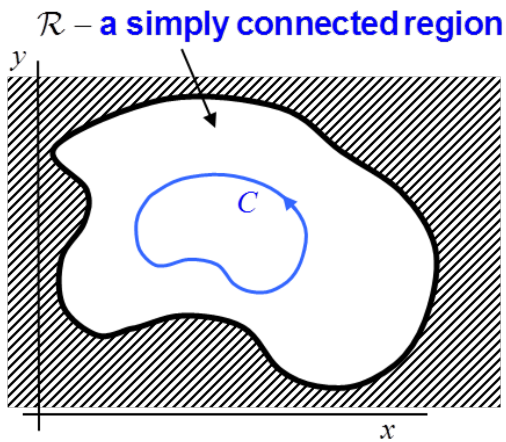
ECE 6382



Fall 2023

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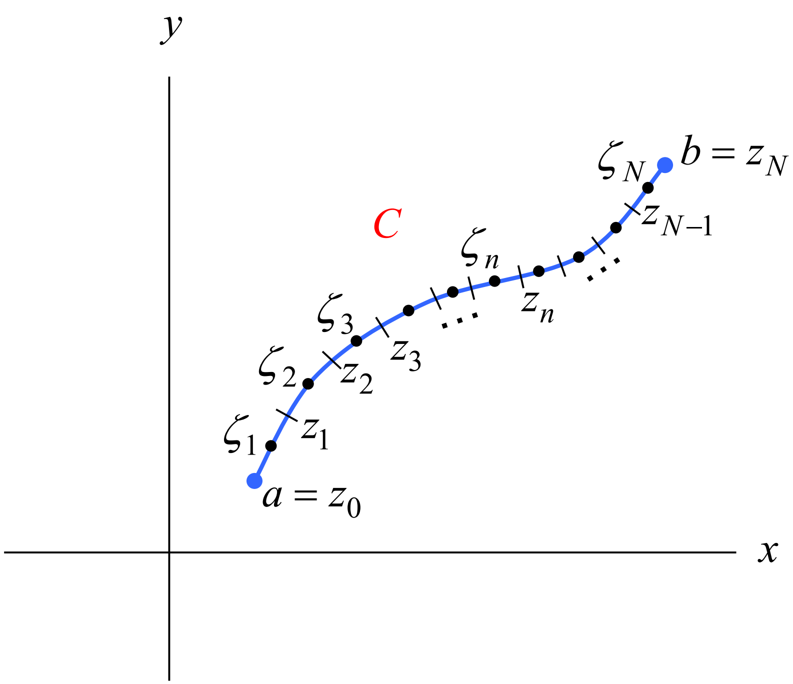
Notes 3

Integration in the Complex Plane

Notes are adapted from D. R. Wilton, Dept. of ECE

Defining Line Integrals in the Complex Plane



$$I \equiv \int_C f(z)\,dz = \int_a^b f(z)\,dz$$

- Define ζ_n on C between z_{n-1} and z_n
- Consider the sums

$$I_N = \sum_{n=1}^{N} f(\zeta_n)\overbrace{(z_n - z_{n-1})}^{\Delta z_n}$$

- Let the number of subdivisions $N \to \infty$ such that $\Delta z_n = (z_n - z_{n-1}) \to 0$ and define

$$I \equiv \int_a^b f(z)\,dz = \lim_{N\to\infty} I_N$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} f(\zeta_n)\overbrace{(z_n - z_{n-1})}^{\Delta z_n}$$

(The result is independent of the details of the path subdivision, for reasonably well-behaved functions.)

Equivalence Between Complex and Real Line Integrals

Denote

$$I \equiv \int_a^b f(z)\,dz = \int_a^b \left[u(x,y) + iv(x,y)\right](dx + idy)$$

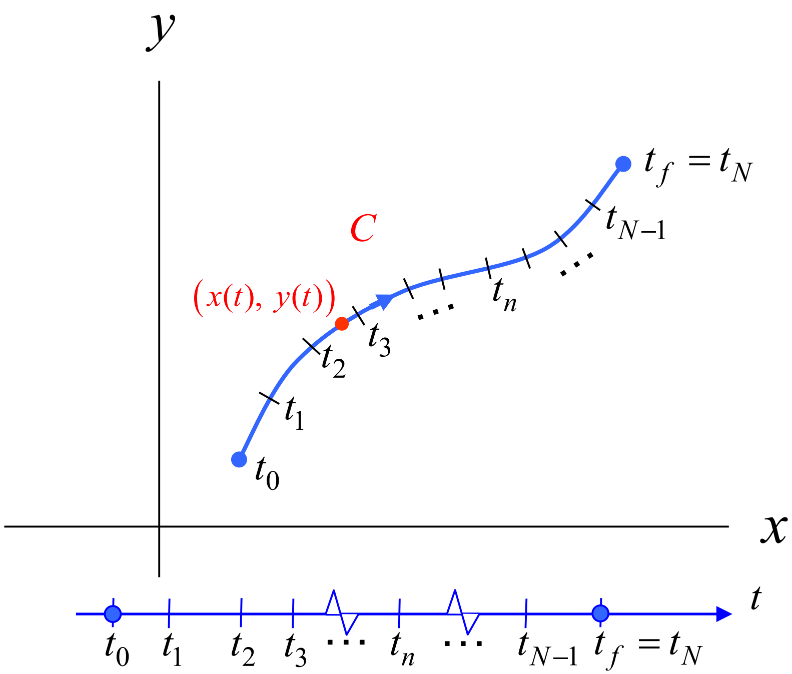
$$= \int_{a=(x_0,y_0)}^{b=(x_N,y_N)} u(x,y)\,dx - v(x,y)\,dy \; + \; i \int_{a=(x_0,y_0)}^{b=(x_N,y_N)} v(x,y)\,dx + u(x,y)\,dy$$

$$= \int_C u\,dx - v\,dy + \; i\int_C v\,dx + u\,dy$$

$$I = \int_C u\,dx - v\,dy + \; i\int_C v\,dx + u\,dy$$

The complex line integral is equivalent to <u>two</u> real line integrals on C.

Review of Line Integral Evaluation



- A line integral written as $\int_C u(x,y)\,dx - v(x,y)\,dy$ is really a shorthand for

$$\int_{t_0}^{t_f} \left(u\frac{dx}{dt} - v\frac{dy}{dt} \right) dt$$

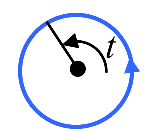
where t is some parameterization of C:

$$C: x = x(t),\ \ y = y(t), \quad t_0 \le t \le t_f$$

◊ Example : parameterizations of the circle $x^2 + y^2 = a^2$

The path C goes counterclockwise around the circle.

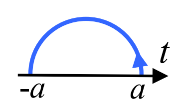
1) $x = a\cos t,\ y = a\sin t, \quad 0 \le t\ (=\theta) \le 2\pi$

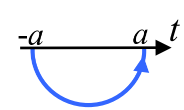


2) $x = t\,,\ y = \sqrt{a^2 - t^2}, \quad t_0 = a, \quad t_f = -a,$

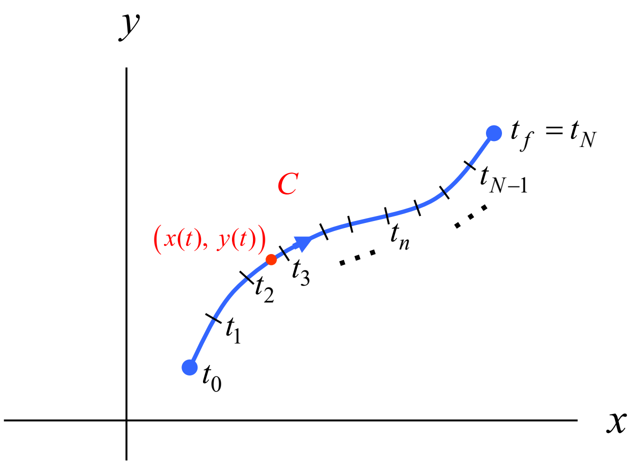
and

$x = t\,,\ y = -\sqrt{a^2 - t^2}, \quad t_0 = -a, \quad t_f = a,$





Review of Line Integral Evaluation (cont.)



- While it may be possible to parameterize C using x or y as the independent parameter, it must be remembered that the other variable (y or x) is in general always a *function* of that parameter!

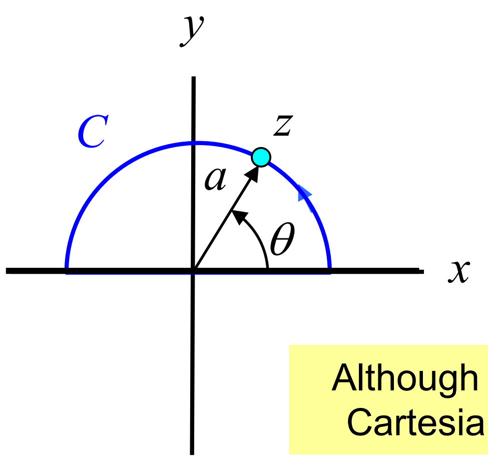
Illustration: $\int_C u(x,y)\,dx - v(x,y)\,dy$

The red color denotes functional dependence.

$$= \int_{x_0}^{x_f} \left[u\big(x, y(x)\big) - v\big(x, y(x)\big)\frac{dy}{dx} \right] dx$$ (if x is the independent parameter)

$$= \int_{y_0}^{y_f} \left[u\big(x(y), y\big)\frac{dx}{dy} - v\big(x(y), y\big) \right] dy$$ (if y is the independent parameter)

Line Integral Example



◊ Evaluate $I = \int_C \frac{1}{z} dz$: where

$$C: \; x = a\cos\theta, \; y = a\sin\theta, \quad 0 \le \theta \le \pi$$

Although it is easier to use polar coordinates (see the next example), we use Cartesian coordinates to illustrate the previous Cartesian line integral form.

