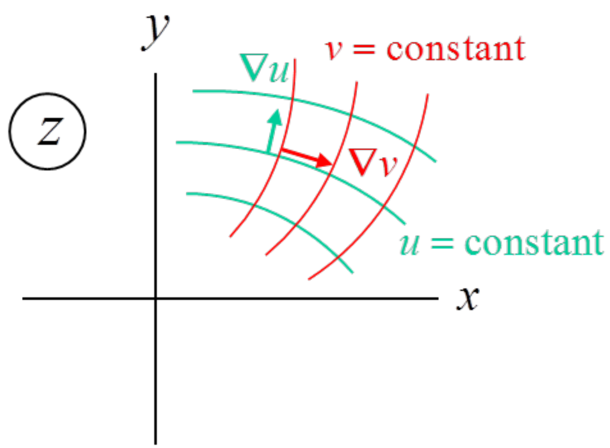
ECE 6382



Fall 2023

David R. Jackson

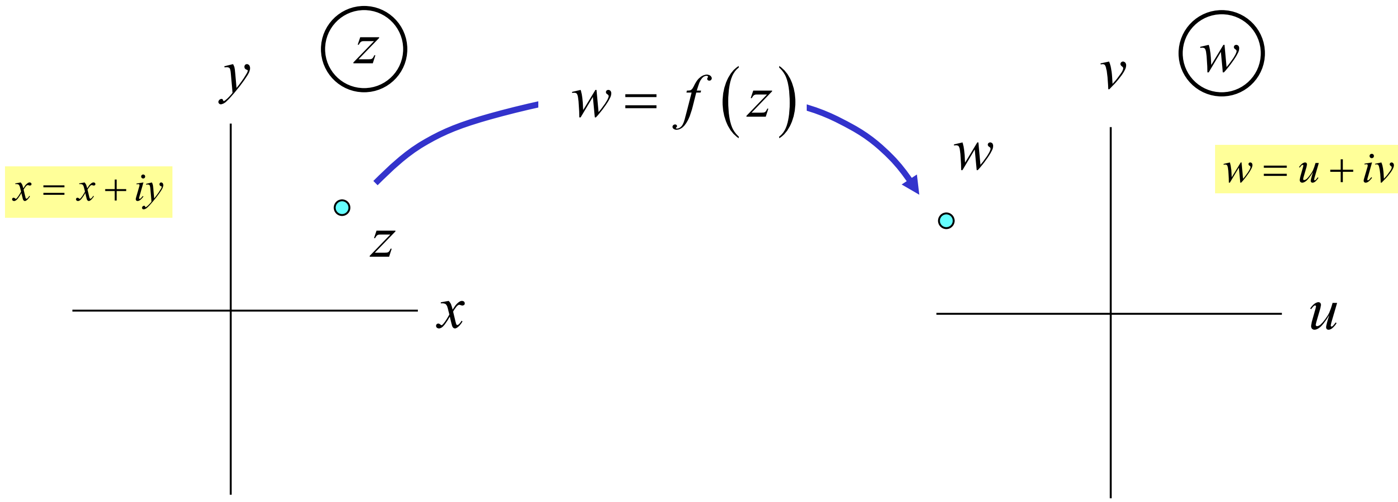
Notes 4

Functions of a Complex Variable as Mappings

Notes are adapted from D. R. Wilton, Dept. of ECE

A Function of a Complex Variable as a Mapping

- **A function of a complex variable,** $w = f(z)$**, is usually viewed as a *mapping* from the complex** z **plane to the complex** w **plane.**



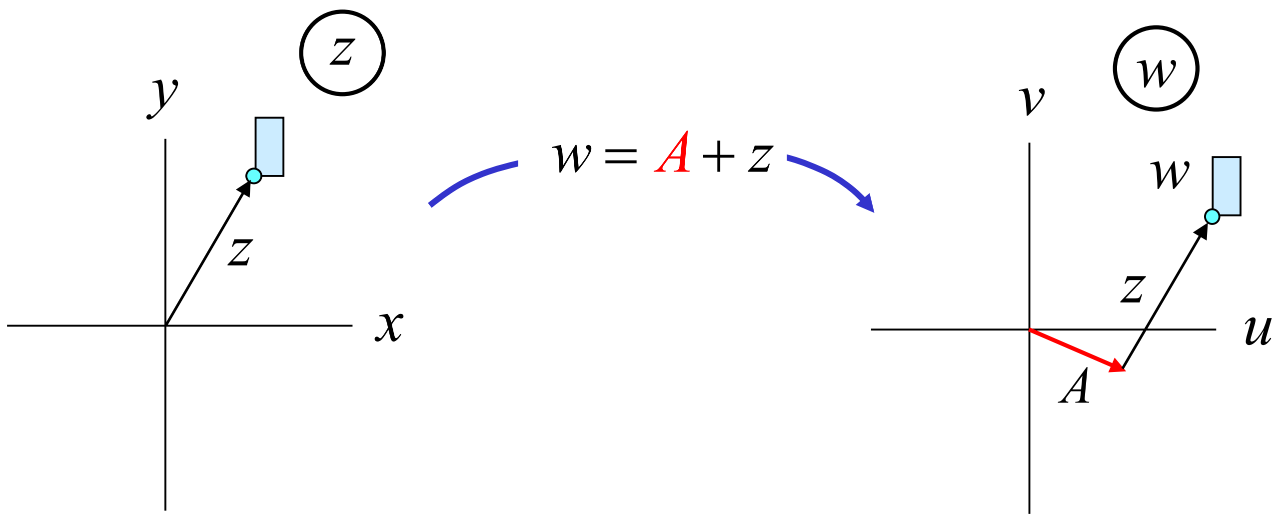
For example, $w = z^3$

Simple Mappings: Translations

- **Translation :**

$$w = A + z$$

where A is a complex constant.



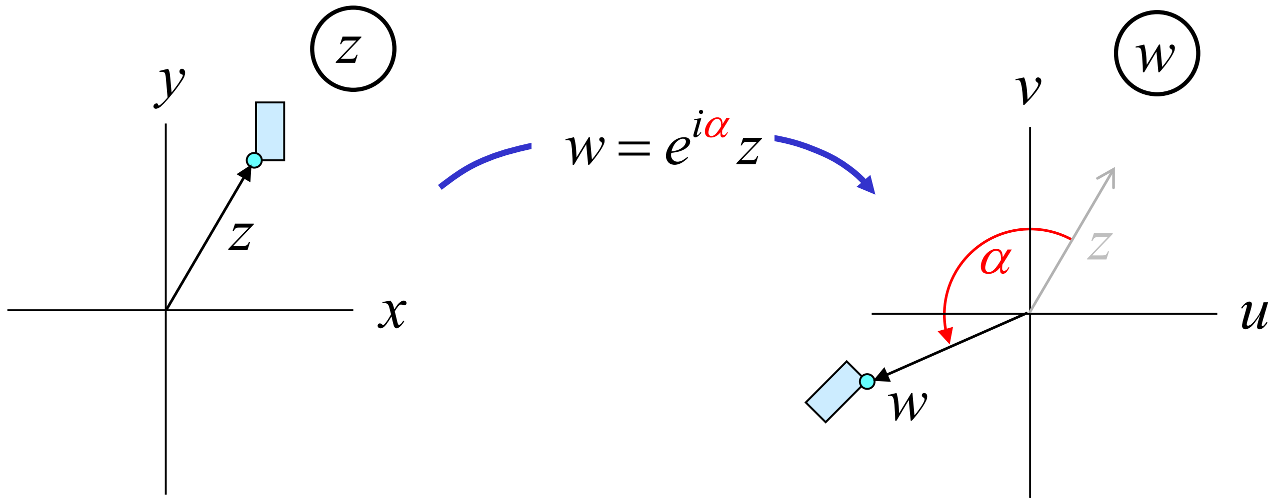
- **The mapping translates every point in the z plane by the "vector" A.**

Simple Mappings: Rotations

- **Rotation :**

$$w = e^{i\alpha} z = e^{i\alpha}\left(re^{i\theta}\right) = re^{i(\alpha+\theta)}$$

where α is a *real* constant.



