ECE 6382

Fall 2023

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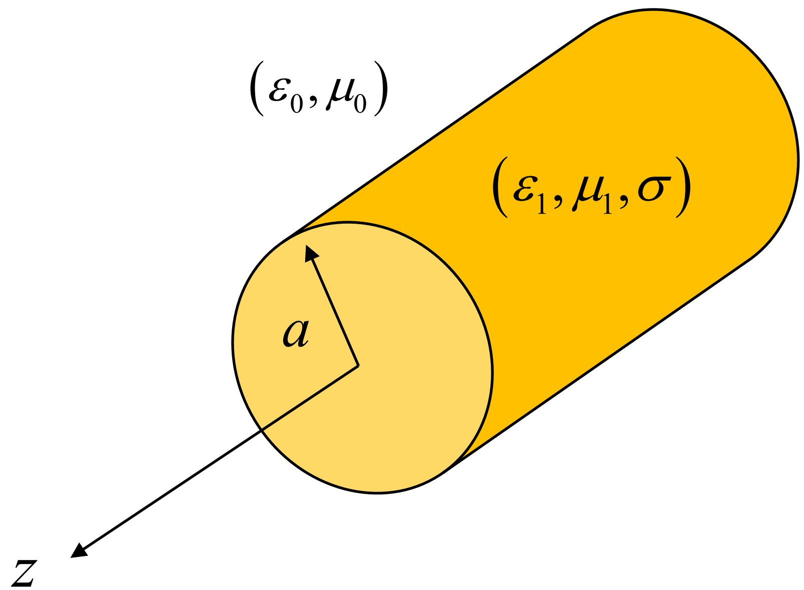
Notes 22

Applications of Bessel Functions

Note: j is used in this set of notes instead of i.

Impedance of Wire

A round wire made of conducting material is examined.



$$k_1 = \omega\sqrt{\mu\varepsilon_c}$$

$$\varepsilon_{c1} = \varepsilon_1 - j\frac{\sigma}{\omega}$$

The wire has a conductivity of σ.

We neglect the z variation of the fields inside the wire ($|k_z| << |k_1|$).

Impedance of Wire (cont.)

Inside the wire:

$$E_z = A J_0\left(k_{\rho 1}\rho\right) e^{-jk_z z}$$

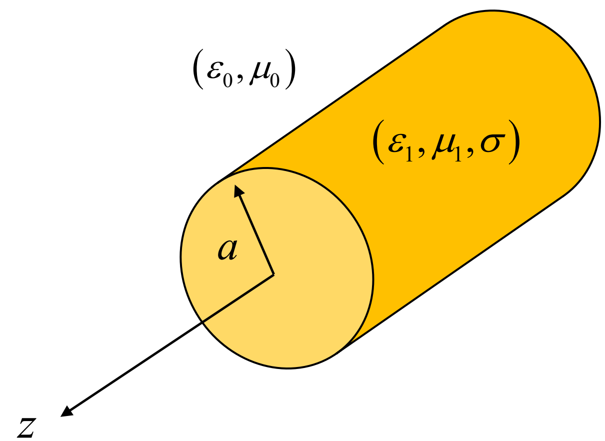
(for some constant A)

(The field must be finite on the z axis, no ϕ variation.)

Recall: $\psi = \begin{Bmatrix} J_\upsilon(k_{\rho 1}\rho) \\ Y_\upsilon(k_{\rho 1}\rho) \end{Bmatrix} \begin{Bmatrix} \sin(\upsilon\phi) \\ \cos(\upsilon\phi) \end{Bmatrix} e^{-jk_z z}$

$$\begin{aligned} k_1 &= \omega\sqrt{\mu_1 \varepsilon_c} \\ &= \omega\sqrt{\mu_1\varepsilon_1\left(1 - j\frac{\sigma}{\omega\varepsilon_1}\right)} \\ &\approx \omega\sqrt{\mu_1\varepsilon_1\left(-j\frac{\sigma}{\omega\varepsilon_1}\right)} \\ &= \sqrt{-j\omega\mu_1\sigma} \\ &= \sqrt{\omega\mu_1\sigma}\left(e^{-j\pi/4}\right) \\ &= \sqrt{2}\sqrt{\frac{\omega\mu_1\sigma}{2}}\left(e^{-j\pi/4}\right) \end{aligned}$$

$$k_{\rho 1} = \sqrt{k_1^2 - k_z^2} \approx k_1$$

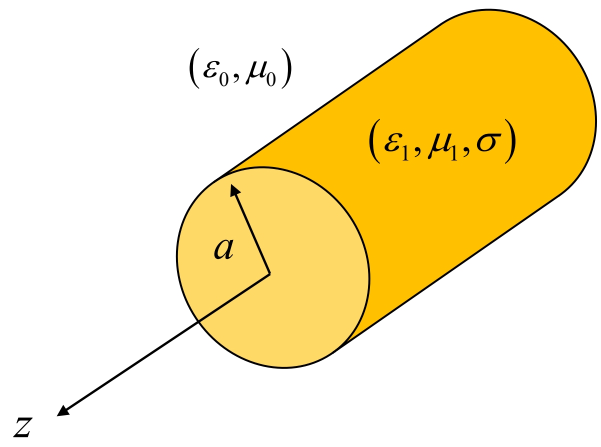


Note:
This assumes that there are no sources inside the wire.

Impedance of Wire (cont.)

Hence, we have

$$E_z = AJ_0\left(\frac{\rho}{\delta}\sqrt{2}\,e^{-j\pi/4}\right)$$



where

$$\delta = \sqrt{\frac{2}{\omega\mu_1\sigma}}$$ (skin depth of metal)

Note:

We can also write

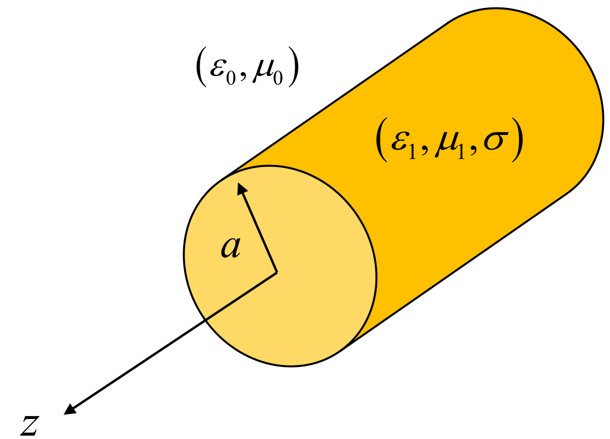
$$E_z = AJ_0\left(\frac{\rho}{\delta}(1-j)\right)$$

We can also write the field as

$$E_z = AJ_0\left(-\frac{\rho}{\delta}\sqrt{2}\,e^{j3\pi/4}\right) = AJ_0\left(\frac{\rho}{\delta}\sqrt{2}\,e^{j3\pi/4}\right)$$ (J_0 is an even function.)

