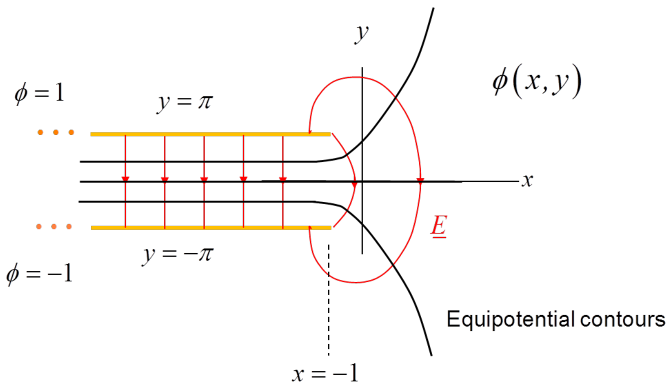
ECE 6382



Fall 2023

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Notes 5

Conformal Mapping

Notes are adapted from D. R. Wilton, Dept. of ECE

Conformal Mapping

This is a method for solving 2D problems involving Laplace's equation.

$$\nabla^2 \phi(x, y) = 0$$

$\phi(x, y) =$ constant on C (Dirichlet boundary condition)

or

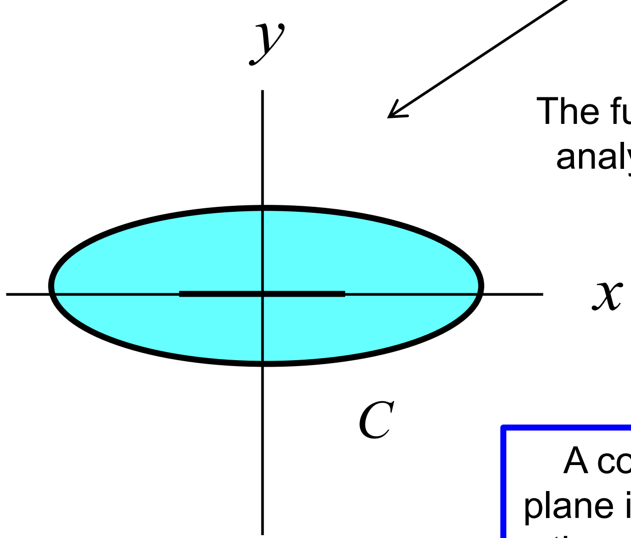
$\dfrac{\partial \phi(x, y)}{\partial n} = 0$ on C (Neumann boundary condition)

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 9th Ed., McGraw-Hill, 2013.

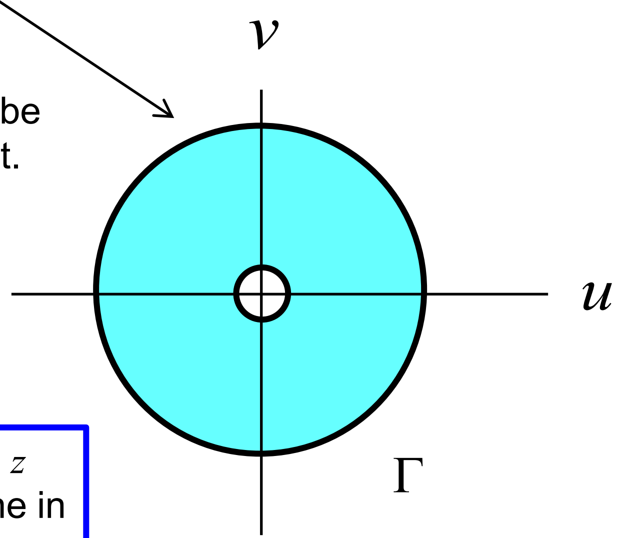
Conformal Mapping (cont.)

$$w = f(z)$$

The function $f(z)$ is assumed to be analytic in the region of interest.



z plane



w plane

A complicated boundary in the z plane is mapped into a simple one in the w plane -- for which we know how to solve the Laplace equation.

$$\phi(x,y)$$

Desired

$$\psi(u,v)$$

Known

Final Result:

$$\phi(x,y) = \psi\big(u(x,y), v(x,y)\big)$$

(We simply take ψ and map it back to the z plane.)

Conformal Mapping (cont.)

The key to being successful with the method of conformal mapping is to find a mapping that works for your problem
(i.e., it maps your problem into one the is simple enough for you to solve).

- J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 9th Ed., McGraw-Hill, 2013.
  - An appendix has many basic conformal mappings.

- H. Kober, *Dictionary of Conformal Representations*, Admiralty, Mathematical and Statistical Section, Dept. of Physical Research, 1945.
  - A very thorough compilation of conformal mappings.

Conformal Mapping (cont.)