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\left(\frac{x-iy}{x-iy}\right) = \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right) = \left(\frac{x}{a^2}\right) + i\left(\frac{-y}{a^2}\right)$$

Hence

$$u(z) = \frac{x}{a^2}$$

$$v(z) = \frac{-y}{a^2}$$

$$I = \int_C u\,dx - v\,dy + \; i\int_C v\,dx + u\,dy$$

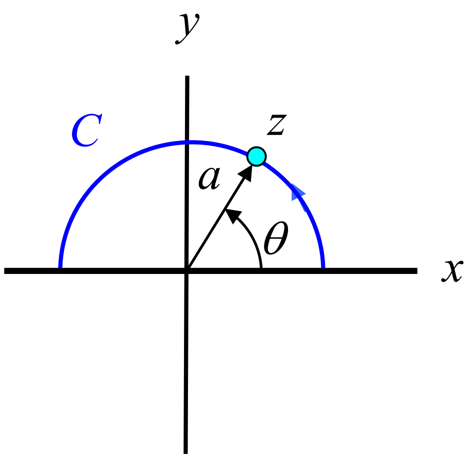
$$= \int_C (u+iv)\,dx + \int_C (-v+iu)\,dy$$

$$= I_1 + I_2$$

The red color denotes functional dependence.

$$I_1 = \int_a^{-a} \left[\frac{x}{a^2} + i\left(\frac{-y(x)}{a^2}\right)\right] dx \qquad I_2 = \int_0^0 \left[\frac{y}{a^2} + i\left(\frac{x(y)}{a^2}\right)\right] dy$$

Line Integral Example (cont.)



$$I_1 = \int_a^{-a} \left[\frac{x}{a^2} + i\left(\frac{-y(x)}{a^2} \right) \right] dx$$

$$= \int_a^{-a} \left[\frac{x}{a^2} + i\left(\frac{-1}{a^2} \right) \sqrt{a^2 - x^2} \right] dx$$

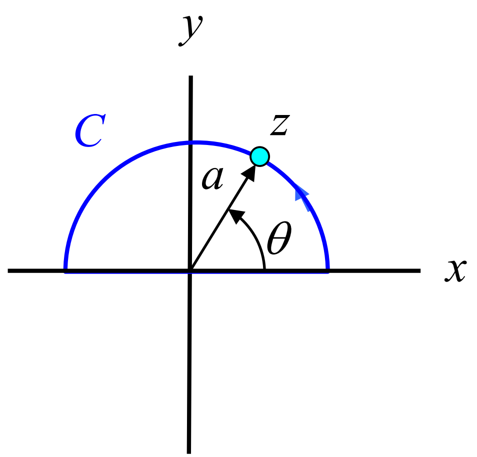
$$= 0 + i\left(\frac{-1}{a^2} \right) \int_a^{-a} \sqrt{a^2 - x^2}\, dx$$

$$= i\left(\frac{-1}{a^2} \right) \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a} \right) \right]_a^{-a}$$

$$= i\left(\frac{-1}{\cancel{a^2}} \right) \left[\frac{\cancel{a^2}}{2} \left(-\frac{\pi}{2} - \frac{\pi}{2} \right) \right]$$

$$= i\left(\frac{\pi}{2} \right)$$

Line Integral Example (cont.)



$$I_2 = \int_0^0 \left[\frac{y}{a^2} + i\left(\frac{x(y)}{a^2} \right) \right] dy$$

$$= 0 + \frac{i}{a^2} \int_0^0 x(y)\, dy$$

$$= \underbrace{\frac{i}{a^2} \int_0^a \sqrt{a^2 - y^2}\, dy}_{0 \le \theta \le \pi/2} + \underbrace{\frac{i}{a^2} \int_a^0 \left(-\sqrt{a^2 - y^2} \right) dy}_{\pi/2 \le \theta \le \pi}$$

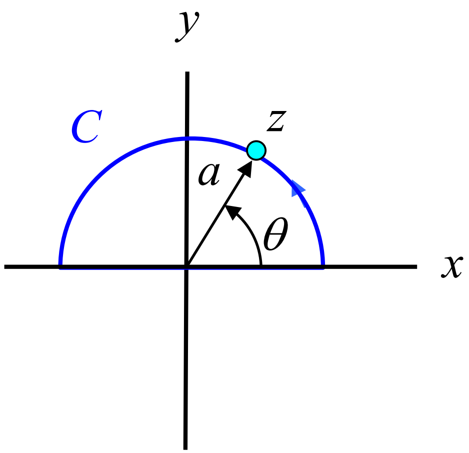
$$= 2\frac{i}{a^2} \int_0^a \sqrt{a^2 - y^2}\, dy$$

$$= 2\frac{i}{a^2} \left[\frac{y\sqrt{a^2 - y^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{y}{a} \right) \right]_0^a$$

$$= 2\frac{i}{a^2} \left[\frac{a^2}{2} \sin^{-1}\left(\frac{a}{a} \right) \right]$$

$$= i\left(\frac{\pi}{2} \right)$$

Line Integral Example (cont.)



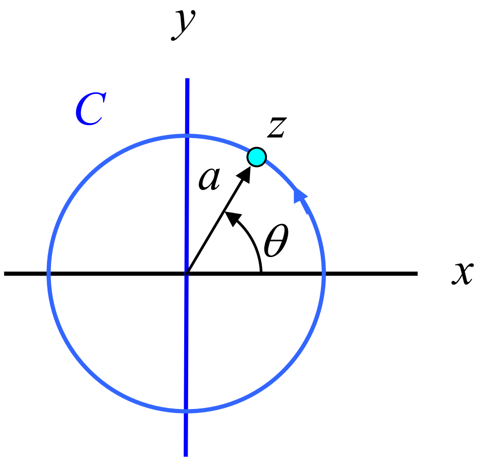
$$I = I_1 + I_2 = i\left(\frac{\pi}{2}\right) + i\left(\frac{\pi}{2}\right)$$

Hence

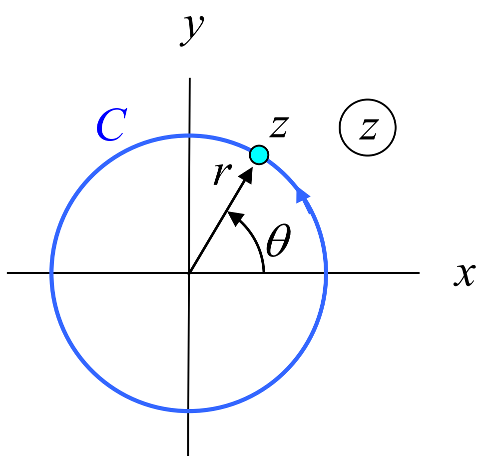
$$I = i\pi$$

Note: By symmetry (compare z and $-z$, and compare dz), we also have:

$$\oint_C \frac{1}{z}\,dz = 2\pi i$$



Line Integral Example



◊ Evaluate $\oint_C z^n dz$: where

$$C: \; x = r\cos\theta, \; y = r\sin\theta, \quad 0 \le \theta \le 2\pi$$

$$\Rightarrow \quad z = r\cos\theta + i\,r\sin\theta = re^{i\theta},$$

$$\Rightarrow \quad dz = rie^{i\theta}d\theta,$$

$$\Rightarrow \quad \oint_C z^n dz = \int_0^{2\pi} \left(re^{i\theta}\right)^n rie^{i\theta} d\theta$$

$$= ir^{n+1}\int_0^{2\pi} e^{i\theta(n+1)} d\theta = \cancel{i}r^{n+1} \left.\frac{e^{i\theta(n+1)}}{\cancel{i}(n+1)}\right|_0^{2\pi} \quad (n \ne -1)$$

$$= r^{n+1}\frac{e^{i2\pi(n+1)}-1}{(n+1)} = \begin{cases} 0, & n \ne -1 \\ 2\pi i, & n = -1 \end{cases}$$

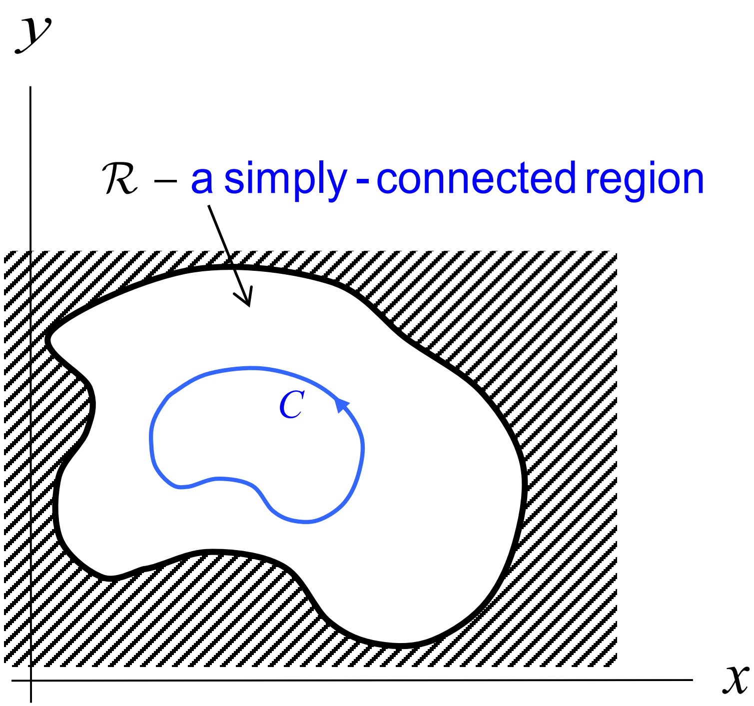
This is a useful result, and it is used to prove the "residue theorem".

Note: For $n = -1$, we can use the result on slide 9 (or just evaluate the integral in θ directly).

Cauchy's Theorem

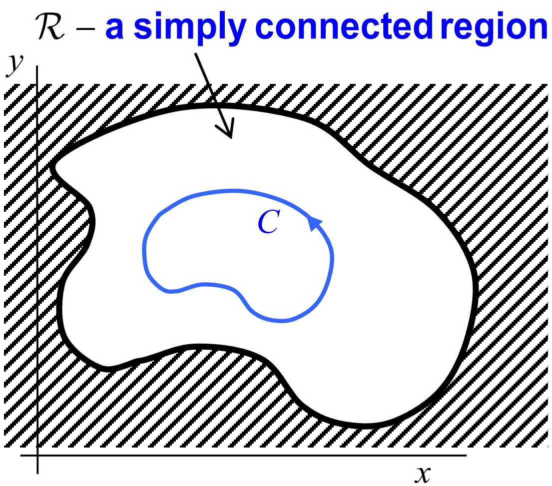
Cauchy's theorem:

If $f(z)$ is analytic in $\mathcal{R}$ then $\oint_C f(z)\,dz = 0$



A "simply-connected" region means that there are no "holes" in the region. (Any closed path can be shrunk down to zero size.)

Proof of Cauchy's Theorem



- First, note that (from slide 3) if $f(z) = w = u + iv$, then
  $$\oint_C f(z)\,dz = \oint_C u\,dx - v\,dy + i \oint_C v\,dx + u\,dy\,;$$
  then use Stokes's theorem (see below).

- Construct 2D vectors $\underline{A} = u\underline{\hat{x}} - v\underline{\hat{y}}, \quad \underline{B} = v\underline{\hat{x}} + u\underline{\hat{y}}, \quad d\underline{r} = dx\,\underline{\hat{x}} + dy\,\underline{\hat{y}}$ in the xy – plane and write the integral above as

$$\oint_C f(z)\,dz = \oint_C \underline{A}\cdot d\underline{r} + i\oint_C \underline{B}\cdot d\underline{r} \overset{\text{Stokes's Theorem}}{=} \int_{\text{interior of } C} \left(\nabla\times\underline{A}\right)\cdot\underline{\hat{z}}\,dS + i\int_{\text{interior of } C} \left(\nabla\times\underline{B}\right)\cdot\underline{\hat{z}}\,dS,$$ but

$$\underline{\hat{z}}\cdot\left(\nabla\times\underline{A}\right) = \underline{\hat{z}}\cdot\begin{vmatrix} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & -v & 0 \end{vmatrix} = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \overset{\text{C.R. cond's}}{=} 0, \qquad \underline{\hat{z}}\cdot\left(\nabla\times\underline{B}\right) = \underline{\hat{z}}\cdot\begin{vmatrix} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v & u & 0 \end{vmatrix} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \overset{\text{C.R. cond's}}{=} 0$$

$$\Rightarrow \boxed{\oint_C f(z)\,dz = 0}$$

Proof of Cauchy's Theorem (cont.)