Impedance of Wire (cont.)

$$E_z = AJ_0\left(\frac{\rho}{\delta}\sqrt{2}\,e^{j3\pi/4}\right)$$



Recall:

$$\mathrm{Ber}_\upsilon(x) \equiv \mathrm{Re}\left(J_\upsilon\left(xe^{j3\pi/4}\right)\right)$$

$$\mathrm{Bei}_\upsilon(x) \equiv \mathrm{Im}\left(J_\upsilon\left(xe^{j3\pi/4}\right)\right)$$

Therefore, we can write

$$E_z = A\left(\mathrm{Ber}_0\left(\frac{\rho}{\delta}\sqrt{2}\right) + j\mathrm{Bei}_0\left(\frac{\rho}{\delta}\sqrt{2}\right)\right)$$

Impedance of Wire (cont.)

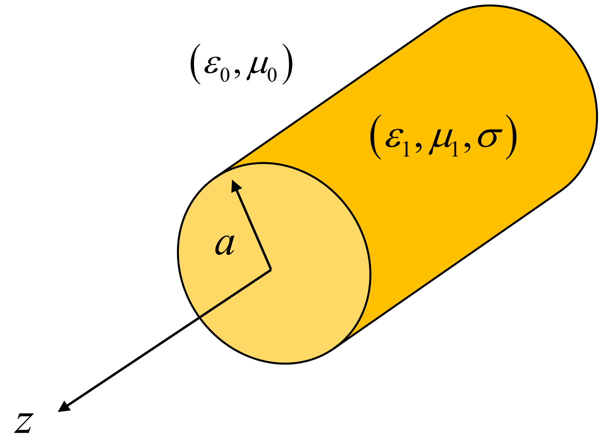
The current flowing in the wire is

$$I = \int_S J_z \, dS$$

$$= \int_0^{2\pi} \int_0^a J_z \, \rho d\rho d\phi$$

$$= 2\pi \int_0^a J_z \, \rho d\rho$$

$$= 2\pi\sigma \int_0^a E_z \, \rho d\rho$$

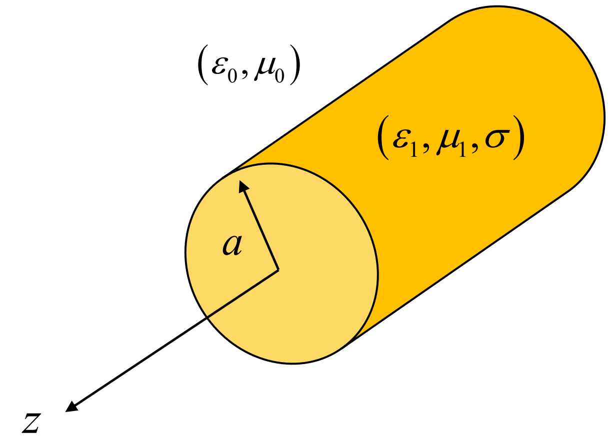


Hence $$I = 2\pi\sigma A \int_0^a J_0\left(\frac{\rho}{\delta}\sqrt{2}\, e^{j3\pi/4}\right) \rho d\rho$$

Impedance of Wire (cont.)

The impedance per unit length defined as:

$$Z_l \equiv \frac{E_z(a)}{I}$$

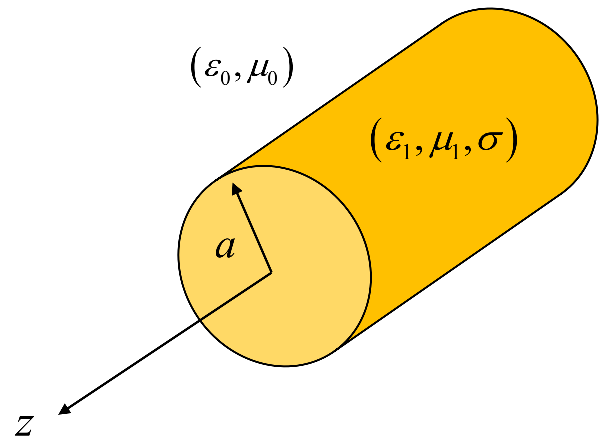


Hence,

$$Z_l = \frac{J_0\left(\frac{a}{\delta}\sqrt{2}\,e^{j3\pi/4}\right)}{2\pi\sigma\int_0^a J_0\left(\frac{\rho}{\delta}\sqrt{2}\,e^{j3\pi/4}\right)\rho d\rho}$$

Note:
This assumes that the wire is fed (excited) from the outside.

Impedance of Wire (cont.)



We have the following helpful integration identity:

$$\int J_0(x)\, x dx = x J_1(x)$$

Hence

$$\int_0^a J_0\left(\frac{\rho}{\delta}\sqrt{2}e^{j3\pi/4}\right)\rho d\rho = \left(\delta \frac{1}{\sqrt{2}} e^{-j3\pi/4}\right)^2 \int_0^L J_0(x)\, x dx = \left(\delta \frac{1}{\sqrt{2}} e^{-j3\pi/4}\right)^2 x J_1(x)\Big|_0^L$$

$$= \left(\delta \frac{1}{\sqrt{2}} e^{-j3\pi/4}\right)^2 L J_1(L)$$

$$= a\left(\delta \frac{1}{\sqrt{2}} e^{-j3\pi/4}\right) J_1(L)$$

$$= a\left(\delta \frac{1}{\sqrt{2}} e^{-j3\pi/4}\right) J_1\left(\frac{a}{\delta}\sqrt{2}e^{j3\pi/4}\right)$$

$$x \equiv \frac{\rho}{\delta}\sqrt{2}\, e^{j3\pi/4}$$

$$dx = d\rho\left(\frac{1}{\delta}\right)\sqrt{2}\, e^{j3\pi/4}$$

$$x dx = \rho d\rho\left(\frac{1}{\delta}\sqrt{2}\, e^{j3\pi/4}\right)^2$$

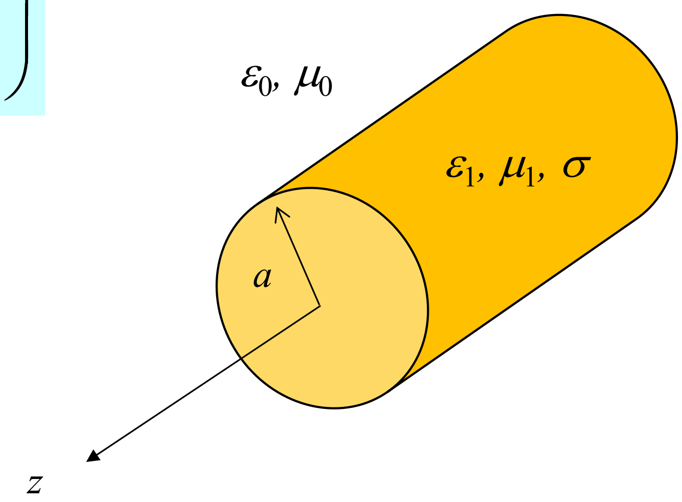
where

$$L \equiv \frac{a}{\delta}\sqrt{2}e^{j3\pi/4}$$

Impedance of Wire (cont.)