Theorem:

If $\psi(u,v)$ satisfies the Laplace equation in the (u,v) plane, then $\phi(x,y)$ satisfies the Laplace equation in the (x,y) plane.

$$\phi(x,y) = \psi\left(u(x,y), v(x,y)\right)$$

Proof:

Assume that

$$\nabla^2 \psi(u,v) = 0$$

or $$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = 0$$

We want to prove that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Conformal Mapping (cont.)

$$\phi(x,y)=\psi\left(u(x,y),v(x,y)\right)$$

$$\Rightarrow \frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\psi}{\partial v}\frac{\partial v}{\partial x}$$ (partial derivative chain rule)

Using the product rule and the partial derivative chain rule:

$$\frac{\partial^2\phi}{\partial x^2}=\left(\frac{\partial\psi}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial u}\right)\frac{\partial u}{\partial x}\right)+\left(\frac{\partial\psi}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial v}\right)\frac{\partial v}{\partial x}\right)$$

$$=\frac{\partial\psi}{\partial u}\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial^2\psi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial^2\psi}{\partial v\partial u}\left(\frac{\partial v}{\partial x}\right)\right)\left(\frac{\partial u}{\partial x}\right)$$

$$+\frac{\partial\psi}{\partial v}\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial^2\psi}{\partial u\partial v}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial^2\psi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)\right)\left(\frac{\partial v}{\partial x}\right)$$

Note: $\frac{\partial^2\psi}{\partial v\partial v}=\frac{\partial^2\psi}{\partial u\partial v}$

Conformal Mapping (cont.)

$$\phi(x,y)=\psi\big(u(x,y),v(x,y)\big)$$

$$\Rightarrow \frac{\partial \phi}{\partial y}=\frac{\partial \psi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \psi}{\partial v}\frac{\partial v}{\partial y}$$

Using the chain rule:

$$\frac{\partial^2 \phi}{\partial y^2}=\left(\frac{\partial \psi}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial u}\right)\frac{\partial u}{\partial y}\right)+\left(\frac{\partial \psi}{\partial v}\frac{\partial^2 v}{\partial y^2}+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial v}\right)\frac{\partial v}{\partial y}\right)$$

$$=\frac{\partial \psi}{\partial u}\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial^2 \psi}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial^2 \psi}{\partial v \partial u}\left(\frac{\partial v}{\partial y}\right)\right)\left(\frac{\partial u}{\partial y}\right)$$

$$+\frac{\partial \psi}{\partial v}\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial^2 \psi}{\partial u \partial v}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial^2 \psi}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)\right)\left(\frac{\partial v}{\partial y}\right)$$

Note: $\frac{\partial^2 \psi}{\partial v \partial v}=\frac{\partial^2 \psi}{\partial u \partial v}$

Conformal Mapping (cont.)

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\partial\psi}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2\psi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial\psi}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2\psi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2+2\frac{\partial^2\psi}{\partial u\partial v}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial\psi}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2\psi}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial\psi}{\partial v}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2\psi}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial^2\psi}{\partial u\partial v}\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right)$$

(The new color coding here shows how to combine terms.)

Use Cauchy-Riemann equations (red, blue, and black terms):

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\qquad\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

ψ satisfies Laplace's equation.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\left(\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right)\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial\psi}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial\psi}{\partial u}\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right)\left(\frac{\partial u}{\partial y}\right)^2$$

$$+\frac{\partial\psi}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial\psi}{\partial v}\frac{\partial^2 v}{\partial y^2}+\underbrace{2\frac{\partial^2\psi}{\partial u\partial v}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right)+2\frac{\partial^2\psi}{\partial u\partial v}\left(-\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)}_{\text{cancels}}$$

Conformal Mapping (cont.)

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\frac{\partial\psi}{\partial u}+\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\frac{\partial\psi}{\partial v}$$

Recall that for any analytic function f

$$\nabla^2 u(x,y)=0$$

$$\nabla^2 v(x,y)=0$$

Hence, we have

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

(proof complete)

Conformal Mapping (cont.)

Example

Illustrates : $\nabla^2\psi(u,v)=0 \;\Rightarrow\; \nabla^2\phi(x,y)=0,$ assuming $w=f(z)$ is analytic

Given:
$$\begin{cases} \psi(u,v)=u \\ w=f(z)=z^2 \end{cases}$$

(This ψ satisfies Laplace's equation.)

(This is an analytic mapping function.)

Verify $\phi(x,y)$ satisfies Laplace's equation.

$$\phi(x,y)=\psi\big(u(x,y),v(x,y)\big)=u(x,y)$$

$$w=u+iv=(x+iy)^2$$

Hence:

$$u=x^2-y^2$$
$$v=2xy$$

$$\Rightarrow \quad \phi(x,y)=x^2-y^2$$

The function ϕ satisfies Laplace's equation.