Some comments:

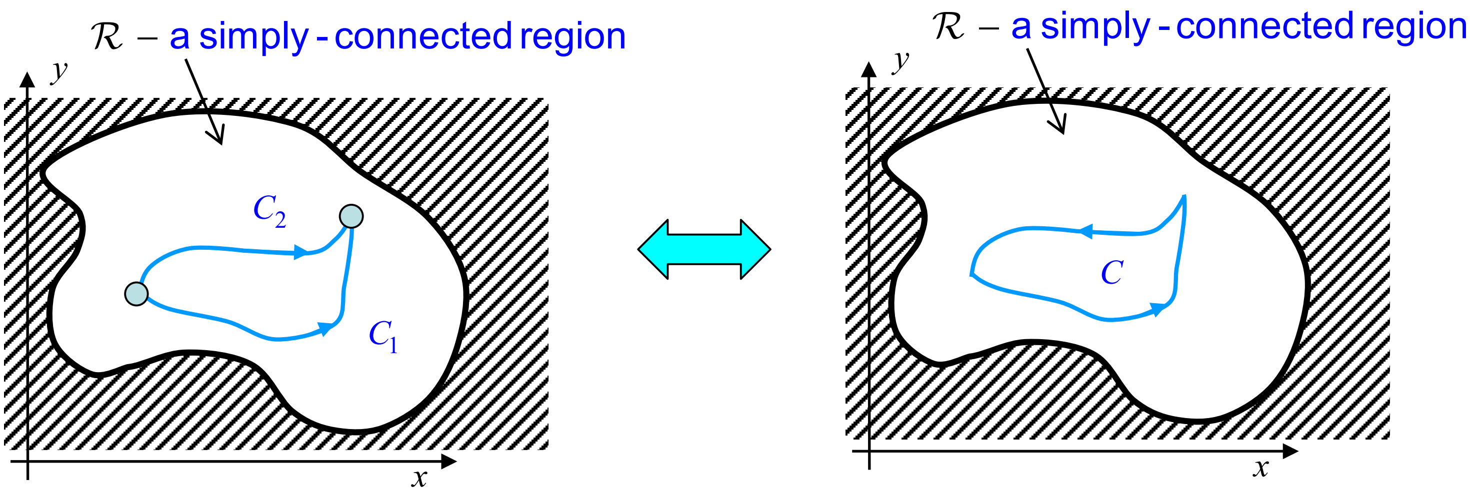
- The proof using Stokes's theorem requires that $u(x,y), v(x,y)$ have continuous first derivatives and that C be smooth.

- The Goursat proof removes these restrictions; hence the theorem is often called the *Cauchy - Goursat theorem*.

Cauchy's Theorem and Path Independence

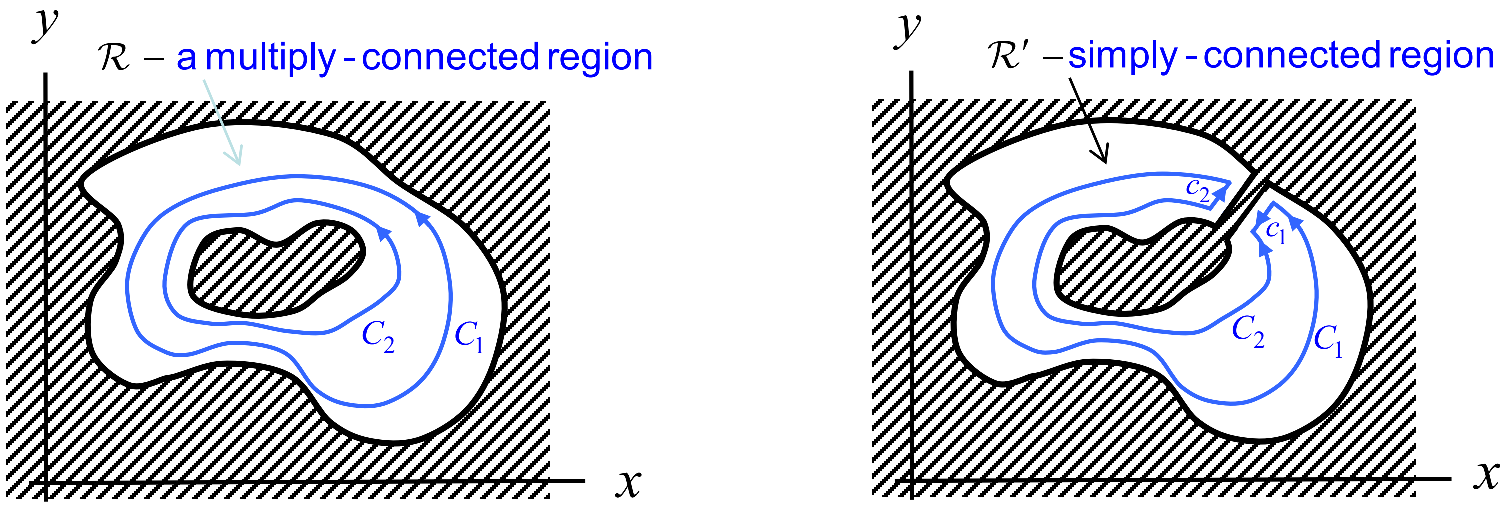
$$f(z) \text{ is analytic} \quad \Rightarrow \quad \oint_C f(z)\,dz = 0$$

- This implies that the line integral between any two points is *independent of the path*, as long as the function is analytic in the region enclosed by the paths.



$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0 \quad \Rightarrow \quad \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Extension of Cauchy's Theorem to Multiply-Connected Regions



- If $f(z)$ is analytic in $\mathcal{R}$ then $\oint_{C_{1,2}} f(z)\,dz \neq 0$ in general.
- Introduce an infinitesimal-width "bridge" to make $\mathcal{R}$ into a simply connected region $\mathcal{R}'$

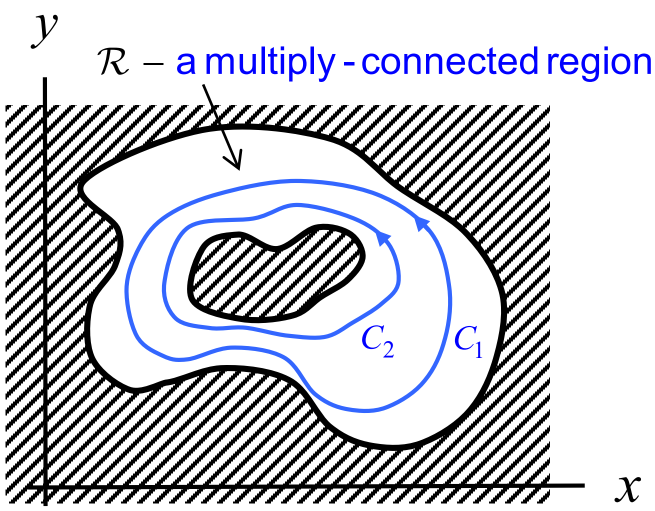
$$\oint_{C_1 - C_2 + \cancel{c_1} + \cancel{c_2}} f(z)\,dz = \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz = 0$$

since integrals along c_1, c_2 are in opposite directions and thus cancel $\Rightarrow \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$

Note: The closed path integrals on C_1 and C_2 are not usually zero!

Extension of Cauchy's Theorem to Multiply-Connected Regions

Summary: Result for a multiply-connected region



Note:
The path cannot be shrunk down farther than the boundary of the "island".

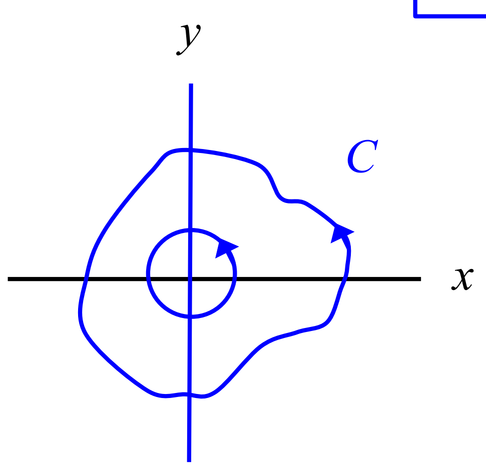
$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

Extension of Cauchy's Theorem to Multiply-Connected Regions (cont.)

◊ Example :

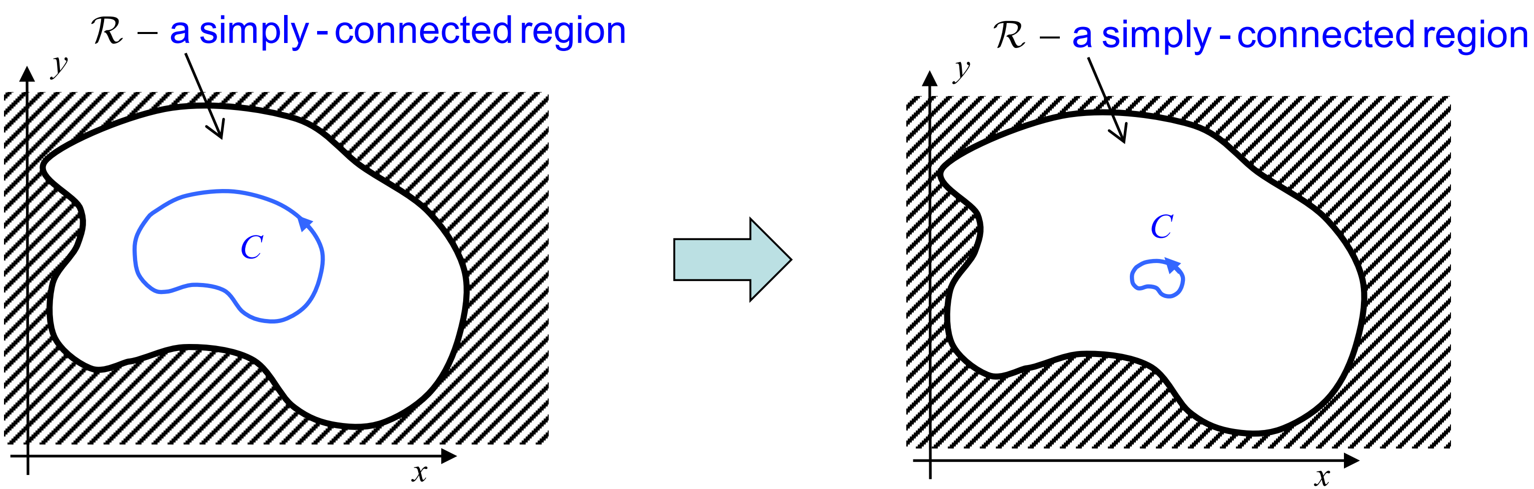
$$\oint_C \frac{1}{z}\,dz = 2\pi i$$

The integral around the <u>arbitrary</u> closed path C must give the same result as the integral around the circle (and we already know the answer for the circle).



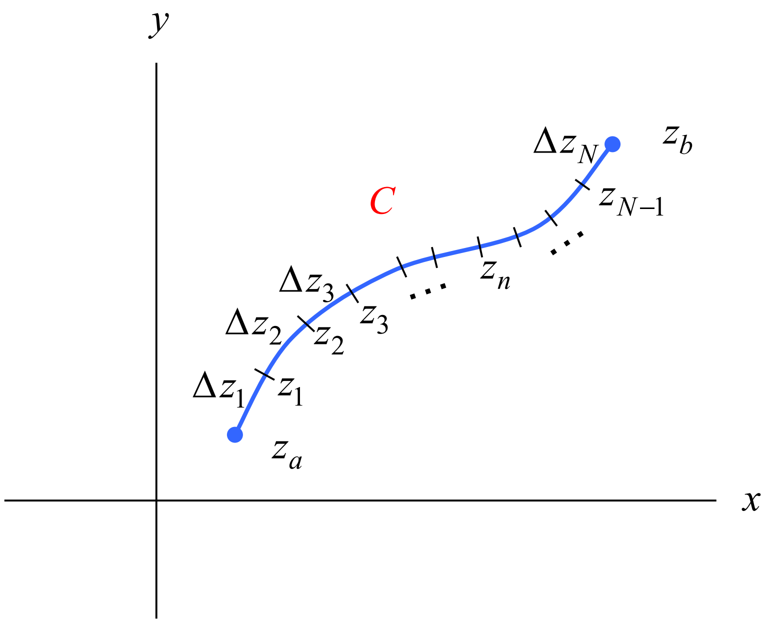
Cauchy's Theorem, Revisited

❖ If a function is analytic everywhere in a simply connected region, we can shrink down the path to zero size, which verifies that the line integral around a closed path in the region must be zero (since the integrand must be continuous and hence finite in the region).