Hence, we have

$$Z_l = \frac{J_0\left(\frac{a}{\delta}\sqrt{2}\,e^{j3\pi/4}\right)}{2\pi\sigma\left(a\delta\frac{1}{\sqrt{2}}e^{-j3\pi/4}\right)J_1\left(\frac{a}{\delta}\sqrt{2}e^{j3\pi/4}\right)}$$



where

$$J_0\left(\frac{a}{\delta}\sqrt{2}e^{j3\pi/4}\right) = \mathrm{Ber}_0\left(\frac{a}{\delta}\sqrt{2}\right) + j\mathrm{Bei}_0\left(\frac{a}{\delta}\sqrt{2}\right)$$

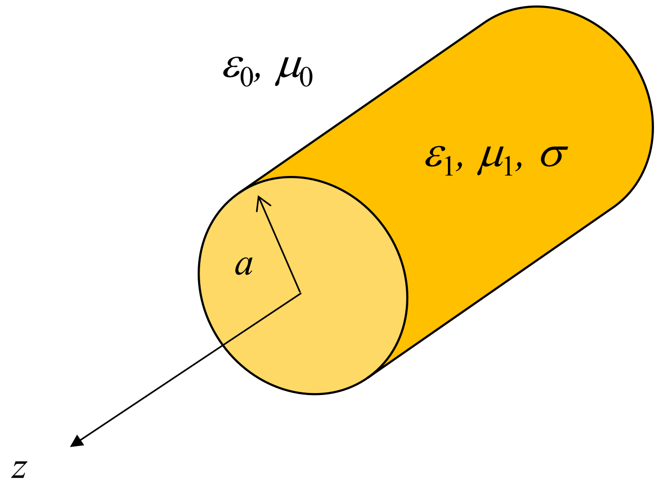
$$J_1\left(\frac{a}{\delta}\sqrt{2}e^{j3\pi/4}\right) = \mathrm{Ber}_1\left(\frac{a}{\delta}\sqrt{2}\right) + j\mathrm{Bei}_1\left(\frac{a}{\delta}\sqrt{2}\right)$$

Impedance of Wire (cont.)

At low frequency ($a << \delta$):

$$Z_l \approx \frac{1}{\sigma\left(\pi a^2\right)}$$

(ECE 3318)



At high frequency ($a >> \delta$):

$$Z_l \approx \frac{Z_s}{2\pi a}$$

(ECE 6340)

where

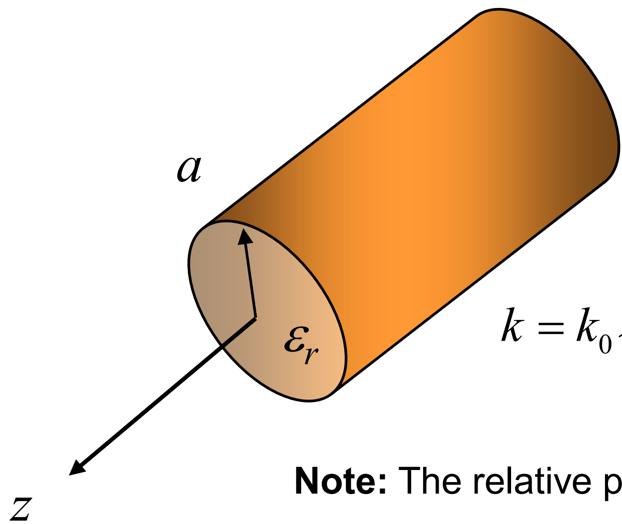
$$Z_s = R_s\left(1 + j\right)$$

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu_1}{2\sigma}}$$

(surface resistance of metal)

Circular Waveguide

The waveguide is homogeneously filled,
so we have independent TE_z and TM_z modes.



$$k = k_0 \sqrt{\varepsilon_r}$$

TM_z mode:

$$E_z = \psi\left(\rho, \phi, z\right)$$

Note: The relative permittivity could be complex (due to loss).

$$\psi = \begin{Bmatrix} J_\upsilon(k_\rho \rho) \\ Y_\upsilon(k_\rho \rho) \end{Bmatrix} \begin{Bmatrix} \sin(\upsilon\phi) \\ \cos(\upsilon\phi) \end{Bmatrix} e^{-jk_z z}$$

$$k_\rho^2 = k^2 - k_z^2$$

Circular Waveguide (cont.)

(1) ϕ variation $\quad \phi \in [0, 2\pi]$

$$\psi(\rho, \phi + 2\pi, z) = \psi(\rho, \phi, z)$$

(uniqueness of solution)

$$\Rightarrow \quad \upsilon = n$$

Choose $\cos(n\phi)$

$$\psi = \begin{Bmatrix} J_n(k_\rho \rho) \\ Y_n(k_\rho \rho) \end{Bmatrix} \cos(n\phi)\, e^{-jk_z z}$$

Circular Waveguide (cont.)

(2) The field should be finite on the z axis ($\rho = 0$)

$\Rightarrow$ $Y_n(k_\rho \rho)$ is not allowed

$$\psi = \cos(n\phi) J_n(k_\rho \rho) e^{-jk_z z}$$

$$k_\rho^2 = k^2 - k_z^2$$

Circular Waveguide (cont.)