- **The mapping rotates every point in the z plane through an angle α .**

Simple Mappings: Dilations

- **Dilation (stretching) :**

$$w = az \;=\; a\left(re^{i\theta}\right) \;=\; (ar)e^{i\theta}$$

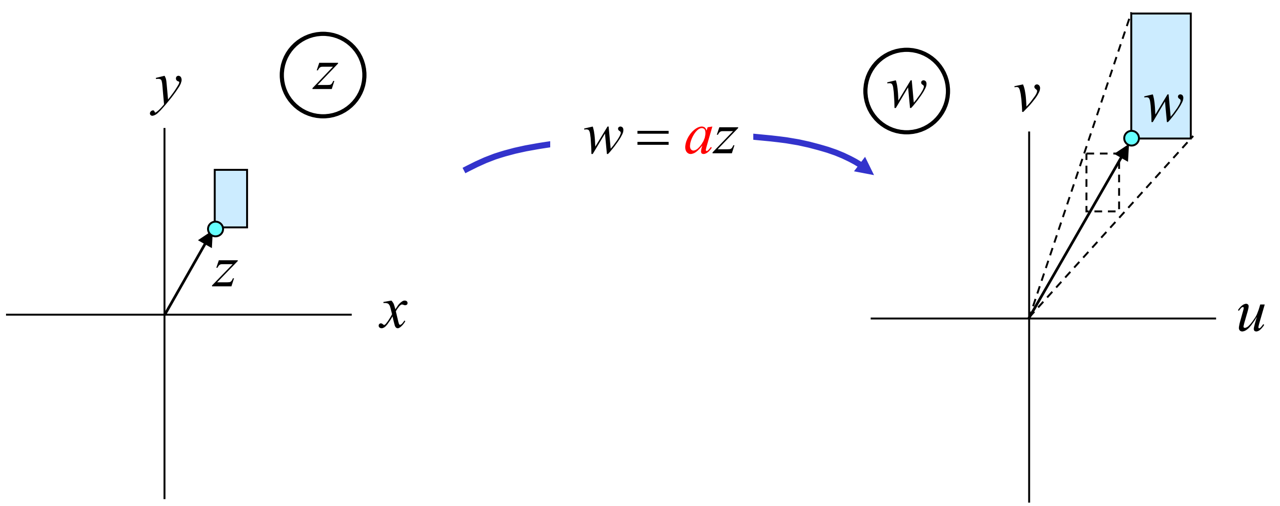
where a **is a *real* constant.**

Note:

$$u = ax,\; v = ay$$

$$\Rightarrow \begin{cases} du = a\,dx \\ dv = a\,dy \end{cases}$$

(All distances are uniformly stretched.)



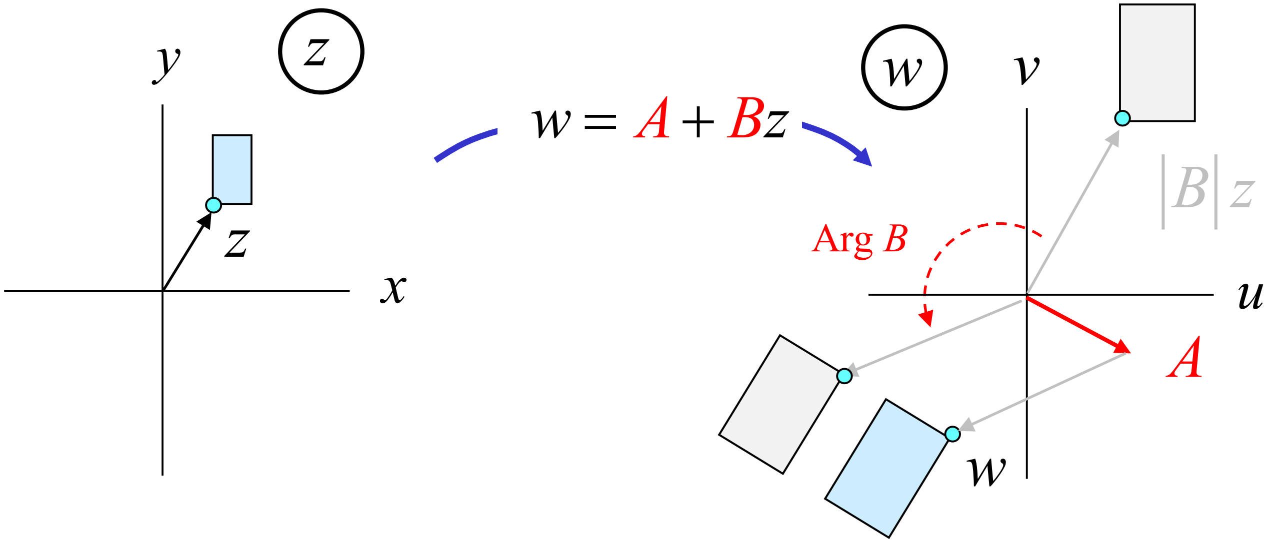
- **The mapping magnifies the magnitude** $|z|$ **of a point** z **in the complex plane by a factor** a**.**

A General Linear Transformation (Mapping) is a Combination of Translation, Rotation, and Dilation

- **Linear transformation :**

$$w = A + Bz \;= A + |B| e^{i \,\mathrm{Arg}\, B} r e^{i\theta} \;= \underbrace{A}_{\text{translation}} + \overbrace{|B| r}^{\text{dilation}} e^{i \overbrace{(\theta + \mathrm{Arg}\, B)}^{\text{rotation}}}$$

where A, B **are complex constants .**



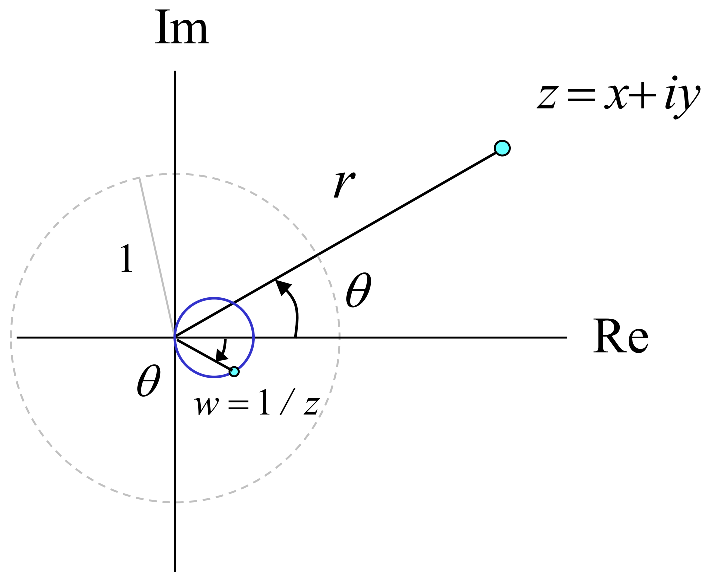
Shapes do not change under a linear transformation!

Simple Mappings: Inversions

□ **Inversion :**

$$w = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

The magnitude becomes the reciprocal, and the phase angle becomes the negative.

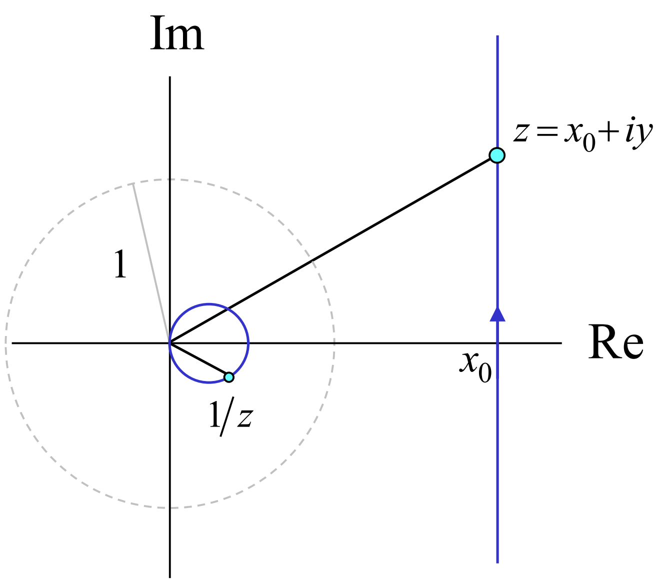


- Points outside the unit circle get mapped to the inside of the unit circle.
- Points inside the unit circle get mapped to the outside of the unit circle.