Shrink the path down.

Fundamental Theorem of the Calculus of Complex Variables



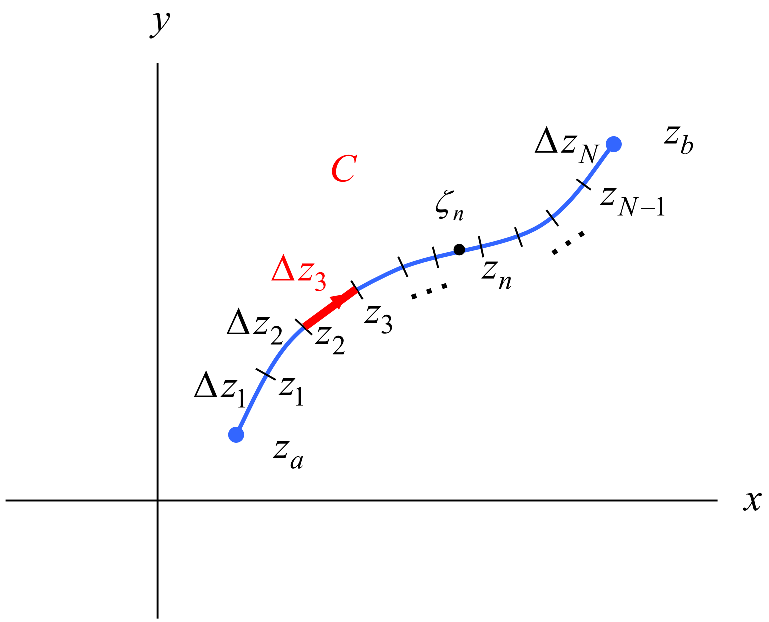
This is an extension of the same theorem in calculus (for real functions) to complex functions.

Assume that f is <u>analytic</u> in a region $\mathcal{R}$ containing the path C.

Suppose we can find $F(z)$ such that $F'(z) = \dfrac{dF}{dz} = f(z)$:

$$\Rightarrow \int_{z_a}^{z_b} f(z)\,dz = F(z_b) - F(z_a)$$

Fundamental Theorem of the Calculus of Complex Variables (cont.)



Proof

Recall:

The integal is path independent

if $f(z)$ is analytic on paths from z_a to z_b.

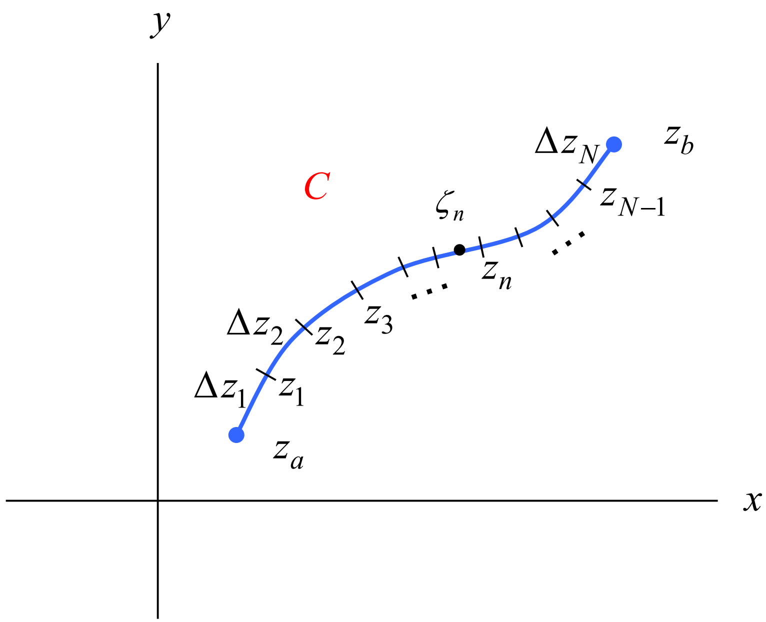
Choose a particular path C.

On this path:

$$\int_{z_a}^{z_b} f(z)\,dz \equiv \lim_{N\to\infty} \sum_{n=1}^{N} f(\zeta_n)\Delta z_n\,, \quad \Delta z_n = z_n - z_{n-1},$$

ζ_n on C between z_{n-1} and z_n (definition of line integral).

Fundamental Theorem of the Calculus of Complex Variables (cont.)



Proof (cont.)

- Assume $F'(z) = \dfrac{dF}{dz} = f(z)$;

Hence $f(\zeta_n) = F'(\zeta_n) \approx \dfrac{\Delta F_n}{\Delta z_n} = \dfrac{F(z_n) - F(z_{n-1})}{\Delta z_n}$

Note: The error in this approximation is proportional to $|\Delta z|$ (see next slide).

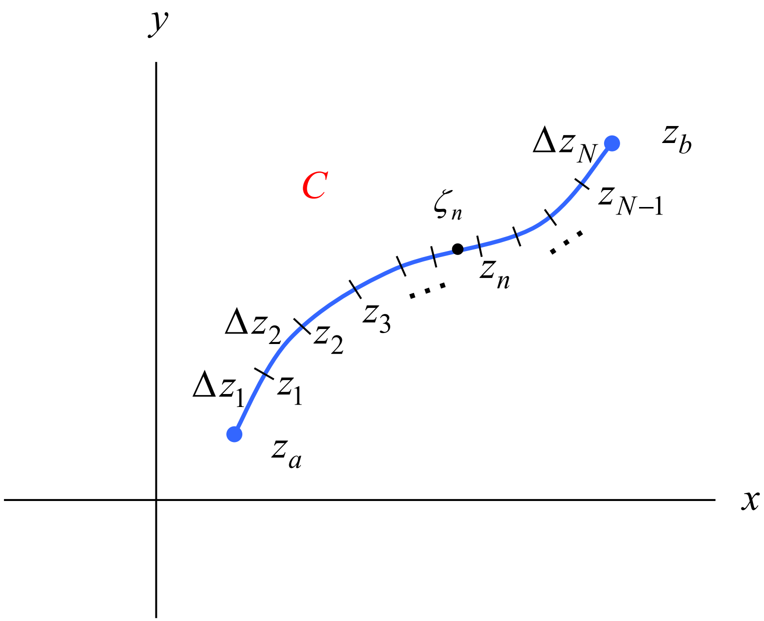
$$\Rightarrow \int_{z_a}^{z_b} f(z)\,dz = \lim_{\Delta z_n \to 0} \sum_{n=1}^{N} f(\zeta_n)\Delta z_n = \lim_{\Delta z_n \to 0} \sum_{n=1}^{N} F'(\zeta_n)\Delta z_n = \lim_{\Delta z_n \to 0} \sum_{n=1}^{N} \frac{\Delta F_n}{\Delta z_n}\Delta z_n$$

$$= \lim_{\Delta z_n \to 0} \left[F(z_1) - F(z_a)\right] + \left[F(z_2) - F(z_1)\right] + \left[F(z_3) - F(z_2)\right] + \cdots - \cdots$$

$$+ \cdots + \left[F(z_{N-1}) - F(z_{N-2})\right] + \left[F(z_b) - F(z_{N-1})\right]$$

$= F(z_b) - F(z_a)$ (path independent if $f(z)$ is analytic on paths from z_a to z_b)

Fundamental Theorem of the Calculus of Complex Variables (cont.)



Examination of Error in Approximation:

$$F'(\zeta_n) \approx \frac{F(z_n) - F(z_{n-1})}{\Delta z_n}$$

Use Taylor series:

$$F(z_n) = F(\zeta_n) + F'(\zeta_n)(z_n - \zeta_n) + \frac{1}{2}F''(\zeta_n)(z_n - \zeta_n)^2 + \ldots$$

$$F(z_{n-1}) = F(\zeta_n) + F'(\zeta_n)(z_{n-1} - \zeta_n) + \frac{1}{2}F''(\zeta_n)(z_{n-1} - \zeta_n)^2 + \ldots$$

$$F(z_n) - F(z_{n-1}) = F(\zeta_n) + F'(\zeta_n)(z_n - \zeta_n) + \frac{1}{2}F''(\zeta_n)(z_n - \zeta_n)^2 + \ldots$$
$$- F(\zeta_n) + F'(\zeta_n)(z_{n-1} - \zeta_n) + \frac{1}{2}F''(\zeta_n)(z_{n-1} - \zeta_n)^2 + \ldots$$

$$\frac{F(z_n) - F(z_{n-1})}{\Delta z_n} = \frac{F'(\zeta_n)(z_n - z_{n-1})}{\Delta z_n} + \frac{1}{\Delta z_n}\left[\frac{1}{2}F''(\zeta_n)(z_n - \zeta_n)^2 - \frac{1}{2}F''(\zeta_n)(z_{n-1} - \zeta_n)^2\right] + \ldots$$

$$\Rightarrow \quad \frac{F(z_n) - F(z_{n-1})}{\Delta z_n} = F'(\zeta_n) + \frac{1}{\Delta z_n}\underbrace{\left[\frac{1}{2}F''(\zeta_n)(z_n - \zeta_n)^2 - \frac{1}{2}F''(\zeta_n)(z_{n-1} - \zeta_n)^2\right]}_{\propto (\Delta z_n)^2} + \ldots$$

Fundamental Theorem of the Calculus of Complex Variables (cont.)

Fundamental Theorem of Calculus:

$$\int_{z_a}^{z_b} f(z)\,dz = F(z_b) - F(z_a)$$

- This permits us to make use of *indefinite integrals* :

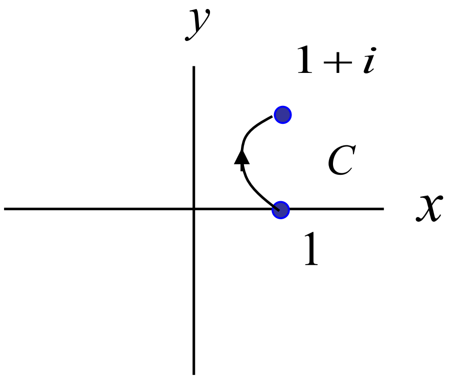
$$\int z^n dz = \frac{z^{n+1}}{n+1}, \qquad \int \sin z \; dz = -\cos z, \qquad \int e^{az} dz = \frac{e^{az}}{a}, \qquad \text{etc.}$$

Example:

$$\int_1^{1+i} \sin(z)\,dz = -\cos(z)\Big|_1^{1+i} = \cos(1) - \cos(1+i)$$

$$= 0.540302 - \left(0.833730 + i\left(0.988898\right)\right)$$

$$= -0.293428 + i\left(-0.988898\right)$$



(This result is path independent since sin is analytic everywhere!)

Fundamental Theorem of the Calculus of Complex Variables (cont.)