(3) B.C.'s:

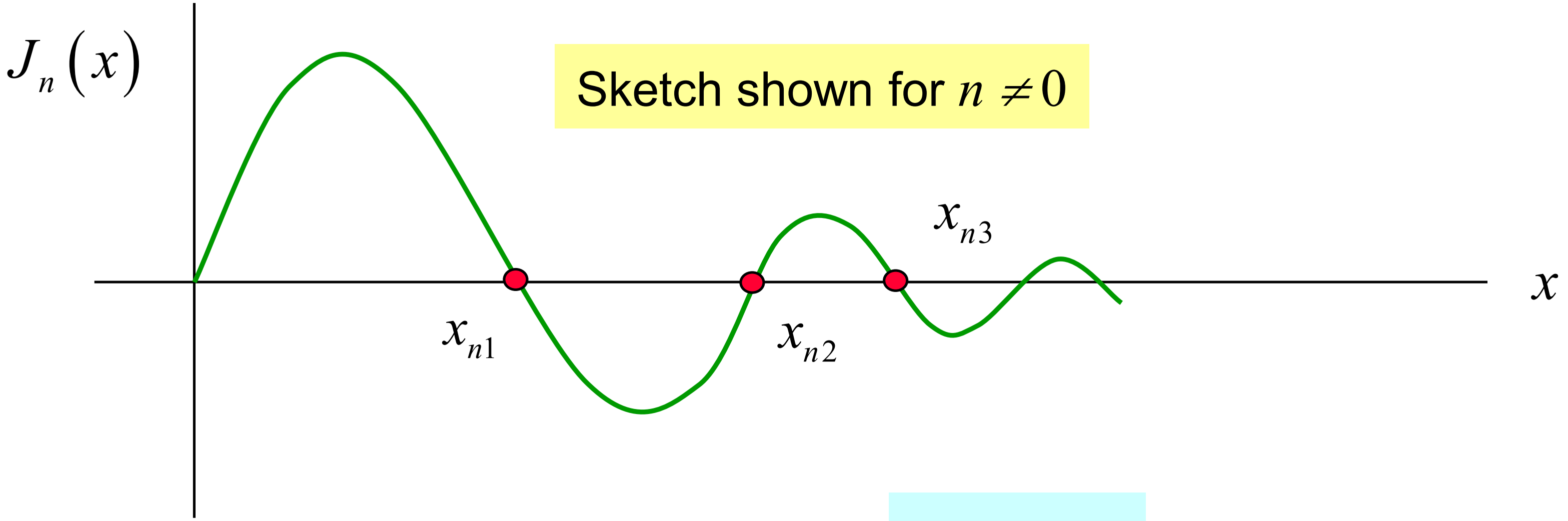
$$E_z\left(a,\phi,z\right)=0$$

Hence

$$J_n(k_\rho a)=0$$

Circular Waveguide (cont.)

$$J_n(k_\rho a) = 0$$



$$k_\rho a = x_{np} \quad \Rightarrow \quad k_\rho = \frac{x_{np}}{a}$$

Note: $x_{n0} = 0$ is not included since (for $n > 0$) $J_n(0) = 0$ (trivial solution).

Circular Waveguide (cont.)

TM$_{np}$ mode:

$$E_z = \cos(n\phi)\,J_n\left(x_{np}\frac{\rho}{a}\right)e^{-jk_z z} \qquad n = 0,1,2\ldots$$

$$k_z = \left(k^2 - \left(\frac{x_{np}}{a}\right)^2\right)^{1/2} \qquad p = 1,2,3,\ldots$$

Cutoff Frequency: TM_z

(We assume a lossless dielectric for the cutoff discussion.)

$$k_z^2 = k^2 - k_\rho^2$$

$$k_z = 0 \quad \Rightarrow \quad k = k_\rho = \frac{x_{np}}{a}$$

$$2\pi f_c \sqrt{\mu\varepsilon} = \frac{x_{np}}{a}$$

$$f_c^{\,\mathrm{TM}} = \left(\frac{c}{2\pi a\sqrt{\varepsilon_r}}\right) x_{np}$$

$$k_z = \begin{cases} \beta = \sqrt{k^2 - \left(\dfrac{x_{np}}{a}\right)^2}\,, & f > f_c^{\,\mathrm{TM}} \\ -j\alpha = -j\sqrt{\left(\dfrac{x_{np}}{a}\right)^2 - k^2}\,, & f < f_c^{\,\mathrm{TM}} \end{cases}$$

Cutoff Frequency: TM_z (cont.)

x_{np} values

$p \setminus n$	0	1	2	3	4	5
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.372	
4	11.792	13.324	14.796			

Ordering of modes by cutoff frequency: TM_{01}, TM_{11}, TM_{21}, TM_{02}, …

TE_z Modes

$$H_z = \psi\left(\rho, \phi, z\right)$$

$$\psi = \cos(n\phi)\, J_n(k_\rho \rho)\, e^{-jk_z z}$$

In this case the boundary condition is different:

$$\psi\left(a, \phi, z\right) \neq 0$$

TE_z Modes (cont.)

Set

$$E_\phi\left(a,\phi,z\right)=0$$

$$\nabla\times\underline{H}=j\omega\varepsilon\underline{E}$$

$$\Rightarrow E_\phi=\frac{1}{j\omega\varepsilon}\left(\frac{\partial H_\rho}{\partial z}-\frac{\partial H_z}{\partial\rho}\right)$$

At the boundary, the first term on the RHS is zero:

$$H_\rho\left(a,\phi,z\right)=0$$

Hence

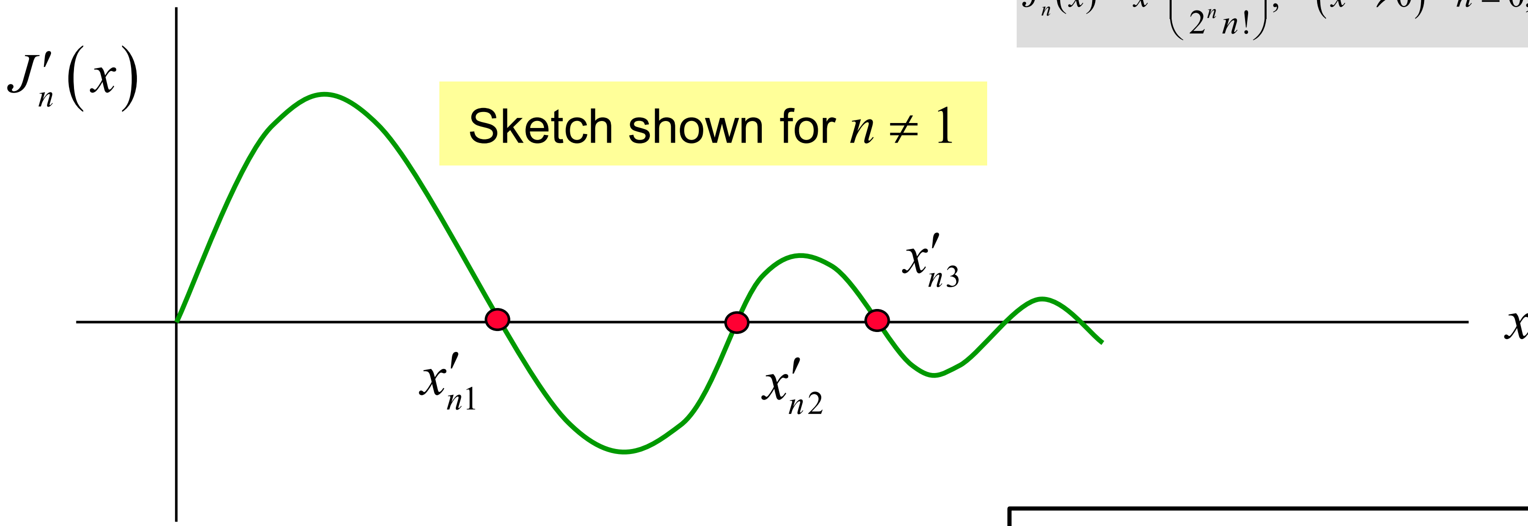
$$J_n'(k_\rho a)=0$$

TE$_z$ Modes (cont.)

