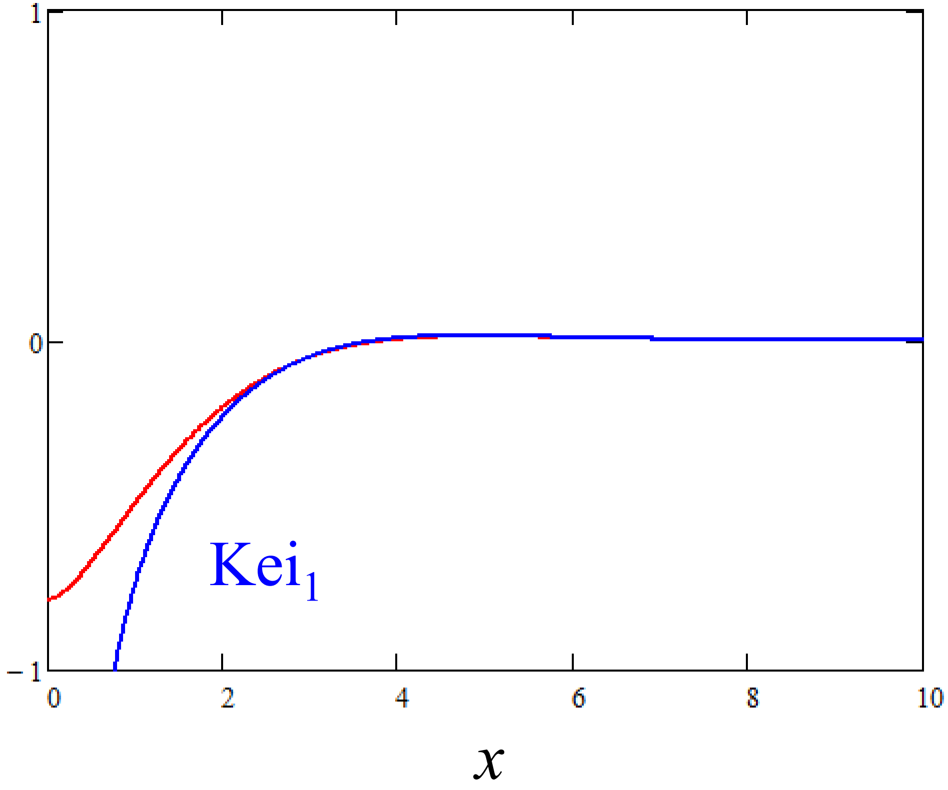
Kelvin Functions (cont.)

The Ker functions decay exponentially.
They are infinite at $x = 0$.



Kelvin Functions (cont.)

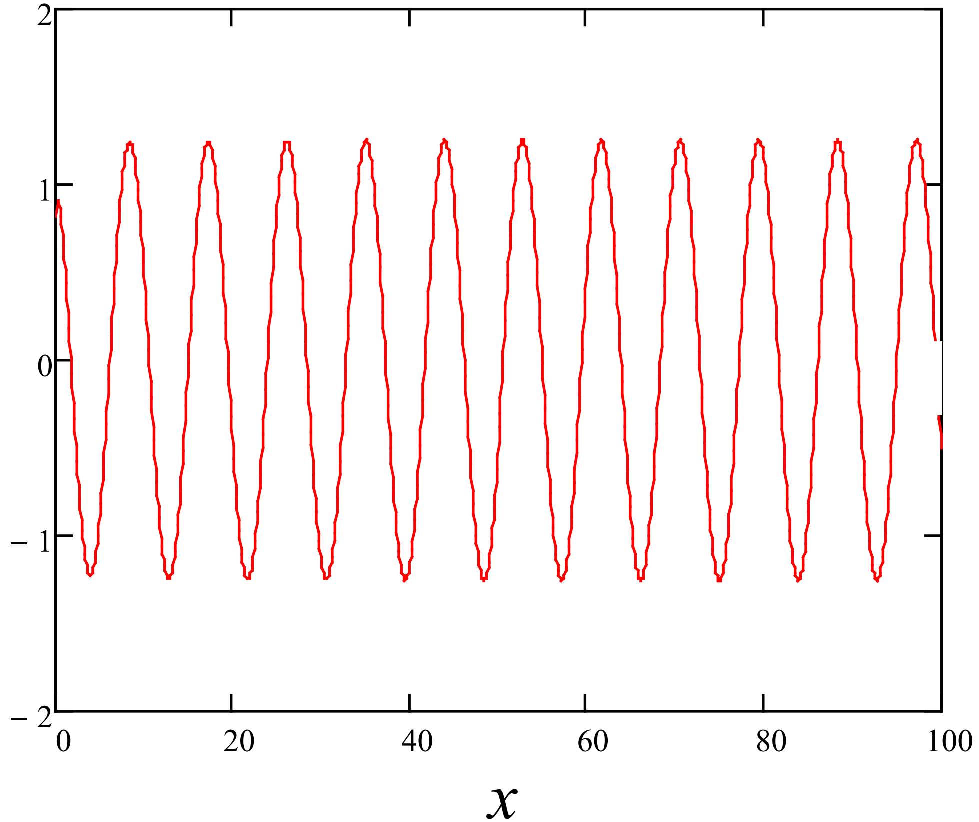
The Kei functions decay exponentially.
They are infinite at $x = 0$.



Kelvin Functions (cont.)

Normalizing makes it more obvious that the Ker and Kei functions decrease exponentially and also oscillate as x increases.

$$\frac{\mathrm{Ker}_0(x)}{\frac{1}{\sqrt{x}}e^{-x/\sqrt{2}}}$$



Kelvin Functions (cont.)

Normalizing makes it more obvious that the Ker and Kei functions decrease exponentially and also oscillate as x increases.

$$\frac{\mathrm{Kei}_0(x)}{\frac{1}{\sqrt{x}}e^{-x/\sqrt{2}}}$$