Consider this integral:

$$\int_{z_0}^{z} f(z)\,dz = F(z) - F(z_0) \quad \Rightarrow \quad F(z) = \int_{z_0}^{z} f(z)\,dz + F(z_0)$$

Consider two different indefinite integrals:

$$F_1(z) = \int_{z_1}^{z} f(\zeta)\,d\zeta + F_1(z_1) \text{ for arbitrary } z_1$$

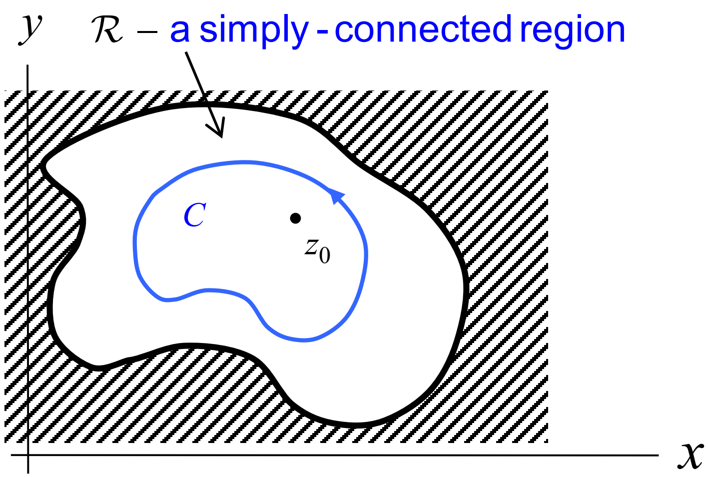
$$F_2(z) = \int_{z_2}^{z} f(\zeta)\,d\zeta + F_2(z_2) \text{ for arbitrary } z_2$$

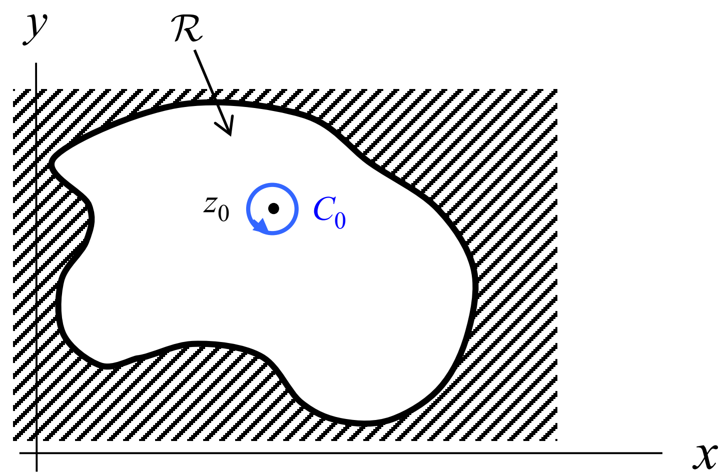
$$\Rightarrow \; F_2(z) - F_1(z) = \int_{z_2}^{z_1} f(\zeta)\,d\zeta + F_2(z_2) - F_1(z_1) = \text{constant}$$

↑ This integral is some constant.

All indefinite integrals can only differ by a (complex) constant.

Cauchy Integral Formula





- $f(z)$ is assumed analytic in $\mathcal{R}$ but we multiply by a factor $\dfrac{1}{(z-z_0)}$ (which is analytic *except* at z_0) and consider the following integral around C:

$$I = \int_C \frac{f(z)}{(z-z_0)} dz$$

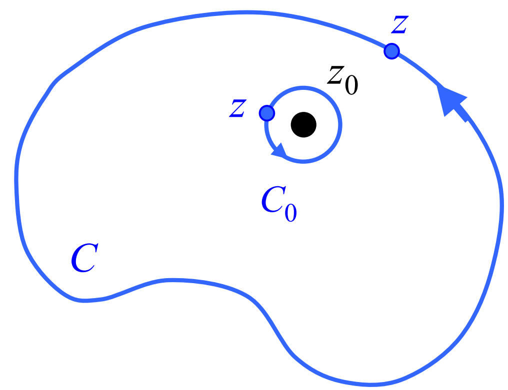
- To evaluate this, shrink that path to a small circular path path C_0 as shown:

$$\int_C \frac{f(z)}{z-z_0} dz = \int_{C_0} \frac{f(z)}{z-z_0} dz$$

Cauchy Integral Formula (cont.)

We have:

$$\int_C \frac{f(z)}{z-z_0}dz = \int_{C_0} \frac{f(z)}{z-z_0}dz$$



- Evaluate the C_0 integral on a small circular path, $z-z_0=re^{i\theta},\ dz=rie^{i\theta}d\theta$:

$$\int_{C_0} \frac{f(z)}{(z-z_0)}dz \overset{r\to 0}{=} f(z_0)\int_0^{2\pi} \frac{r\,i\,e^{i\theta}\,d\theta}{r\,e^{i\theta}} = 2\pi i\, f(z_0) \quad \text{for } r\to 0$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0) \quad \Rightarrow \quad \boxed{f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz}$$

Cauchy Integral Formula

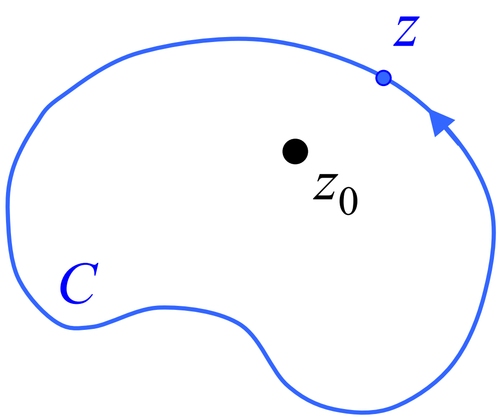
- Note the *remarkable* result :

The value of $f(z)$ at z_0 is completely determined by its values on C !

Cauchy Integral Formula (cont.)

Summary of Cauchy Integral Formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$



Another way to write it:

$$f(z) = -\frac{1}{2\pi i} \oint_C \frac{f(z')}{z - z'}\, dz'$$

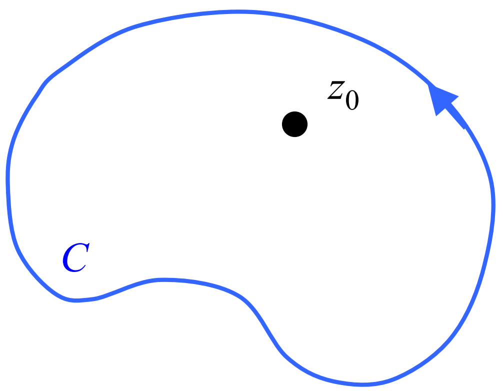
The Cauchy integral formula is useful for many purposes in complex variable theory.

Cauchy Integral Formula (cont.)

- Note that if z_0 is outside C, the integrand is analytic inside C; hence by the Cauchy's theorem, we have

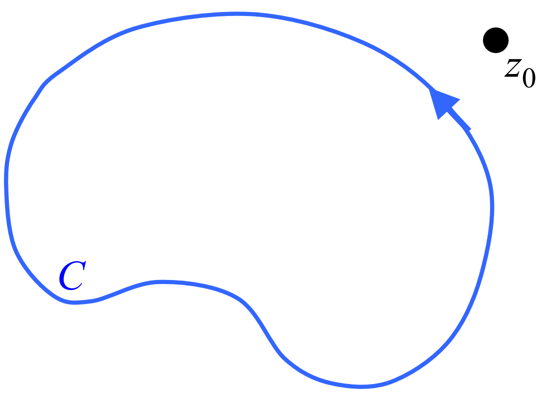
$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = 0$$

z_0 is inside C



$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

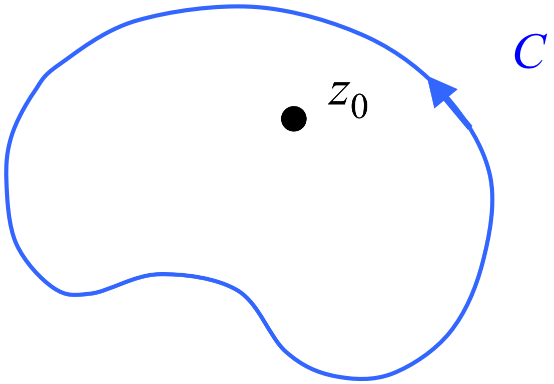
z_0 is outside C



$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = 0$$

Cauchy Integral Formula (cont.)

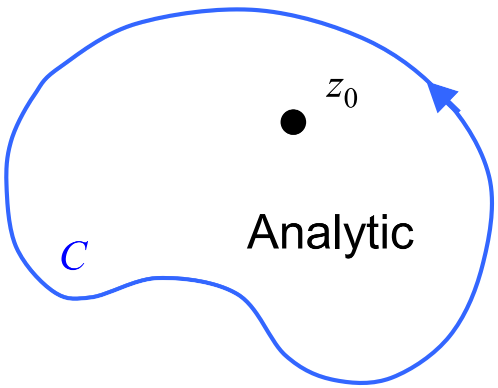
Cauchy Integral Formula: Summary of Both Cases:



$$\oint_C \frac{f(z)}{z - z_0}\, dz = \begin{cases} 2\pi i\, f(z_0), & z_0 \text{ inside } C \\ 0, & z_0 \text{ outside } C \end{cases}$$

Application:
This gives us a numerical way to determine if a point is inside of a region: just set $f(z) = 1$.

Derivative Formulas



Since f is analytic inside C, we can start with the Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz \quad \text{(Note } z_0 \text{ is inside } C.\text{)}$$

$$f(z_0 + \Delta z) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0 - \Delta z}\,dz$$

$$\Rightarrow \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i \Delta z}\oint_C \left(\frac{f(z)}{z - z_0 - \Delta z} - \frac{f(z)}{z - z_0}\right) dz$$

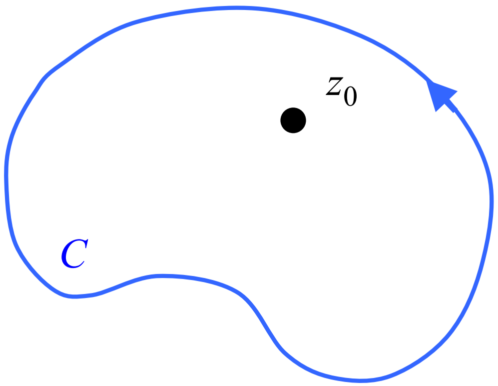
$$= \frac{1}{2\pi i \cancel{\Delta z}}\oint_C f(z)\left(\frac{\cancel{\Delta z}}{(z - z_0 - \Delta z)(z - z_0)}\right) dz$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{(z - z_0 - \Delta z)(z - z_0)}\right) dz$$

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^2}\,dz$$

Note: We have proved that you can differentiate w.r.t. z_0 under the integral sign!

Derivative Formulas (cont.)



$f(z)$ analytic in a simply connected region containing C

Similarly (derivation omitted):

$$f''(z_0) = \frac{2}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3}\, dz$$

In general:

$$f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_C f(z) \frac{d^n}{dz_0^n}\left(\frac{1}{z-z_0}\right) dz$$

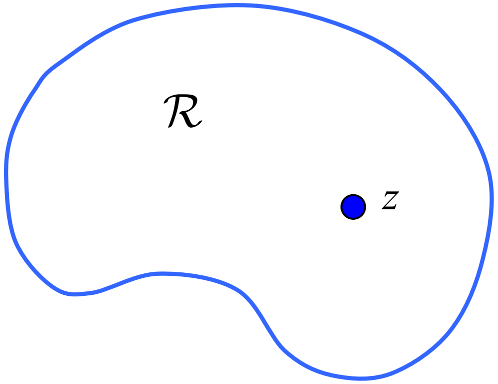
⇓

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

If f is analytic in a region, then all of the derivatives exist, and hence all of the derivatives are analytic as well.

Note: All derivatives can be determined from f on the boundary!