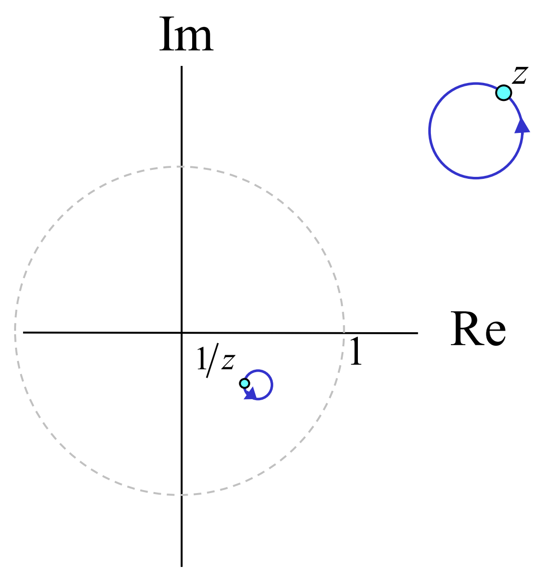
Simple Mappings: Inversions

- **Inversion:**

$$w = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$



Inverson: a straight line maps to a circle



Inversion: circle-preserving property

Inversions have a "circle preserving" property, i.e., circles always map to circles (Straight lines are a special case where the radius of the circle is infinity.)

Circle Property of Inversion Mapping: Proof

$$w = \frac{1}{z}$$

(This maps circles into circles.)

$$z = \frac{1}{w} \quad \Rightarrow \quad x + iy = \frac{1}{u + iv} \quad \Rightarrow \quad x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}$$

Consider a circle: $(x - x_0)^2 + (y - y_0)^2 = a^2$

This is in the form $x^2 + y^2 + a_1 x + a_2 y + a_3 = 0$

$a_1 \equiv -2x_0$

$a_2 \equiv -2y_0$

$a_3 \equiv x_0^2 + y_0^2 - a^2$

Hence

$$\left(\frac{u}{u^2 + v^2}\right)^2 + \left(\frac{-v}{u^2 + v^2}\right)^2 + a_1\left(\frac{u}{u^2 + v^2}\right) + a_2\left(\frac{-v}{u^2 + v^2}\right) + a_3 = 0$$

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 9th Ed., McGraw-Hill, 2013.

Circle Property of Inversion Mapping: Proof (cont.)

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2+a_1\left(\frac{u}{u^2+v^2}\right)+a_2\left(\frac{-v}{u^2+v^2}\right)+a_3=0$$

Multiply by u^2+v^2:

$$\left(\frac{u^2}{u^2+v^2}\right)+\left(\frac{v^2}{u^2+v^2}\right)+a_1(u)+a_2(-v)+a_3\left(u^2+v^2\right)=0$$

or

$$1+a_1(u)+a_2(-v)+a_3\left(u^2+v^2\right)=0$$

This is in the form of a circle (see next slide).

Circle Property of Inversion Mapping: Proof (cont.)

$$1 + a_1(u) + a_2(-v) + a_3(u^2 + v^2) = 0$$

Divide by a_3:

$$(u^2 + v^2) + a_1'(u) + a_2'(-v) + a_0' = 0$$

$a_1' \equiv a_1 / a_3$

$a_2' \equiv a_2 / a_3$

$a_0' \equiv 1 / a_3$

Complete the square:

$$(u + a_1'/2)^2 + (v - a_2'/2)^2 + \left(a_0' - \frac{a_1'^2}{4} - \frac{a_2'^2}{4}\right) = 0$$

This is in the form of a circle:

$$(u - u_0)^2 + (v - v_0)^2 = R^2$$

$u_0 \equiv -a_1'/2$

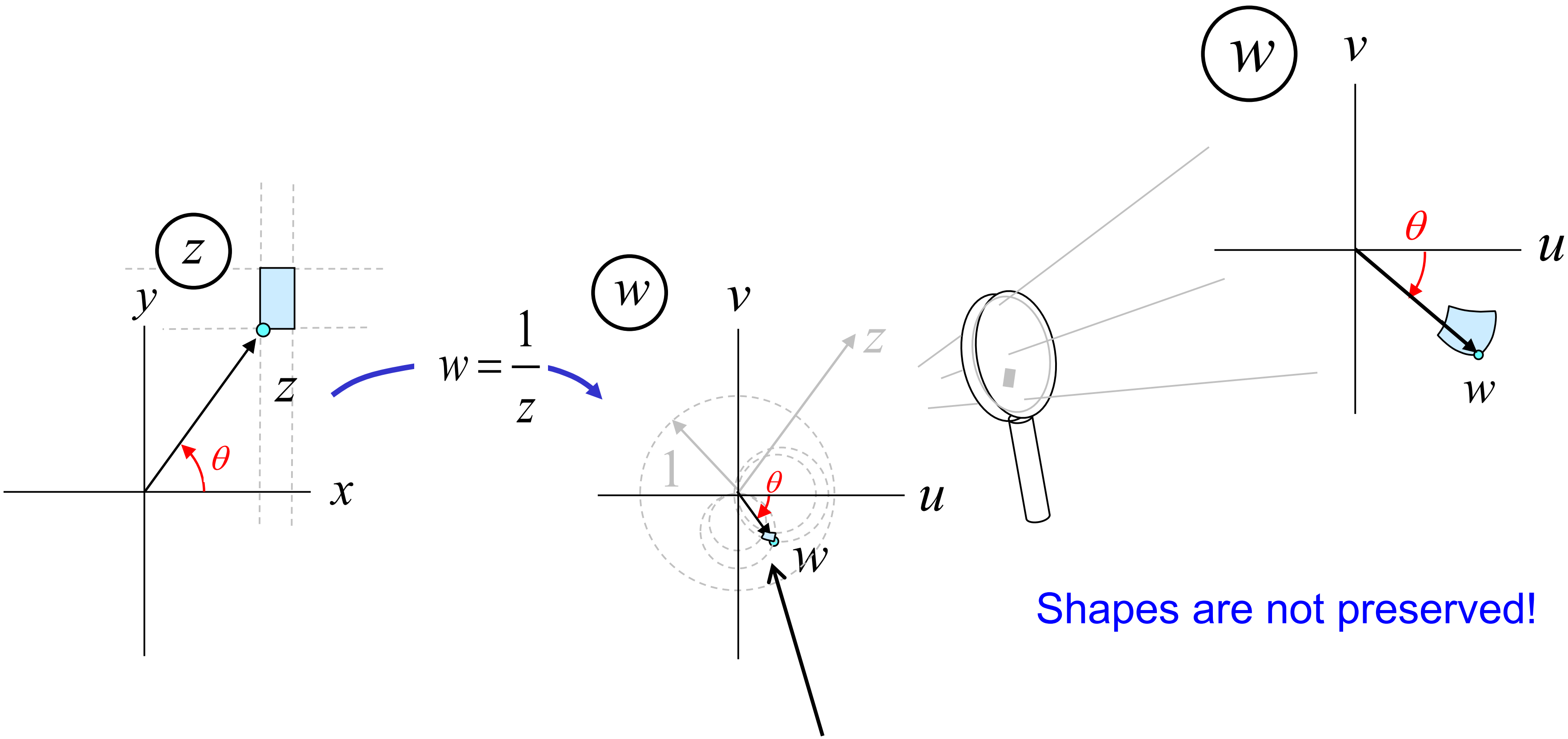
$v_0 \equiv +a_2'/2$

$R^2 \equiv \frac{a_1'^2}{4} + \frac{a_2'^2}{4} - a_0'$

$$R^2 = \frac{a_1^2}{4a_3^2} + \frac{a_2^2}{4a_3^2} - \frac{1}{a_3} = \frac{x_0^2 + y_0^2}{(x_0^2 + y_0^2 - a^2)^2} - \frac{1}{x_0^2 + y_0^2 - a^2} = \frac{x_0^2 + y_0^2 - (x_0^2 + y_0^2 - a^2)}{(x_0^2 + y_0^2 - a^2)^2} = \frac{a^2}{(x_0^2 + y_0^2 - a^2)^2}$$

Simple Mappings: Inversions (cont.)