$$J_n'(k_\rho a) = 0$$

Recall:

$$J_n(x) \sim x^n \left(\frac{1}{2^n n!}\right), \quad (x \to 0) \quad n = 0,1,2,....$$



Note:
$p = 0$ is not included
(see next slide).

$$k_\rho a = x_{np}'$$

$$k_\rho = \frac{x_{np}'}{a} \qquad p = 1,2,3,.....$$

TE_z Modes (cont.)

$$\psi = \cos(n\phi) J_n\left(x'_{np}\frac{\rho}{a}\right)e^{-jk_z z} \qquad p = 1, 2, \ldots$$

If $\quad p = 0, \quad x'_{np} = 0 \quad$ (but p cannot be zero for $n = 1$)

$$p = 0 \begin{cases} n \neq 0 & J_n\left(x'_{np}\frac{\rho}{a}\right) = J_n(0) = 0 \quad \text{(trivial soln.)} \\ n = 0 & J_0\left(x'_{np}\frac{\rho}{a}\right) = J_0(0) = 1 \end{cases}$$

$$\Rightarrow \quad \psi = e^{-jk_z z} = e^{-jkz} \qquad (k_\rho = 0)$$

This generates other field components that are zero; the resulting field has only H_z and violates the magnetic Gauss law.

Cutoff Frequency: TE$_z$

(We assume a lossless dielectric for the cutoff discussion.)

$$k_z^2 = k^2 - k_\rho^2$$

$$k_z = 0 \quad \Rightarrow \quad k_\rho = k = \frac{x'_{np}}{a}$$

$$2\pi f_c \sqrt{\mu\varepsilon} = \frac{x'_{np}}{a}$$

$$f_c^{\mathrm{TE}} = \left(\frac{c}{2\pi a \sqrt{\varepsilon_r}} \right) x'_{np}$$

$$k_z = \begin{cases} \beta = \sqrt{k^2 - \left(\dfrac{x'_{np}}{a} \right)^2}, & f > f_c^{\mathrm{TE}} \\ -j\alpha = -j\sqrt{\left(\dfrac{x'_{np}}{a} \right)^2 - k^2}, & f < f_c^{\mathrm{TE}} \end{cases}$$

Cutoff Frequency: TE_z

x'_{np} values

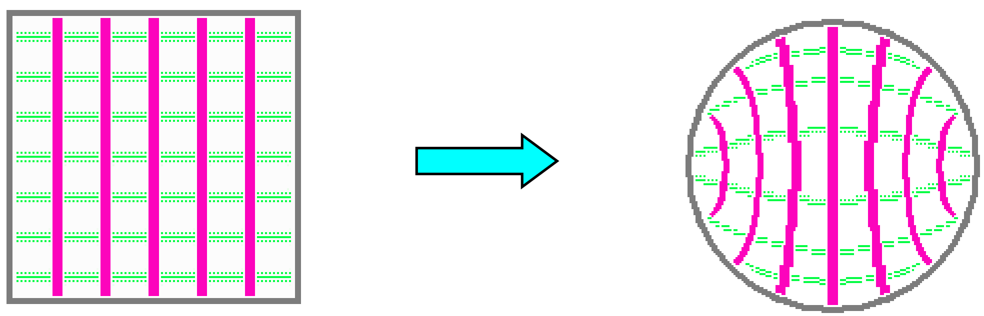
$p \setminus n$	0	1	2	3	4	5
1	3.832	1.841	3.054	4.201	5.317	5.416
2	7.016	5.331	6.706	8.015	9.282	10.520
3	10.173	8.536	9.969	11.346	12.682	13.987
4	13.324	11.706	13.170			

TE_{11}, TE_{21}, TE_{01}, TE_{31}, …….

TE$_{11}$ Mode

The dominant mode of circular waveguide is the TE$_{11}$ mode.

Electric field
Magnetic field



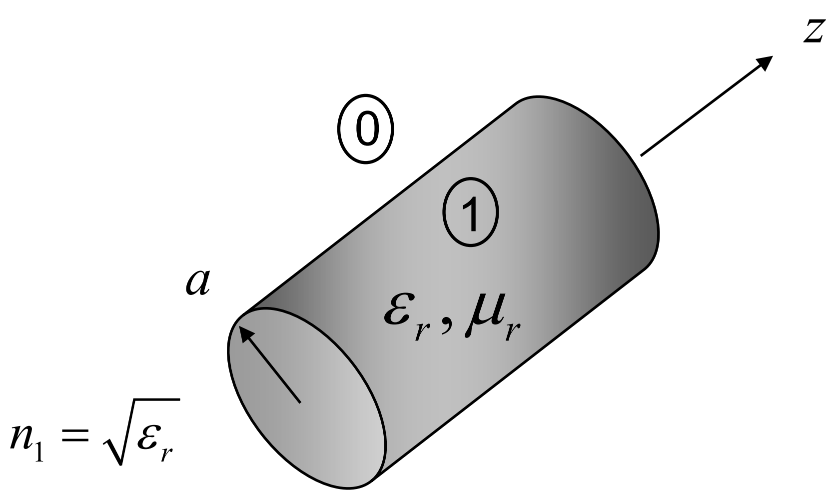
(from Wikipedia)

TE$_{10}$ mode of rectangular waveguide

TE$_{11}$ mode of circular waveguide

The TE$_{11}$ mode can be thought of as an evolution of the TE$_{10}$ mode of rectangular waveguide as the boundary changes shape.

Dielectric Rod



Unknown wavenumber:

$$k_0 < k_z < k_1$$

Modes are hybrid* unless:

$$\frac{\partial}{\partial \phi} = 0 \quad (n = 0)$$

Note:

We can have TE_{0p}, TM_{0p} modes

*This means that we need both E_z and H_z.

Dielectric Rod (cont.)