Morera's Theorem



If a function $f(z)$ is continuous in a simply-connected region $\mathcal{R}$ and

$$\oint_C f(z)\,dz = 0$$

for **every** closed contour C within $\mathcal{R}$, then $f(z)$ is analytic throughout $\mathcal{R}$.

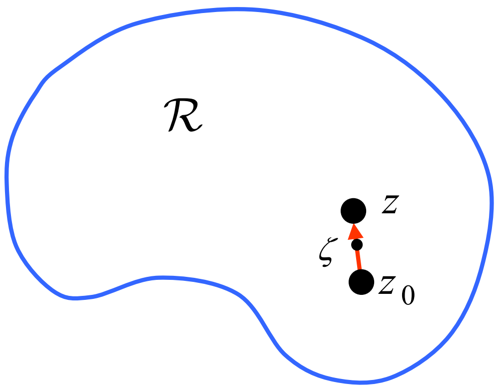
- **Cauchy's Theorem:** If $f(z)$ is analytic in a simply-connected region $\mathcal{R}$ then

$$\oint_C f(z)\,dz = 0, \quad C \text{ in } \mathcal{R}.$$

- **Morera's Theorem:** If a function $f(z)$ is continuous in a simply-connected region $\mathcal{R}$ and $\oint_C f(z)\,dz = 0$ for every closed contour C within $\mathcal{R}$, then $f(z)$ is analytic throughout $\mathcal{R}$.

- These theorems are converses of one another!

Proof of Morera's Theorem



Assume that the function $f(z)$ is continuous in a region $\mathcal{R}$ and

$$\oint_C f(z)\,dz = 0$$

for **every** closed contour C within $\mathcal{R}$.

Note: f will be analytic if we can prove its derivative exists!

$$\oint_C f(z)\,dz = 0 \;\Rightarrow\; \int_{z_0}^{z} f(\zeta)\,d\zeta \text{ is path independent, so define } F(z) \equiv \int_{z_0}^{z} f(\zeta)\,d\zeta .$$

Note that

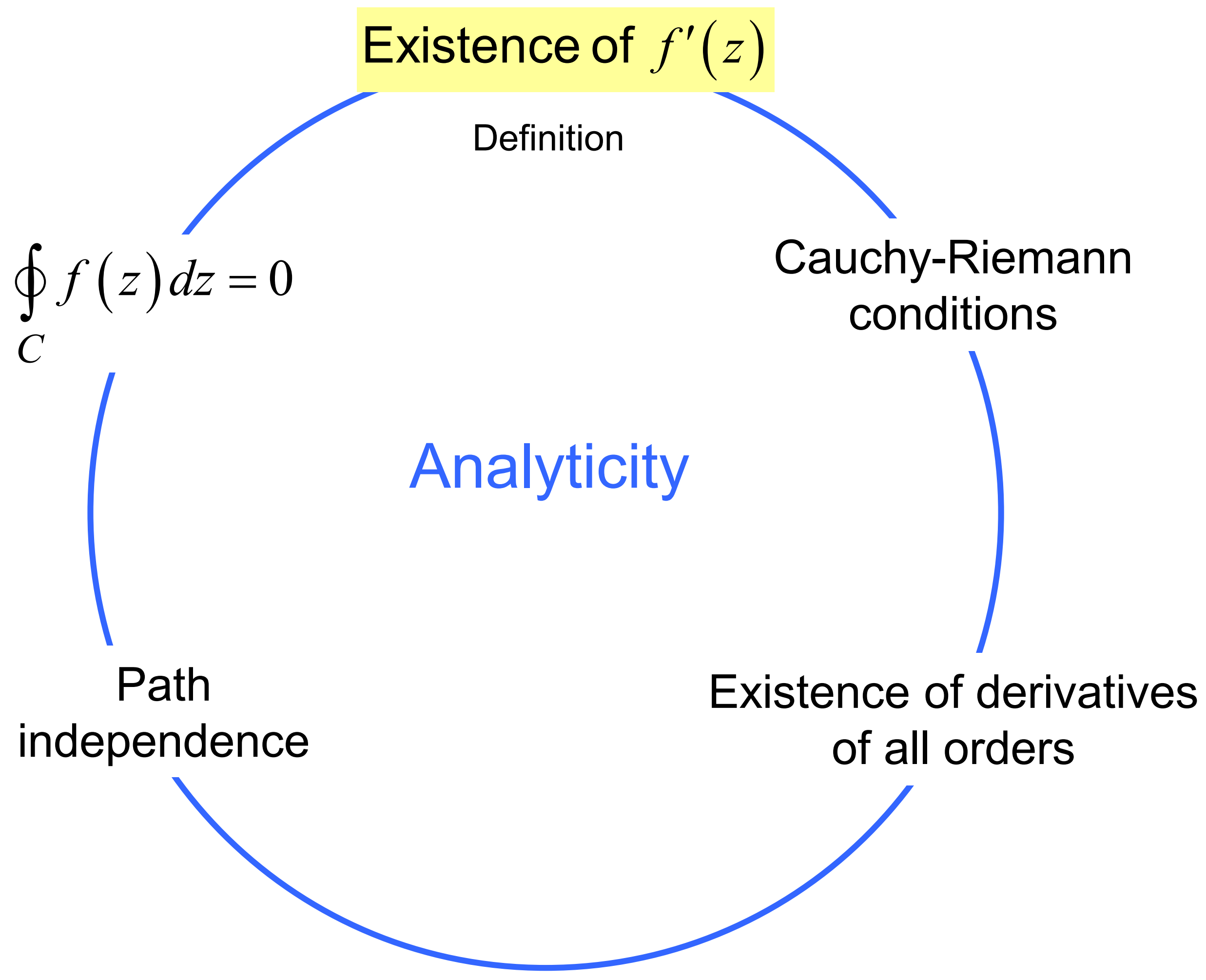
$$F(z) - F(z_0) = \int_{z_0}^{z} f(\zeta)\,d\zeta \quad\Rightarrow\quad \frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\int_{z_0}^{z} f(\zeta)\,d\zeta$$

Note : We can chose a small straight-line path between the two points since the integral is path independent. Along this small path, f is almost constant (from continuity of f).

$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\int_{z_0}^{z} f(\zeta)\,d\zeta \;\overset{z\to z_0}{\rightarrow}\; \frac{f(z_0)}{z-z_0}\int_{z_0}^{z} d\zeta = \frac{f(z_0)}{z-z_0}(z-z_0) = f(z_0)$$

$\Rightarrow$ $F'(z_0) = f(z_0)$. Hence F is analytic at any z_0 in $\mathcal{R}$ (its derivative exists). But then so are all its derivatives. Thus F' is analytic at z_0. But $F' = f$. Hence, f is analytic at z_0.

Properties of Analytic Functions



Cauchy's Inequality

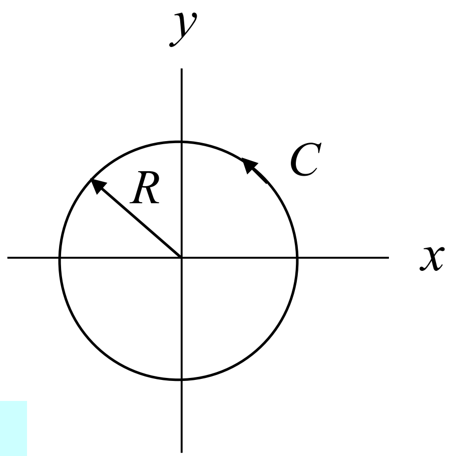
- Suppose $f(z)$:

  (a) is bounded ($|f(z)| < M$) on the circle of radius R

  (b) has a convergent power series representation:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{(see note below)}$$

inside (and on) the circle of radius R.



Then $$|a_n| \le \frac{M}{R^n}.$$

Consider this integral:

$$\frac{1}{2\pi} \int_{|z|=R} \frac{f(z)}{z^{m+1}} dz = \frac{1}{2\pi} \sum_{n=0}^{\infty} a_n \int_{|z|=R} z^{n-m-1} dz = \frac{1}{\cancel{2\pi}} \cancel{2\pi} i a_m$$

(This is from the previous line integral example, where n-m-1 = -1.)

$$\Rightarrow \quad |a_m| = \frac{1}{2\pi} \left| \int_{|z|=R} \frac{f(z)}{z^{m+1}} dz \right| \le \frac{1}{2\pi} \int_{|z|=R} \frac{|f(z)|}{|z^{m+1}|} |dz| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{R^{m+\cancel{1}}} \cancel{R} d\theta = \frac{M}{R^m}$$

$$\overset{m \to n}{\Rightarrow} \quad \boxed{|a_n| \le \frac{M}{R^n}}, \qquad M \equiv \max_{|z|=R} |f(z)|$$

> **Note:** If a function is analytic within and on the circle, then it must have a convergent power (Taylor) series expansion within and on the circle (proven later).

Liouville's Theorem

If $f(z)$ is analytic and bounded in the <u>entire</u> complex plane, it is a <u>constant</u>.

Because it is analytic in the entire complex plane, $f(z)$ will have a power (Taylor) series that converges everywhere (proven later).

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Proof

By the Cauchy Inequality,

$$|a_n| \le \frac{M}{R^n}, \quad \text{for any } R \ \ (\text{so let } R \to \infty)$$

$$\Rightarrow a_n = 0, \ \ n \neq 0$$

$$\Rightarrow f(z) = a_0$$

$\Rightarrow$ No "interesting" (i.e., non-constant) function $f(z)$ is analytic and bounded everywhere!

- z^n is analytic everywhere but unbounded at infinity
- $\sin z, \ e^z$ are analytic everywhere but unbounded at infinity
- $\dfrac{1}{(z - z_0)^2}$ is bounded at infinity but is not analytic at $z = z_0$

The Fundamental Theorem of Algebra

(This theorem is due to Gauss*.)

The N th degree polynomial, $P_N(z)=\sum_{n=0}^{N} a_n z^n,\; a_N \neq 0,\; N>0,$ has N (complex) roots.

As a corollary, an N^{th} degree polynomial can be written in factored form as

$$P_N(z) = a_N(z-z_1)(z-z_2)\cdots(z-z_N)$$

Carl Friedrich Gauss

(his signature)

* The theorem was first proven in Gauss's doctoral dissertation in 1799 using an algebraic method. The present proof, based on Liouville's theorem, was given by him later, in 1816.

http://en.wikipedia.org/wiki/Carl_Friedrich_Gauss

The Fundamental Theorem of Algebra (cont.)

Proof (by contradiction)

- First assume the polynomial has *no* roots. Then $\dfrac{1}{P_N(z)}$ is analytic everywhere and bounded at ∞ since $\lim\limits_{|z|\to\infty} \dfrac{1}{|P_N(z)|} = \lim\limits_{|z|\to\infty} \dfrac{1}{|a_N||z|^N} \to 0.$
- But by Liouville's theorem, we then have that $\dfrac{1}{P_N(z)}$ is a constant, contrary to our assumption $N > 0.$
- Hence $P_N(z)$ must have at least one root, say at $z = z_1$. We can then write

  $P_N(z) = (z - z_1)P_{N-1}(z)$ (see note below).
- Repeat the above procedure $N-1$ times (a total of N times) until we arrive at the conclusion that

$$P_N(z) = (z - z_1)(z - z_2)\cdots(z - z_N)P_0$$

where $P_0(z) = P_0$ is a constant.

Note:

To verify this, we can use the method of "polynomial division" to construct the polynomial $P_{N\text{-}1}(z)$ in terms of a_n. An example is given on the next slide.

The Fundamental Theorem of Algebra (cont.)