- **Geometrical construction of the inversion:** $w = \dfrac{1}{z} = \dfrac{1}{re^{i\theta}} = \dfrac{1}{r}\, e^{-i\theta}$



Shapes are not preserved!

Note the circular boundaries for the region!

Bilinear (a.k.a. Fractional or Mobius) Transformation

$$w = \frac{A + Bz}{C + Dz} \qquad (A,\ B,\ C,\ D \text{ are complex constants})$$

- **Note that if** $D \neq 0$,

$$w = \frac{A + Bz}{C + Dz} = \frac{A - BC/D + B/D\left(C + Dz\right)}{C + Dz} = \frac{A - BC/D}{C + Dz} + B/D$$

Steps in $z \Rightarrow w$:

$$z \Rightarrow C + Dz \Rightarrow \frac{1}{C + Dz} \Rightarrow \frac{A - BC/D}{C + Dz} \Rightarrow \frac{A - BC/D}{C + Dz} + B/D$$

- **This is a sequence of : linear transformation; inversion; dilation and rotation; translation.**
- **Since each transformation preserves circles, bilinear transformations also have the circle-preserving property : circles in the** z **plane are mapped into circles in the** w **plane (with straight lines thought of as circles of infinite radius).**

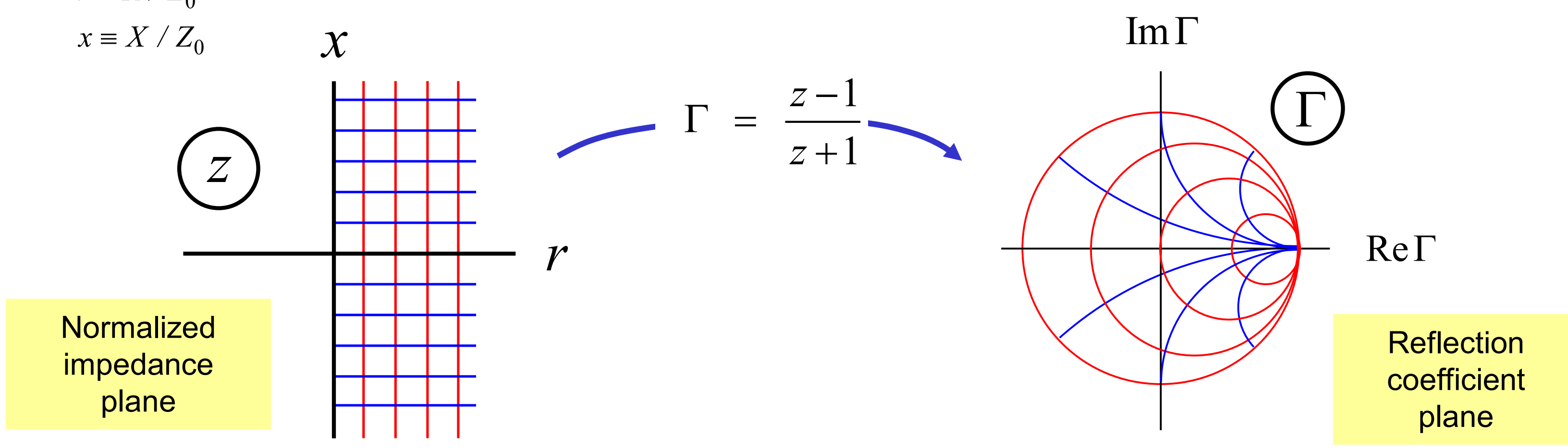
Bilinear Transformation Example: The Smith Chart

□ **Let** $z = r + jx = \dfrac{Z(d)}{Z_0}$ **where** $Z(d) = R(d) + jX(d)$ **is the impedance at** $z = -d$ **on a transmission line of characteristic impedance** Z_0, **and** $\Gamma(d)$ **is the generalized reflection coefficient :**

$$\Gamma(d) = \frac{Z(d)-Z_0}{Z(d)+Z_0} = \frac{z(d)-1}{z(d)+1} \quad \textbf{or simply} \quad \Gamma = \frac{z-1}{z+1}$$

$r \equiv R / Z_0$

$x \equiv X / Z_0$



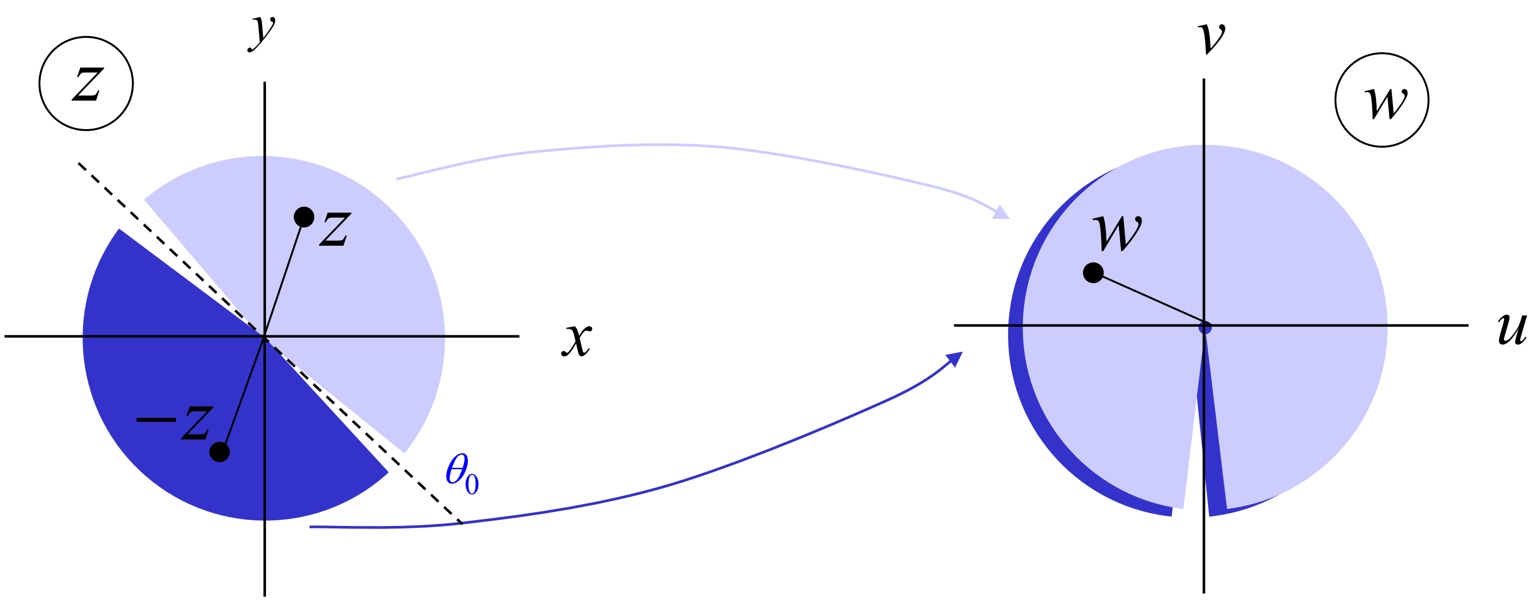
Horizontal and vertical ines (contant reactance and resistance) are mapped into circles.

For an interpretation of Möbius transformations as projections on a sphere, see http://www.youtube.com/watch?v=JX3VmDgiFnY.