Representation of fields inside the rod:

$$E_{z1} = A J_n(k_{\rho 1}\rho)\sin(n\phi)e^{-jk_z z}$$

$$H_{z1} = B J_n(k_{\rho 1}\rho)\cos(n\phi)e^{-jk_z z}$$

$\rho < a$

where

$$k_{\rho 1}^2 = k_1^2 - k_z^2$$

(k_z is unknown)

Dielectric Rod (cont.)

To see choice of sin/cos, examine the field components (for example E_ρ):

From the Appendix:

$$E_\rho = -\frac{j\omega\mu}{k^2 - k_z^2}\frac{1}{\rho}\left(\frac{\partial H_z}{\partial \phi}\right) - \frac{jk_z}{k^2 - k_z^2}\left(\frac{\partial E_z}{\partial \rho}\right)$$

Dielectric Rod (cont.)

Representation of potentials outside the rod: $\rho > a$

Use

$$H_n^{(2)}(k_{\rho 0}\rho) = H_n^{(2)}(-j\alpha_{\rho 0}\rho)$$

where

$$k_{\rho 0} = \left(k_0^2 - k_z^2\right)^{1/2} = -j\alpha_{\rho 0}$$

$$\alpha_{\rho 0} = \sqrt{k_z^2 - k_0^2}$$

Note: $\alpha_{\rho 0}$ is interpreted as a positive real number in order to have decay radially in the air region.

Dielectric Rod (cont.)

Useful identity:

$$H_n^{(2)}(-jx) = (-1)^{n+1} H_n^{(1)}(+jx)$$

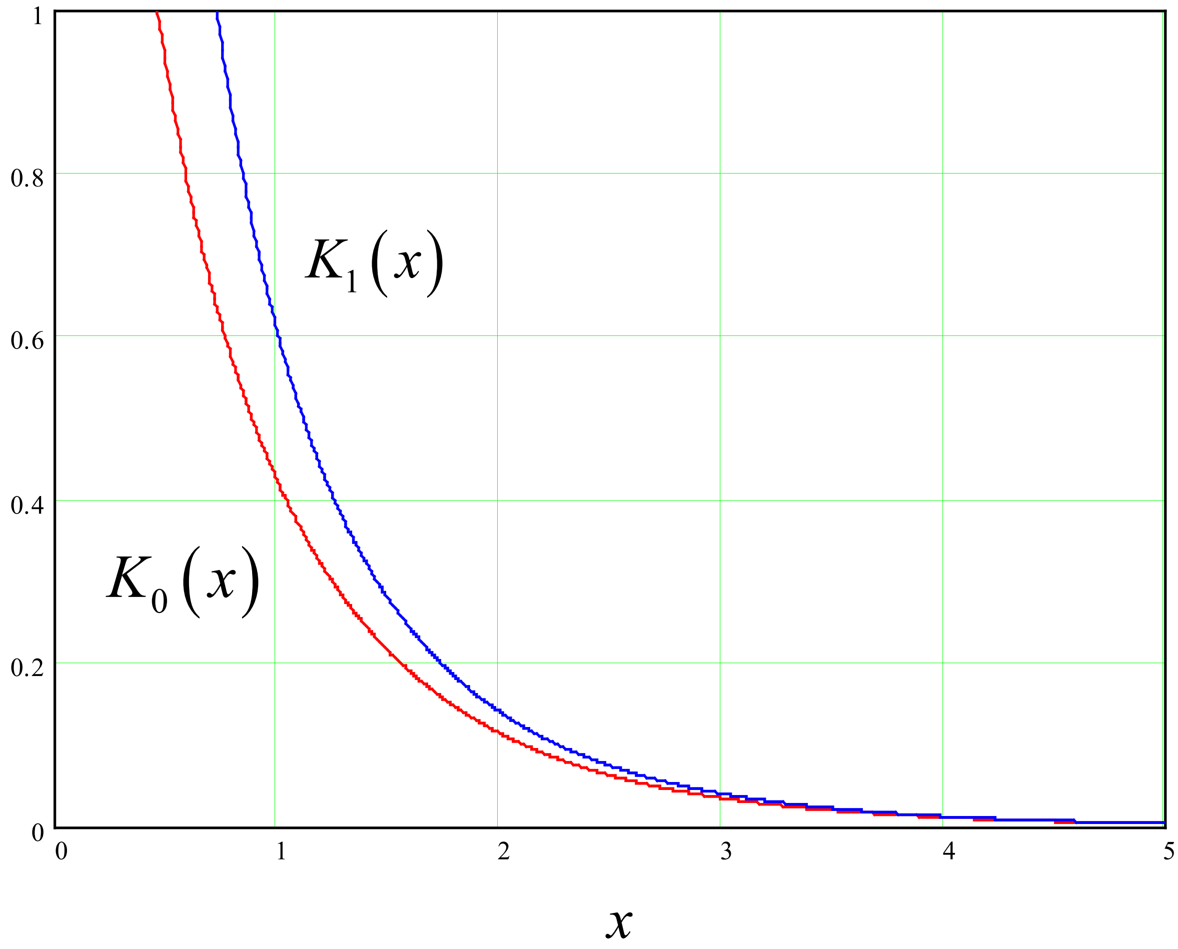
Another useful identity:

$$H_n^{(1)}(jx) = \frac{2}{\pi} j^{-(n+1)} K_n(x)$$

$K_n(x)$ = modified Bessel function of the second kind.

Dielectric Rod (cont.)

The modified Bessel functions decay exponentially.



Dielectric Rod (cont.)

Hence, we choose the following forms in the air region ($\rho > a$):

$$E_{z0} = C K_n(\alpha_{\rho 0}\rho)\sin(n\phi)\, e^{-jk_z z}$$

$$H_{z0} = D K_n(\alpha_{\rho 0}\rho)\cos(n\phi)\, e^{-jk_z z}$$

$$\alpha_{\rho 0} = \sqrt{k_z^2 - k_0^2}$$

Dielectric Rod (cont.)

Match E_z, H_z, E_ϕ, H_ϕ at $\rho = a$:

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example: $E_{z1} = E_{z0}$ ⇨ $AJ_n\left(k_{\rho 1} a\right) = CK_n\left(\alpha_{\rho 0} a\right)$

or $AJ_n\left(k_{\rho 1} a\right) + B(0) + C\left(-K_n\left(\alpha_{\rho 0} a\right)\right) + D(0) = 0$

so $M_{11} = J_n\left(k_{\rho 1} a\right), \quad M_{13} = -K_n\left(\alpha_{\rho 0} a\right), \quad M_{12} = M_{14} = 0$

$$k_{\rho 1}^2 = k_1^2 - k_z^2 \; , \quad \alpha_{\rho 0} = \sqrt{k_z^2 - k_0^2}$$

Recall:

$E_{z1} = A J_n\left(k_{\rho 1}\rho\right)\sin\left(n\phi\right)e^{-jk_z z}$

$E_{z0} = CK_n(\alpha_{\rho 0}\rho)\sin(n\phi)e^{-jk_z z}$

Dielectric Rod (cont.)