◊ Example (polynomial division)

Assume a third - order polynomial $P(z) = z^3 + a_2 z^2 + a_1 z + a_0$ has a root at $z = z_1$.

Show that $P(z) = (z - z_1)(z^2 + b_1 z + b_0)$.

- Equate coefficients other than z^0 :

$$z^2: \quad a_2 = b_1 - z_1 \rightarrow b_1 = a_2 + z_1$$

$$z^1: \quad a_1 = b_0 - b_1 z_1 \rightarrow b_0 = a_1 + b_1 z_1$$

- The difference between $P(z)$ and $(z - z_1)(z^2 + b_1 z + b_0)$ must then be a constant.
- Since both terms vanish at $z = z_1$, this constant must be zero.

Numerical Integration in the Complex Plane

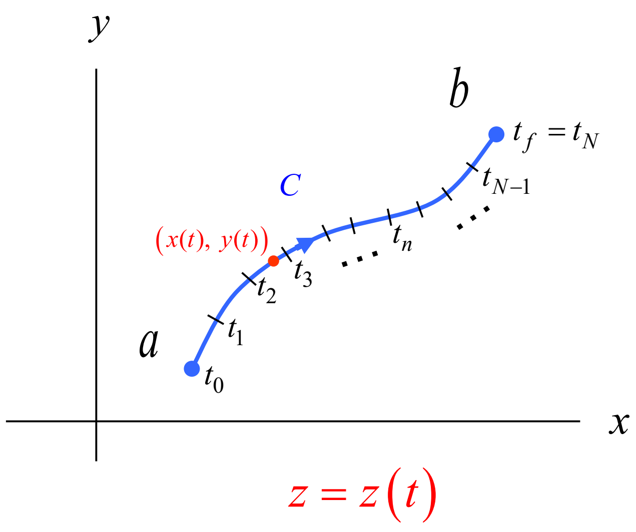
Here we give some tips about numerically integrating in the complex plane.

$$I \equiv \int_a^b f(z)\,dz$$

One way is to parameterize the integral using $z = z(t)$:

We assume a given function and a given path.

$$I = \int_{t_a}^{t_b} f(z(t))\left(\frac{dz}{dt}\right) dt \qquad \begin{aligned} t_a &= t_0 \\ t_b &= t_f \end{aligned}$$



so

$$I = \int_{t_a}^{t_b} F(t)\,dt$$

where

$$F(t) \equiv f(z(t))\left(\frac{dz}{dt}\right) = F_R(t) + iF_I(t)$$

$$I = \int_{t_a}^{t_b} F_R(t)\,dt + i\int_{t_a}^{t_b} F_I(t)\,dt$$

Numerical Integration in the Complex Plane (cont.)

Summary

$$I = \int_a^b f(z)\,dz = \int_{t_a}^{t_b} F(t)\,dt = \int_{t_a}^{t_b} F_R(t)\,dt + i\int_{t_a}^{t_b} F_I(t)\,dt$$

where

$$F(t) \equiv f(z(t))\left(\frac{dz}{dt}\right) = F_R(t) + iF_I(t)$$

Note:
It is not always necessary to treat the real and imaginary parts of F separately; we can just allow F to be complex in most software.

The integrals in t can be preformed in the usual way, using any convenient scheme for integrating functions of a real variable (Simpson's rule, Gaussian Quadrature, Romberg method, etc.).

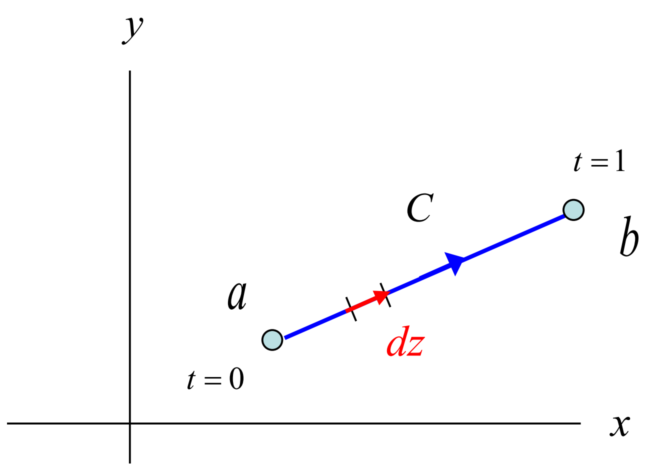
Note: The variable t can be x, or θ, or any other convenient variable.

Numerical Integration of Analytic Functions

Numerically Integrating Functions in a Region where they are <u>Analytic</u>

If the function f is analytic, then the integral is path independent. We can choose a <u>straight-line</u> path!

We explore how this works.



Note: If the path is <u>piecewise</u> linear, we simply add up the results from each linear part of the path.

$$z = a + (b-a)t, \quad 0 \le t \le 1$$

$$\frac{dz}{dt} = b - a$$

$$dz = (b-a)\,dt \;\Rightarrow\; |dz| = |b-a|\,dt, \;\; \arg(dz) = \arg(b-a)$$

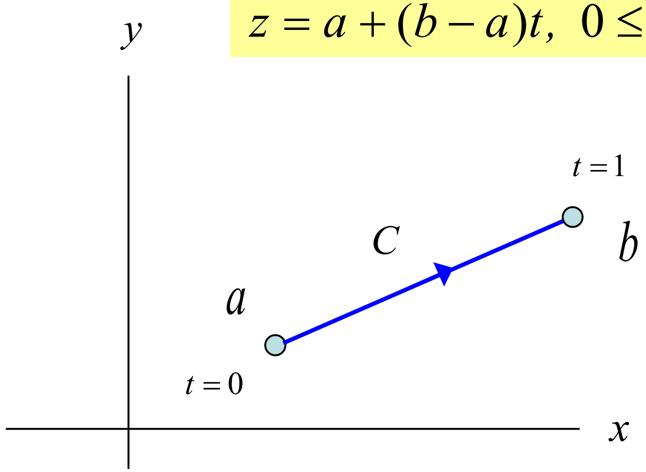
$$I \equiv \int_a^b f(z)\,dz = \int_0^1 f(z(t))\frac{dz}{dt}dt = (b-a)\int_0^1 f(z(t))\,dt$$

Numerical Integration of Analytic Functions (cont.)

Summary:

$$I = (b-a)\int_0^1 f\left(z(t)\right)dt$$

$$z = a + (b-a)t,\ \ 0 \le t \le 1$$



We can use any method that we wish to evaluate this integral along the real axis.

We can also break up the integral into real and imaginary parts if we wish*:

$$I = (b-a)\int_0^1 \operatorname{Re} f\left(z(t)\right)dt + i(b-a)\int_0^1 \operatorname{Im} f\left(z(t)\right)dt$$

* This is usually not necessary for numerical integrations, since most software can handle complex functions.

Note: The term b-a is complex!

Numerical Integration of Analytic Functions (cont.)

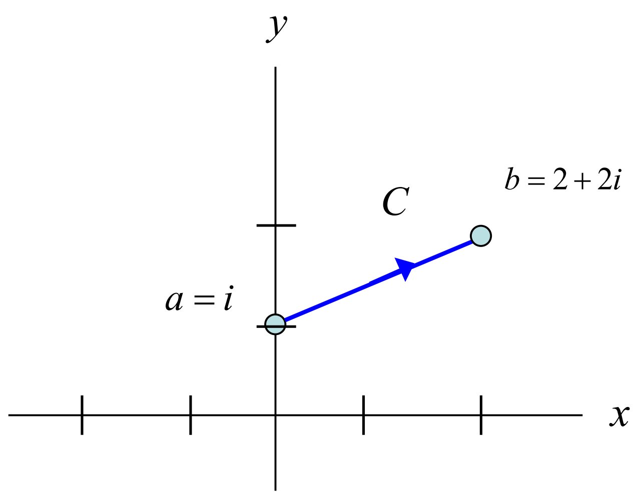
Example:

$$I = \int_a^b \sin(z)\,dz \qquad a = i,\; b = 2 + 2i$$

$$z = a + (b-a)t,\; 0 \le t \le 1$$

$$I = (b-a)\int_0^1 f(z(t))\,dt$$

$$I = (2+i)\int_0^1 \sin\left(i + (2+i)t\right)dt$$



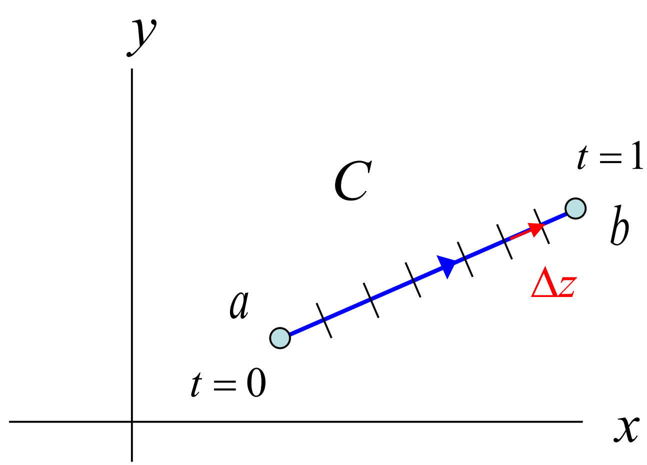
This integral can be evaluated using any numerical integration package.

Numerical Integration of Analytic Functions (cont.)

Uniform Partitioning

$$z = a + (b-a)t, \quad 0 \le t \le 1$$

$$I = (b-a)\int_0^1 f\left(z(t)\right)dt$$



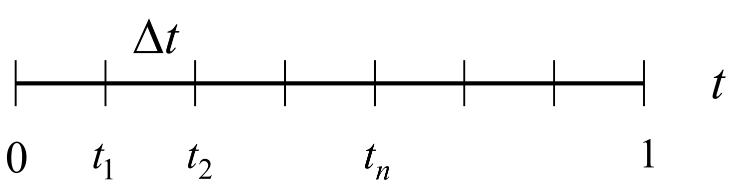
Note: If we partition uniformly in t, then we are really partitioning uniformly along the line with Δz.

$$\Delta z = (b-a)\Delta t = (b-a)/N$$

(We don't have to partition uniformly, but we can if we wish.)