The Squaring Transformation

$$w = f(z) = z^2 = r^2\, e^{i2\theta}$$

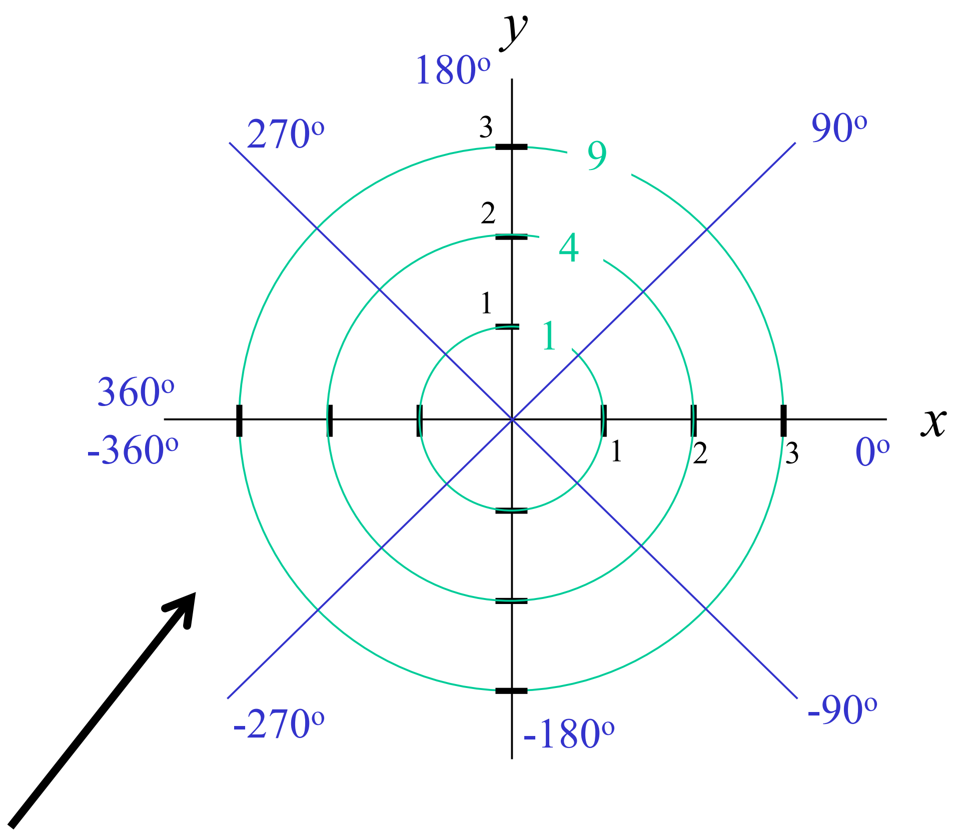
$$(z, -z) \to w$$

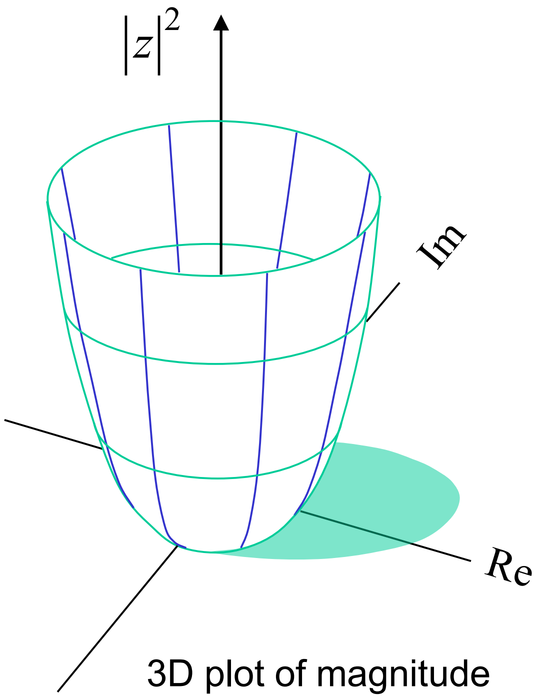


- **The transformation maps *half* the z-plane into the *entire* w-plane.**
- **The entire z-plane covers the w-plane twice.**
- **The transformation is said to be *two-to-one.***

Another Representation of the Squaring Transformation

$$w = f(z) = z^2 = r^2\, e^{i2\theta}$$





3D plot of magnitude

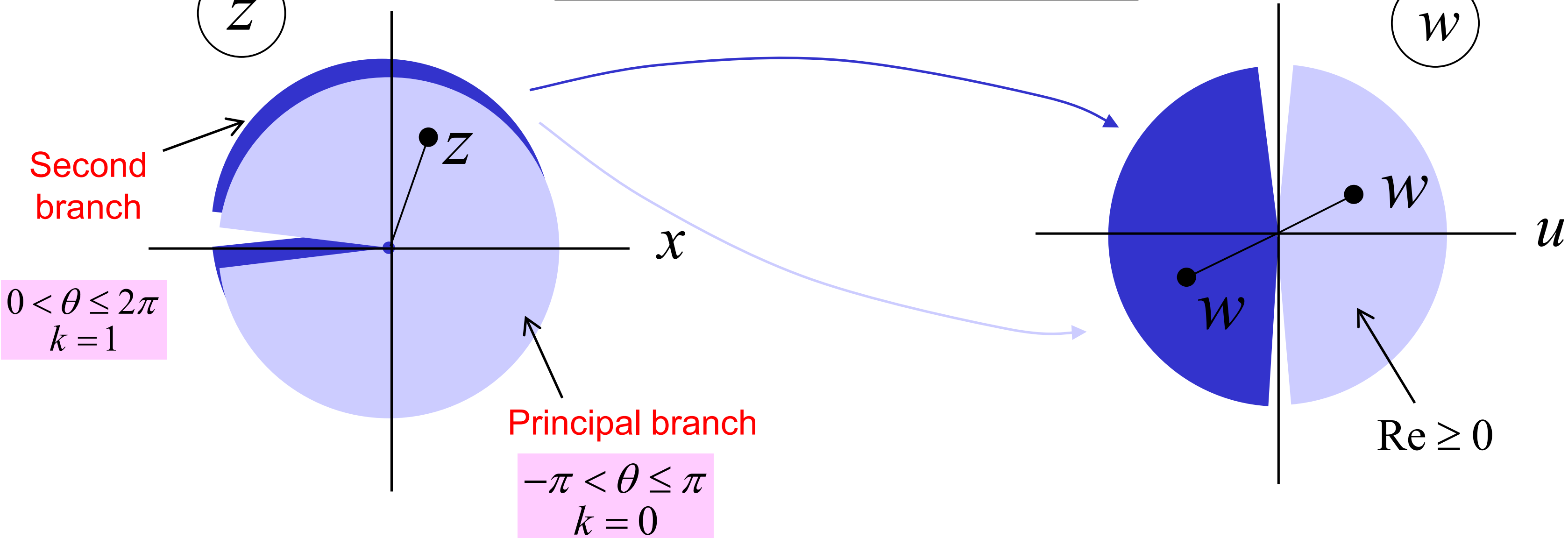
Constant amplitude and phase contours of $w = f(z) = z^2$

The Square Root Transformation

$$w = f(z) = z^{1/2} = \sqrt{r}e^{i\frac{\theta}{2}} = \sqrt{r}e^{i\frac{\theta_p + 2\pi k}{2}}, \quad k = 0,1$$

$$-\pi < \theta_p \le \pi$$

Note: The value of $z^{1/2}$ on one branch is the negative of the value on the other branch.



❖ We say that there are two “branches” (i.e., values) of the square root function.

❖ Note that for the principal branch, the square root function is <u>not continuous</u> on the negative real axis. (There is a “branch cut” there.)