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To have a non-trivial solution, we require that

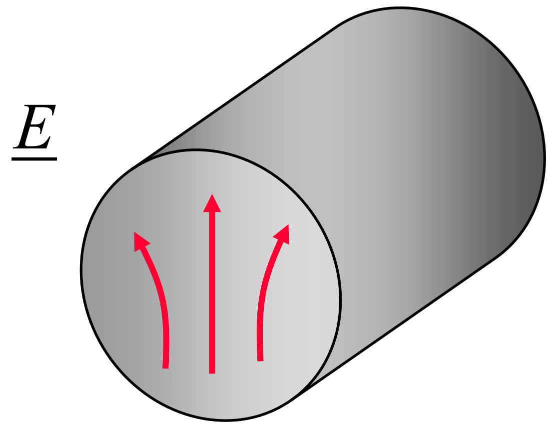
$$\det\left[M(k_z, \omega) \right] = 0$$

There will be an infinite number of solutions ($p = 1, 2, \ldots$), for each assumed value of n.

This is a transcendental equation for the unknown k_z (for a given frequency ω).

Dielectric Rod (cont.)

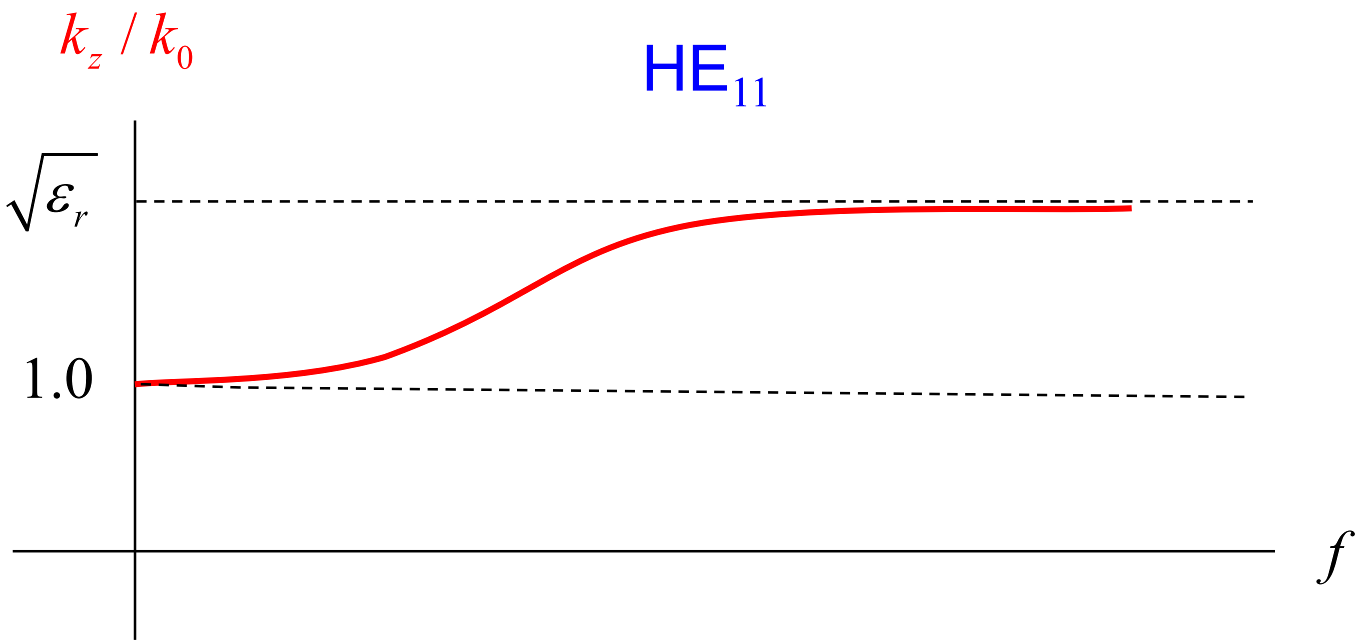
Dominant mode (lowest cutoff frequency): HE_{11} $(f_c = 0)$



This is the mode that is used in fiber-optic guides (single-mode fiber).

Dielectric Rod (cont.)

Sketch of normalized wavenumber

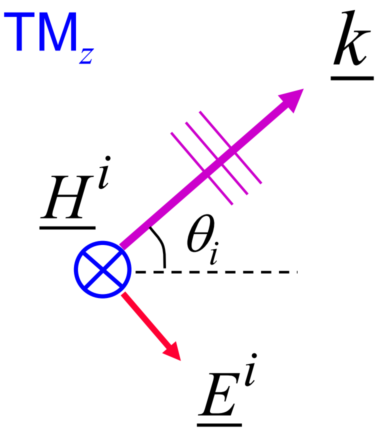


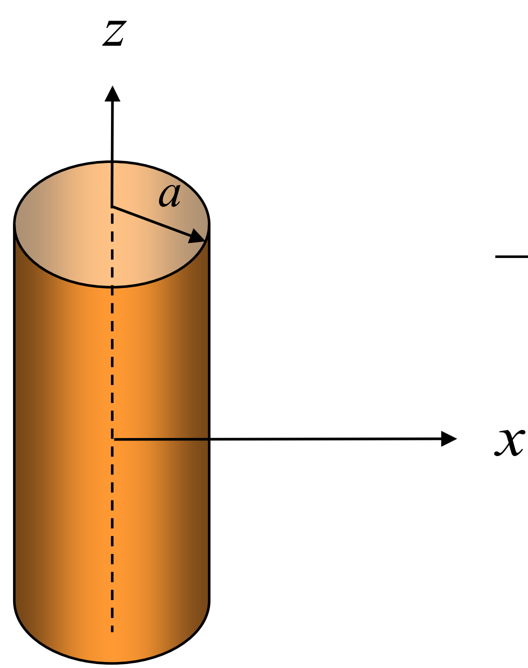
$$\alpha_{\rho 0} = \sqrt{k_z^2 - k_0^2} = k_0 \sqrt{\left(k_z / k_0\right)^2 - 1}$$

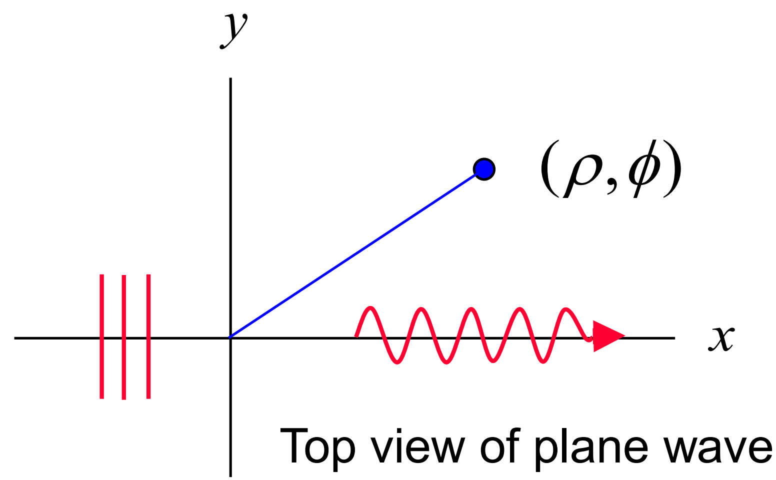
At higher frequencies, the fields are more tightly bound to the rod.