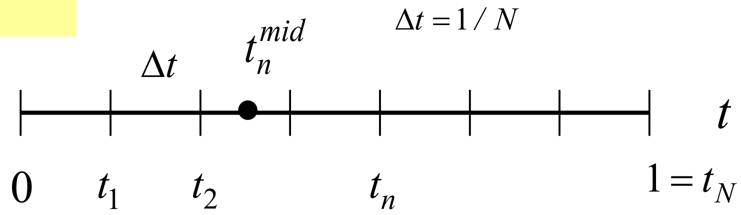
N segments

$$\Delta t = 1/N$$



Numerical Integration of Analytic Functions (cont.)

$$I \equiv (b-a)\int_0^1 f\left(z(t)\right)dt$$

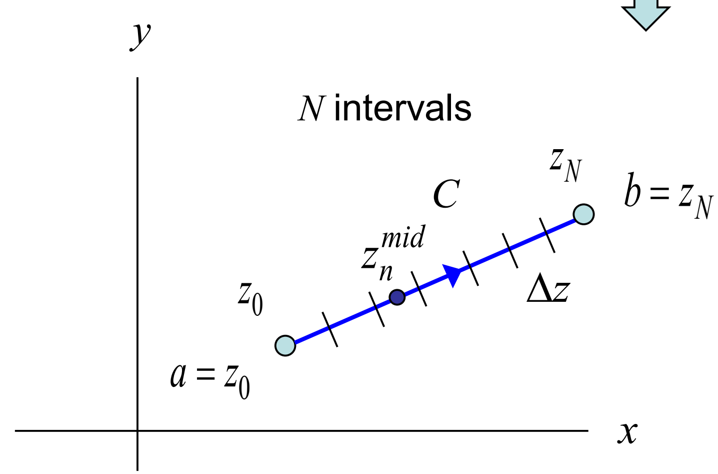


Uniform sampling (Midpoint rule):

$$I \approx (b-a)\sum_{n=1}^{N} f\left(z_n^{mid}\right)\Delta t$$

$$z = a + (b-a)t$$

$$z_n^{mid} = a + (b-a)t_n^{mid}, \quad t_n^{mid} = \Delta t\left(n - \frac{1}{2}\right)$$



Using $\Delta z = (b-a)\Delta t \;\Rightarrow\; \Delta t = \Delta z/(b-a)$

we then have

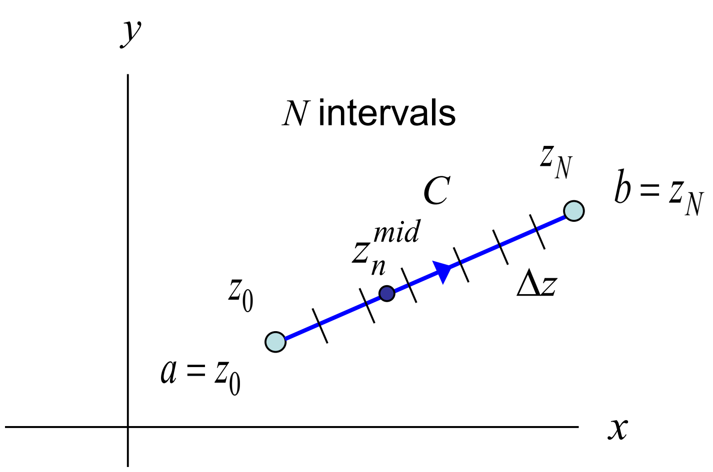
$$I \approx \cancel{(b-a)}\sum_{n=1}^{N} f\left(z_n^{mid}\right)\frac{\Delta z}{\cancel{b-a}}$$

Numerical Integration of Analytic Functions (cont.)

$$I \approx \Delta z \sum_{n=1}^{N} f\left(z_n^{mid}\right)$$

where

$$\begin{aligned} z_n^{mid} &= a + (b-a)t_n^{mid} \\ &= a + (b-a)\Delta t\left(n - \frac{1}{2}\right) \\ &= a + (b-a)\frac{1}{N}\left(n - \frac{1}{2}\right) \\ &= a + \Delta z\left(n - \frac{1}{2}\right) \end{aligned}$$



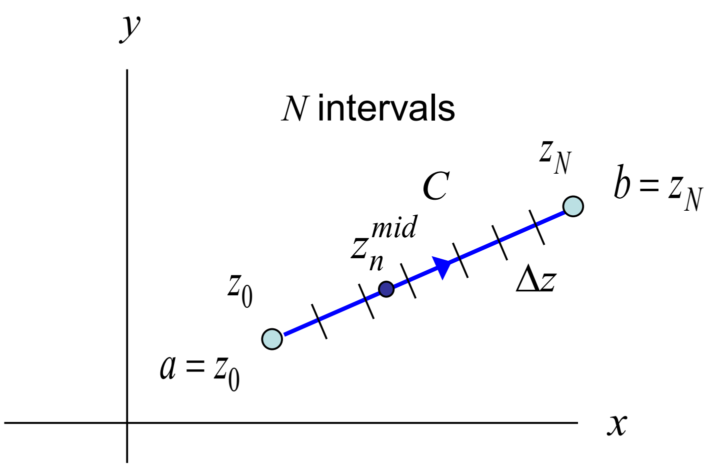
Numerical Integration of Analytic Functions (cont.)

"Complex Midpoint rule"

$$I \approx \Delta z \sum_{n=1}^{N} f\left(z_n^{mid}\right) \qquad \text{Error} \propto \left|\Delta z\right|^2$$

$$\Delta z = \frac{b-a}{N}$$

$$z_n^{mid} = a + \Delta z\left(n - \frac{1}{2}\right)$$



This is the same formula that we usually use for integrating a function along the real axis using the midpoint rule!
(We essentially have a just rotation of the axis.)

Numerical Integration of Analytic Functions (cont.)

"Complex Simpson's rule"

$$I \approx \frac{\Delta z}{3}\left(f\left(z_0\right)+4f\left(z_1\right)+2f\left(z_2\right)+4f\left(z_3\right)+2f\left(z_4\right)+\ldots 4f\left(z_{N-1}\right)+f\left(z_N\right)\right)$$

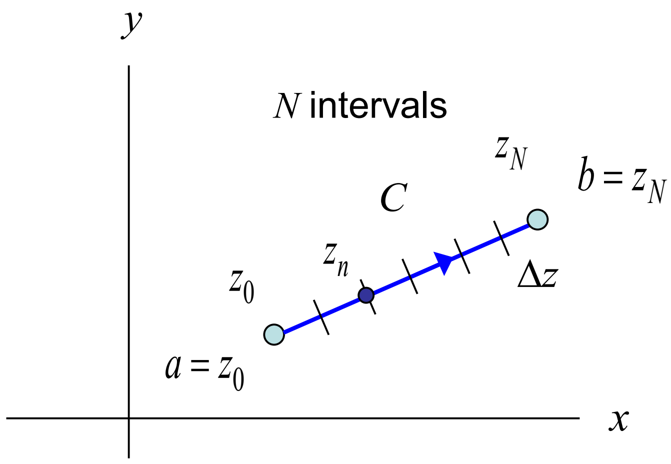
$\text{Error} \propto \left|\Delta z\right|^4$

$N =$ number of segments
$=$ even number

Number of sample points $= N+1$

$$z_n = a + \left(\Delta z\right) n, \quad n = 0,1,\ldots N$$

$$\Delta z = \frac{b-a}{N}$$



Numerical Integration of Analytic Functions (cont.)

6-point Gaussian Quadrature

$$\int_{-1}^{1} F(x)\,dx \approx \sum_{i=1}^{6} F(x_i)\,w_i$$

Use

$$x' = \left(\frac{B-A}{2}\right)x + \left(\frac{A+B}{2}\right) \qquad x \in (-1,1) \;\Rightarrow\; x' \in (A,B)$$

$$dx' = \left(\frac{B-A}{2}\right)dx$$

$$\Rightarrow \int_{A}^{B} f(x')\,dx' = \left(\frac{B-A}{2}\right)\int_{-1}^{1} F(x)\,dx \approx \left(\frac{B-A}{2}\right)\sum_{i=1}^{6} F(x_i)\,w_i = \left(\frac{B-A}{2}\right)\sum_{i=1}^{6} f(x_i')\,w_i$$

$$f(x') = F(x)$$

$$x_i' = \left(\frac{B-A}{2}\right)x_i + \left(\frac{A+B}{2}\right)$$

Sample points	Weights
$x_1 = -0.9324695$	$w_1 = 0.1713245$
$x_2 = -0.6612094$	$w_2 = 0.3607616$
$x_3 = -0.2386192$	$w_3 = 0.4679139$
$x_4 = 0.2386192$	$w_4 = 0.4679139$
$x_5 = 0.6612094$	$w_5 = 0.3607616$
$x_6 = 0.9324695$	$w_6 = 0.1713245$

Numerical Integration of Analytic Functions (cont.)

6-point Gaussian Quadrature (cont.)

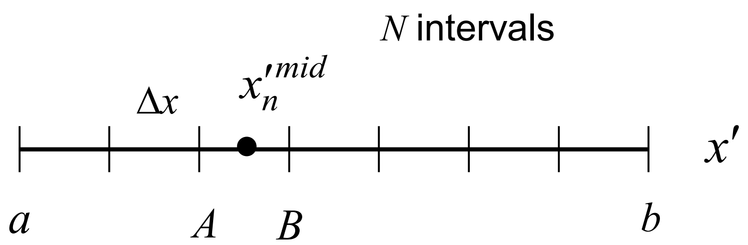
Use N intervals on larger (global) interval (a,b):

$$I = \int_a^b f(x')\,dx' \approx \sum_{n=1}^{N} I_n$$

$$\int_A^B f(x')\,dx' = \left(\frac{B-A}{2}\right)\sum_{i=1}^{6} f(x_i')\,w_i \quad \Rightarrow \quad I_n \approx \left(\frac{\Delta x'}{2}\right)\sum_{i=1}^{6} f\left(x_n'^{mid} + x_i\frac{\Delta x}{2}\right)w_i$$

Recall: $x_i' = \left(\frac{B-A}{2}\right)x_i + \left(\frac{A+B}{2}\right)$

Sample points	Weights
$x_1 = -0.9324695$	$w_1 = 0.1713245$
$x_2 = -0.6612094$	$w_2 = 0.3607616$
$x_3 = -0.2386192$	$w_3 = 0.4679139$
$x_4 = 0.2386192$	$w_4 = 0.4679139$
$x_5 = 0.6612094$	$w_5 = 0.3607616$
$x_6 = 0.9324695$	$w_6 = 0.1713245$



Numerical Integration of Analytic Functions (cont.)

"Complex Gaussian Quadrature"

$$I = \sum_{n=1}^{N} I_n$$

$$I_n \approx \left(\frac{\Delta z}{2}\right) \sum_{i=1}^{6} f\left(z_n^{mid} + x_i \frac{\Delta z}{2}\right) w_i$$

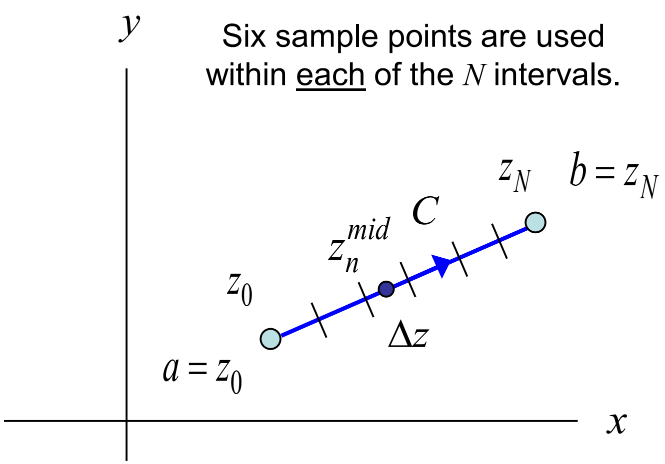
(6-point Gaussian Quadrature)

$$z_n^{mid} = a + (\Delta z)(n - 1/2)$$

$$\text{Error} \propto |\Delta z|^{12}$$

N intervals

Six sample points are used within each of the N intervals.



$$\Delta z = \frac{b-a}{N}$$

Sample points	Weights
$x_1 = -0.9324695$	$w_1 = 0.1713245$
$x_2 = -0.6612094$	$w_2 = 0.3607616$
$x_3 = -0.2386192$	$w_3 = 0.4679139$
$x_4 = 0.2386192$	$w_4 = 0.4679139$
$x_5 = 0.6612094$	$w_5 = 0.3607616$
$x_6 = 0.9324695$	$w_6 = 0.1713245$

Numerical Integration of Analytic Functions (cont.)

Here is an example of a piecewise linear path, used to calculate the electromagnetic field of a dipole source over the earth (Sommerfeld problem).

Gaussian quadrature (e.g., 6-point Gaussian quadrature) could be used on each of the linear segments of the path C (breaking each one up into N intervals as necessary to get good convergence).