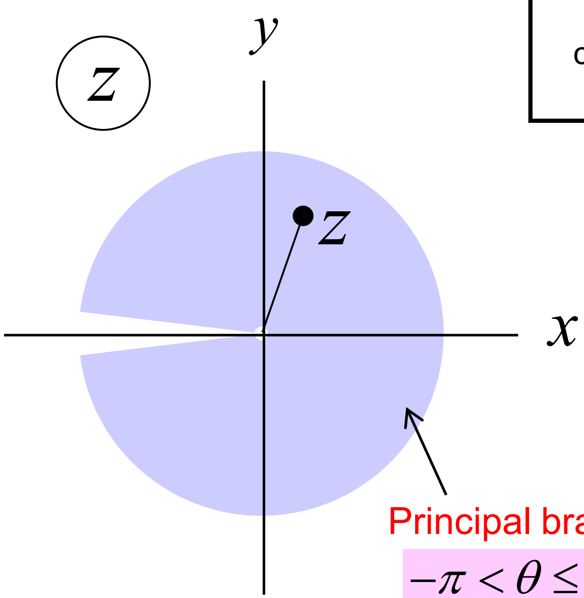
- **The transformation is said to be one - *to* - *two***

The Square Root Transformation (cont.)

$$w = f(z) = z^{1/2} = \sqrt{r}e^{i\frac{\theta}{2}} = \sqrt{r}e^{i\frac{\theta_p + 2\pi k}{2}}, \quad k = 0,1$$

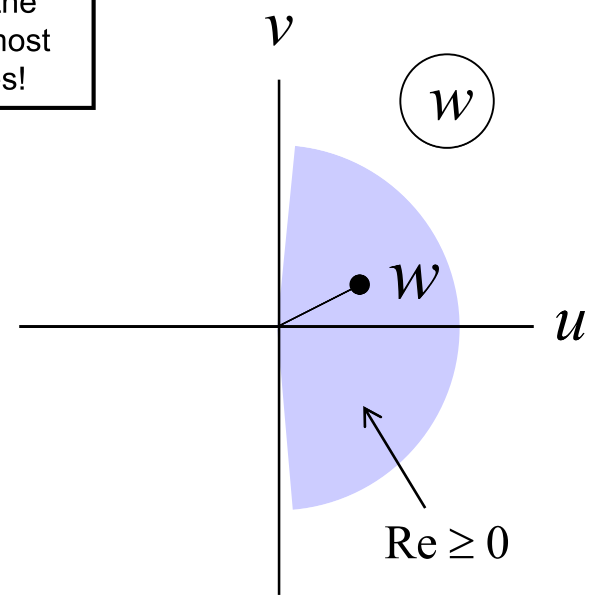
$$-\pi < \theta_p \le \pi$$

The principal branch is the choice in MATLAB and most programming languages!



Principal branch

$$-\pi < \theta \le \pi$$
$$k = 0$$



$\sqrt{1} = 1$

$\sqrt{-1} = i$

$\sqrt{i} = \frac{1+i}{\sqrt{2}}$

$\sqrt{-i} = \frac{1-i}{\sqrt{2}}$

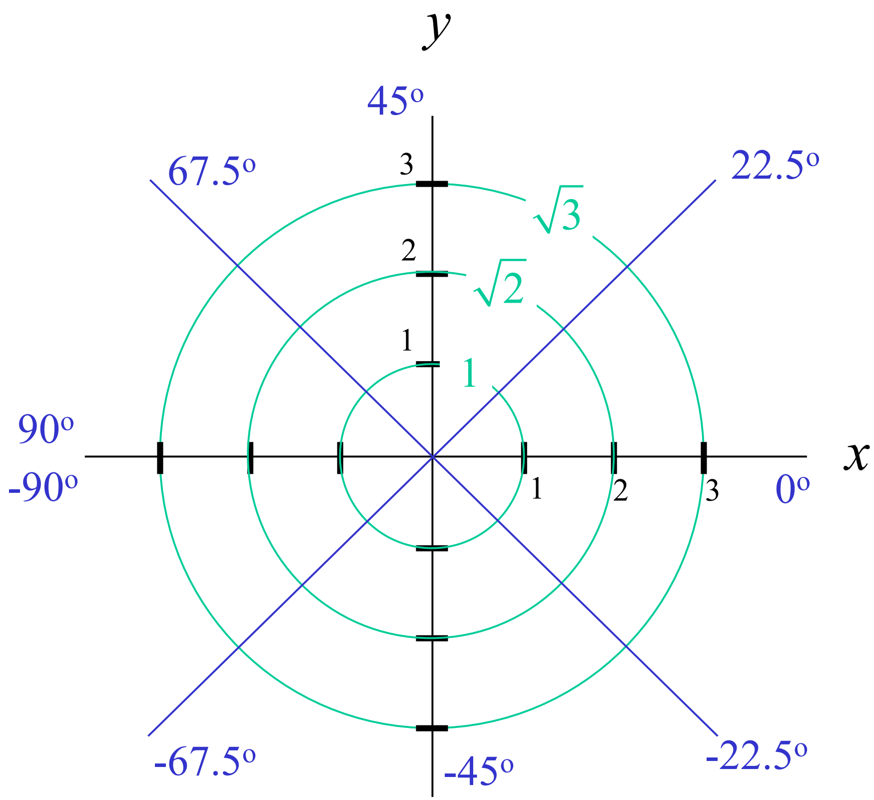
The principal square root is denoted as $\sqrt{z}$

Note: $\mathrm{Re}\sqrt{z} \ge 0$

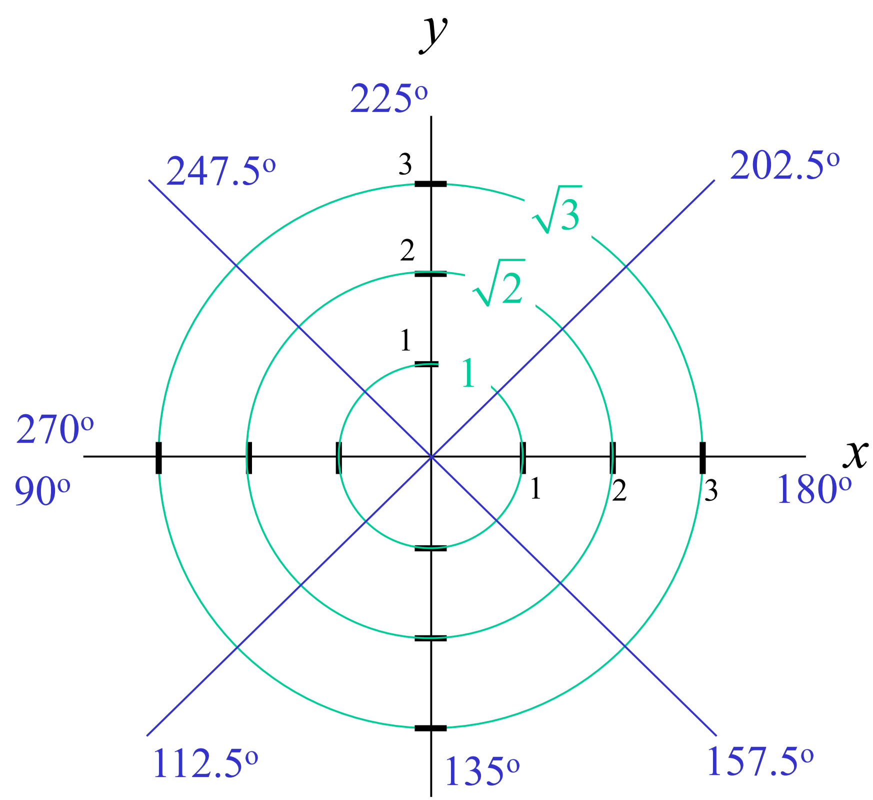
The Square Root Transformation (cont.)