Scattering by Cylinder

A TM_z plane wave is incident on a PEC cylinder.







$$\underline{H}^i = \underline{\hat{y}}\, H_{y0}\, e^{-j(k_x x + k_z z)}$$

$$k_x = k_0 \cos\theta_i$$

$$k_z = k_0 \sin\theta_i$$

Scattering by Cylinder (cont.)

From the plane-wave properties, we have

$$E_z^i = -\eta_0 H_{y0} \cos\theta_i \, e^{-j(k_x x + k_z z)}$$

The total field is written as the sum of incident and scattered parts:

For $\rho \geq a$:

$$E_z = E_z^i + E_z^s$$

Note:
For any wave of the form $\exp(-jk_z z)$, all field components can be put in terms of E_z and H_z. This is why it is convenient to work with E_z.
Please see the Appendix.

Scattering by Cylinder (cont.)

We first put E_z^i into cylindrical form using the Jacobi-Anger identity*:

$$E_z^i = -\eta_0 H_{y0} \cos\theta_i \, e^{-jk_z z} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{j^n} \right) J_n(k_\rho \rho) \, e^{jn\phi}$$

where $k_\rho = k_x = \sqrt{k_0^2 - k_z^2} = k_0 \cos\theta_i$

Recall :

$$e^{-jkx} = \sum_{n=-\infty}^{\infty} (-j)^n J_n(k\rho) e^{jn\phi}$$

Let $k \to k_x \to k_\rho$

Assume the following form for the scattered field:

$$E_z^s = -\eta_0 H_{y0} \cos\theta_i \, e^{-jk_z z} \sum_{n=-\infty}^{+\infty} a_n \left(\frac{1}{j^n} \right) H_n^{(2)}(k_\rho \rho) \, e^{jn\phi}$$

*This was derived previously using the generating function.

Scattering by Cylinder (cont.)

At $\rho = a$ $\quad E_z(a, \phi, z) = 0$

Hence

$$E_z^s(a, \phi, z) = -E_z^i(a, \phi, z)$$

This yields

$$J_n(k_\rho a) = -a_n H_n^{(2)}(k_\rho a)$$

or

$$a_n = -\frac{J_n(k_\rho a)}{H_n^{(2)}(k_\rho a)}$$

Scattering by Cylinder (cont.)

We then have

$$E_z^s = -\eta_0 H_{y0} \cos\theta_i \, e^{-jk_z z} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{j^n}\right)\left(\frac{-J_n\left(k_\rho a\right)}{H_n^{(2)}(k_\rho a)}\right) H_n^{(2)}(k_\rho \rho)\, e^{jn\phi}$$

and

$$H_z^s = 0 \quad (\text{TM}_z)$$

The other components of the scattered field can be found from the formulas in the Appendix.

Appendix

For any wave of the form $\exp(-jk_z z)$,
all field components can be put in terms of E_z and H_z
(derivation omitted).

$$E_x = \frac{-j\omega\mu}{k^2 - k_z^2}\frac{\partial H_z}{\partial y} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial E_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{k^2 - k_z^2}\frac{\partial H_z}{\partial x} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial E_z}{\partial y}$$

$$H_x = \frac{j\omega\varepsilon_c}{k^2 - k_z^2}\frac{\partial E_z}{\partial y} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-j\omega\varepsilon_c}{k^2 - k_z^2}\frac{\partial E_z}{\partial x} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial H_z}{\partial y}$$

Appendix (cont.)

These may be written more compactly as

$$\underline{E}_t = \frac{j\omega\mu}{k^2 - k_z^2}\left(\underline{\hat{z}} \times \nabla_t H_z\right) - \frac{jk_z}{k^2 - k_z^2}\left(\nabla_t E_z\right)$$

$$\underline{H}_t = \frac{-j\omega\varepsilon}{k^2 - k_z^2}\left(\underline{\hat{z}} \times \nabla_t E_z\right) - \frac{jk_z}{k^2 - k_z^2}\left(\nabla_t H_z\right)$$

where

$$\nabla_t \Psi \equiv \underline{\hat{x}} \frac{\partial \Psi}{\partial x} + \underline{\hat{y}} \frac{\partial \Psi}{\partial y}$$

Appendix (cont.)

In cylindrical coordinates we have

$$\nabla_t \Psi = \underline{\hat{\rho}} \frac{\partial \Psi}{\partial \rho} + \underline{\hat{\phi}} \frac{1}{\rho} \frac{\partial \Psi}{\partial \phi}$$

This allows us to calculate the field components in terms of E_z and H_z in cylindrical coordinates.

Appendix (cont.)

In cylindrical coordinates we then have

$$E_\rho = -\frac{j\omega\mu}{k^2 - k_z^2}\frac{1}{\rho}\left(\frac{\partial H_z}{\partial \phi}\right) - \frac{jk_z}{k^2 - k_z^2}\left(\frac{\partial E_z}{\partial \rho}\right)$$

$$E_\phi = \frac{j\omega\mu}{k^2 - k_z^2}\left(\frac{\partial H_z}{\partial \rho}\right) - \frac{jk_z}{k^2 - k_z^2}\frac{1}{\rho}\left(\frac{\partial E_z}{\partial \phi}\right)$$

$$H_\rho = \frac{j\omega\varepsilon}{k^2 - k_z^2}\frac{1}{\rho}\left(\frac{\partial E_z}{\partial \phi}\right) - \frac{jk_z}{k^2 - k_z^2}\left(\frac{\partial H_z}{\partial \rho}\right)$$

$$H_\phi = -\frac{j\omega\varepsilon}{k^2 - k_z^2}\left(\frac{\partial E_z}{\partial \rho}\right) - \frac{jk_z}{k^2 - k_z^2}\frac{1}{\rho}\left(\frac{\partial H_z}{\partial \phi}\right)$$