$$w = f(z) = z^{1/2} = \sqrt{r}e^{i\frac{\theta}{2}} = \sqrt{r}e^{i\frac{\theta_p + 2\pi k}{2}}, \quad k = 0,1$$

$$-\pi < \theta_p \leq \pi$$



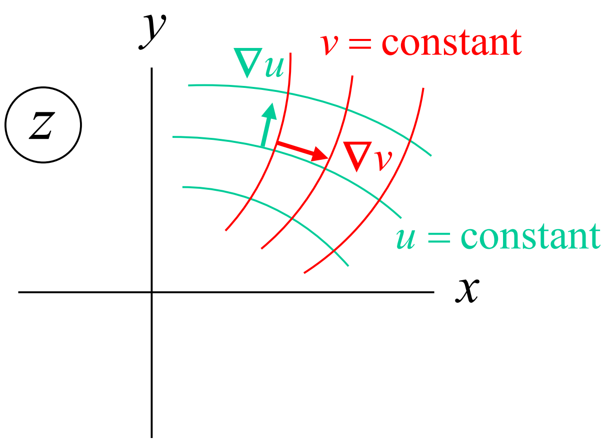
Principal branch, $k = 0$



Other branch, $k = 1$

Constant u and v Contours are Orthogonal

- Consider contours in the z plane on which the real quantities $u(x,y)$ and $v(x,y)$ are constant.



$$w = u(x,y) + iv(x,y) = f(z) \text{ (analytic)}$$

- The directions normal to these contours are along the gradient direction :

$$\nabla u = \frac{\partial u}{\partial x}\hat{\underline{x}} + \frac{\partial u}{\partial y}\hat{\underline{y}}$$

$$\nabla v = \frac{\partial v}{\partial x}\hat{\underline{x}} + \frac{\partial v}{\partial y}\hat{\underline{y}}$$

- The gradients, and therefore the contours, are orthogonal (perpendicular) by the C.R. conditions :

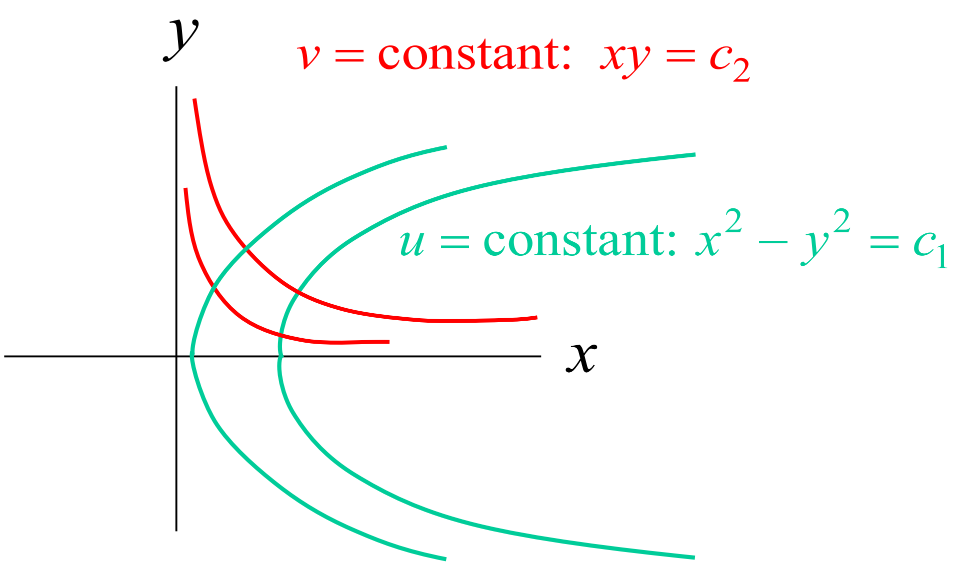
$$\nabla u \cdot \nabla v = \left(\frac{\partial u}{\partial x}\hat{\underline{x}} + \frac{\partial u}{\partial y}\hat{\underline{y}}\right)\cdot\left(\frac{\partial v}{\partial x}\hat{\underline{x}} + \frac{\partial v}{\partial y}\hat{\underline{y}}\right) \overset{\text{C.R. cond's}}{=} \left(\frac{\partial u}{\partial x}\hat{\underline{x}} + \frac{\partial u}{\partial y}\hat{\underline{y}}\right)\cdot\left(-\frac{\partial u}{\partial y}\hat{\underline{x}} + \frac{\partial u}{\partial x}\hat{\underline{y}}\right) = -\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0$$

Constant u and v Contours are Orthogonal (cont.)

Example: $w = z^2$

$$w = (x+iy)^2 = \left(x^2 - y^2\right) + i(2xy)$$

so $$\begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases}$$



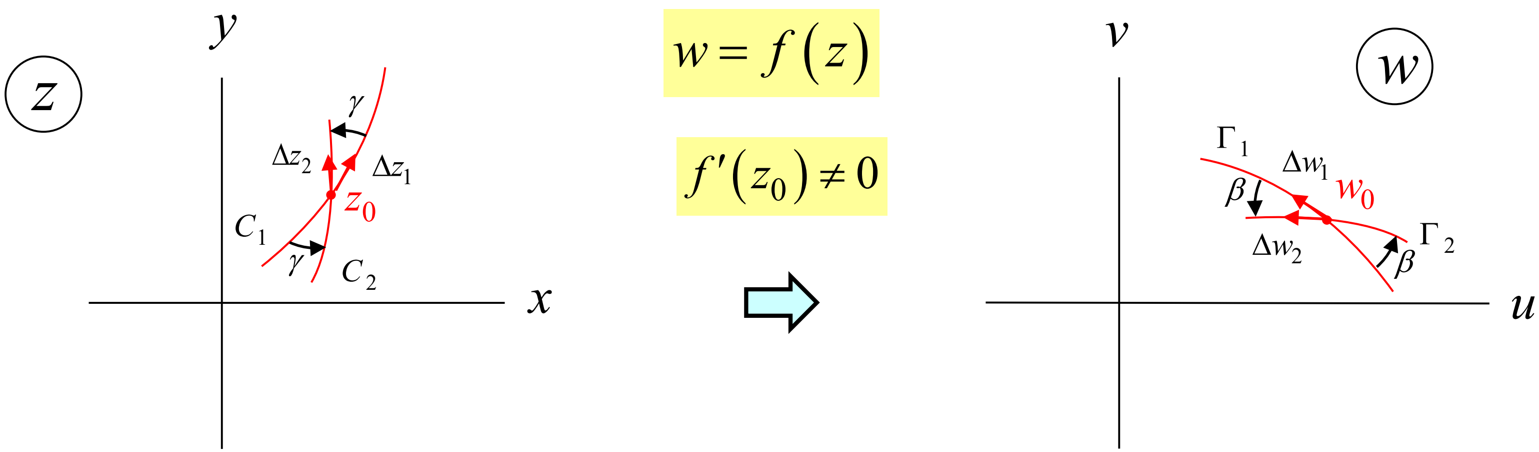
Also, recall that

$$\nabla^2 u(x,y) = 0$$

$$\nabla^2 v(x,y) = 0$$

Mappings of Analytic Functions are *Conformal* (Angle-Preserving)

- Consider a pair of intersecting paths C_1, C_2 in the z plane mapped onto the $w = u + iv$ plane.



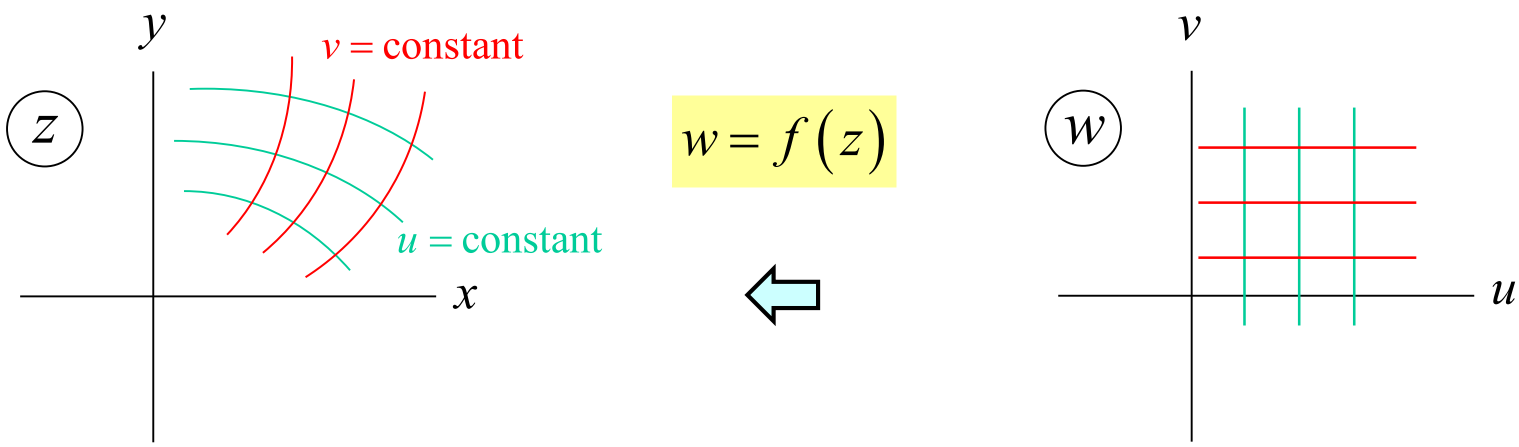
$$\begin{cases} \Delta w_1 \approx f'(z_0)\Delta z_1 \quad \Rightarrow \quad \arg(\Delta w_1) \approx \arg f'(z_0) + \arg(\Delta z_1), \quad \Delta z_1 \text{ along } C_1 \text{ from } z_0 \\ \Delta w_2 \approx f'(z_0)\Delta z_2 \quad \Rightarrow \quad \arg(\Delta w_2) \approx \arg f'(z_0) + \arg(\Delta z_2), \quad \Delta z_2 \text{ along } C_2 \text{ from } z_0 \end{cases}$$

$$\Rightarrow \arg(\Delta w_2) - \arg(\Delta w_1) \approx \arg(\Delta z_2) - \arg(\Delta z_1)$$

This assumes that f' is not zero.

Hence $\beta = \gamma$

Constant u and v Contours are Orthogonal (Revisited)



Since the contours $u = constant$ and $v = constant$ are (obviously) orthogonal in the w plane, they must remain orthogonal in the z plane.

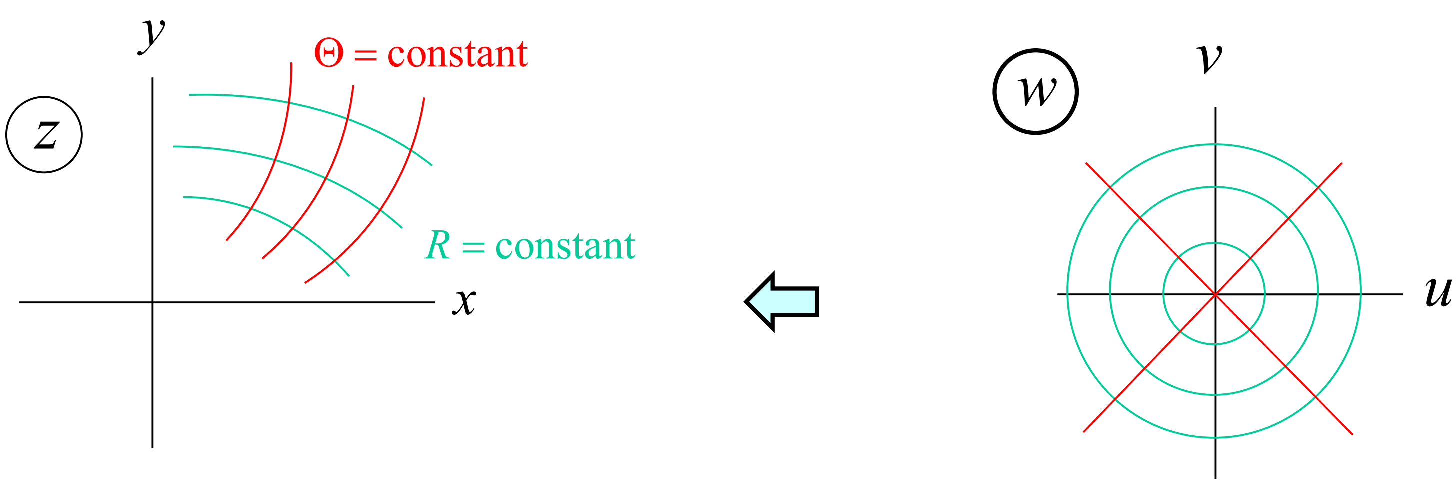
$$\text{Assumption: } \frac{dz}{dw} \neq 0$$

Constant $|w|$ and $\arg(w)$ Contours are also Orthogonal

- If $w = Re^{i\Theta}$ the constant R and Θ contours are (obviously) orthogonal in the w plane.
- If $z = f^{-1}(w)$ is a mapping back to the z plane, the mapping preserves the orthogonality.

Assumption : $\frac{dz}{dw} \neq 0$

Note:
The constant Θ (red) and constant R (green) curves are obviously orthogonal in the w plane.



The Logarithm Function

$$w = \ln(z)$$

$$z = |z| e^{i\theta} = |z| e^{i(\theta_p + 2\pi k)}$$

$$\Rightarrow \ln(z) = \ln|z| + i(\theta_p + 2\pi k), \quad k = 0, \pm 1, \pm 2, \cdots$$

There are an infinite number of branches (values) for the ln function!

Arbitrary Powers of Complex Numbers

$$w = z^a$$ (a may be complex)

Use

$$z = e^{\ln z} \quad \left(z = |z| e^{i\theta} = |z| e^{i(\theta_p + 2\pi k)} \right)$$

$$\Rightarrow \quad z^a = \left(e^{\ln z}\right)^a = e^{a \ln z} = e^{a \ln|z| + ai(\theta_p + 2\pi k)} = e^{a \ln|z|} e^{ia\theta_p} \, e^{i2\pi ak}$$

This has an <u>infinite</u> number of branches unless $ak = \text{integer}$ for some value of $k = q$, i.e, a is real and rational:

$$a = \frac{p}{q} \quad (p, q \text{ are integers})$$

(In this case there are q branches.)

Arbitrary Powers of Complex Numbers (cont.)

Example: $f(z) = z^{2/3} \quad (a = 2/3)$

Recall:

$$z = |z| e^{i(\theta_p + 2\pi k)}$$

$$z^a = e^{a \ln|z|} e^{ia\theta_p} e^{i2\pi ak}$$

$$z^{2/3} = \left(e^{\frac{2}{3}\ln|z|} e^{i\frac{2}{3}\theta_p} \right) e^{i2\pi\left(\frac{2}{3}k\right)}$$

$k = 0 \quad \frac{2}{3}k = 0 \quad \Rightarrow \quad z^{2/3} = e^{\frac{2}{3}\ln|z|} e^{i\frac{2}{3}\theta_p}$

$k = 1 \quad \frac{2}{3}k = \frac{2}{3} \quad \Rightarrow \quad z^{2/3} = e^{\frac{2}{3}\ln|z|} e^{i\frac{2}{3}\theta_p} e^{i(2\pi)\frac{2}{3}}$

$k = 2 \quad \frac{2}{3}k = \frac{4}{3} \quad \Rightarrow \quad z^{2/3} = e^{\frac{2}{3}\ln|z|} e^{i\frac{2}{3}\theta_p} e^{i(2\pi)\frac{4}{3}}$

$k = 3 \quad \frac{2}{3}k = 2 \quad \Rightarrow \quad z^{2/3} = e^{\frac{2}{3}\ln|z|} e^{i\frac{2}{3}\theta_p} e^{i(2\pi)2} = e^{\frac{2}{3}\ln|z|} e^{i\frac{2}{3}\theta_p} \leftarrow$ starts repeating!

$k = 4 \quad \frac{2}{3}k = \frac{8}{3} = 2 + \frac{2}{3} \quad \Rightarrow \quad \cdots$ repeats!

$\vdots$

For $z^{p/q}$ the repetition period is $k = q$ (if p and q have no common factors). For irrational powers, the repetition period is infinite; i.e., values never repeat!