Conformal Mapping (cont.)

Example

Illustrates: $\nabla^2\psi(u,v)=0 \;\Rightarrow\; \nabla^2\phi(x,y)=0,$ assuming $w=f(z)$ is analytic

Given:
$$\begin{cases} \psi(u,v)=u^2-v^2 & \text{(This satisfies Laplace's equation.)} \\ w=f(z)=e^z & \text{(This is an analytic mapping function.)} \end{cases}$$

Verify $\phi(x,y)$ satisfies Laplace's equation.

$$\phi(x,y)=\psi\big(u(x,y),v(x,y)\big)=u^2(x,y)-v^2(x,y)$$

$$w=u+iv=e^{x+iy}=e^x(\cos y+i\sin y)$$

Hence:
$$u=e^x\cos y$$
$$v=e^x\sin y$$

$$\Rightarrow \quad \phi(x,y)=e^{2x}\cos^2 y-e^{2x}\sin^2 y=e^{2x}\cos(2y)$$

The function ϕ satisfies Laplace's equation.

Conformal Mapping (cont.)

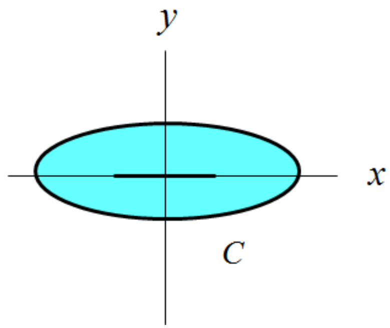
Theorem:

If $\psi(u,v)$ satisfies Dirichlet or Neumann boundary conditions in the (u,v) plane, then $\phi(x,y)$ satisfies the same boundary conditions in the (x,y) plane.

Proof:

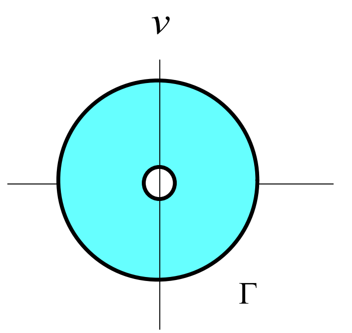
Assume that

$$\psi(u,v) = c \text{ on } \Gamma$$



Then we immediately have that

$$\phi(x,y) = c \text{ on } C$$



since

$$\phi(x,y) = \psi\left(u(x,y), v(x,y)\right)$$

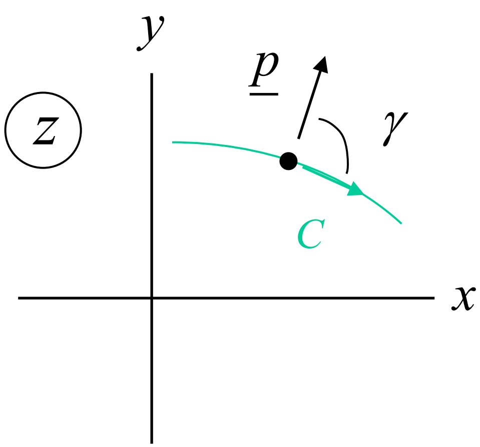
Conformal Mapping (cont.)

Next, assume that

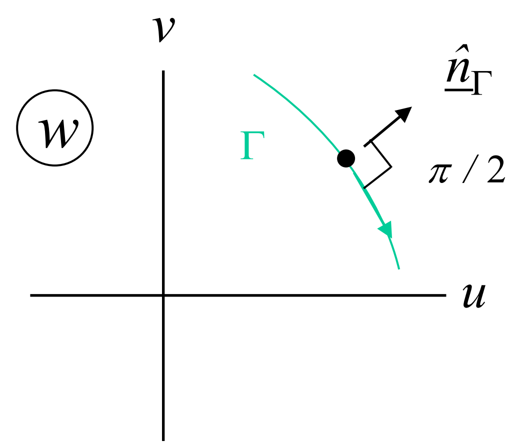
$$\frac{\partial \psi}{\partial n_\Gamma} = 0 \text{ on } \Gamma$$

$\underline{p} \equiv$ mapping of n_Γ

$\underline{p} \equiv$ mapping of $\underline{\hat{n}}_\Gamma$



$$\frac{\partial \phi}{\partial l} = 0 \text{ in direction of } \underline{\hat{p}}$$



Because of the angle-preserving (conformal) property of analytic functions, we have:

$$\gamma = \pi / 2 \Rightarrow \underline{p} \propto \underline{\hat{n}}_C$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial n_C} = 0 \text{ on } C$$

Conformal Mapping (cont.)

Relation Between Charge Densities in the Two Planes

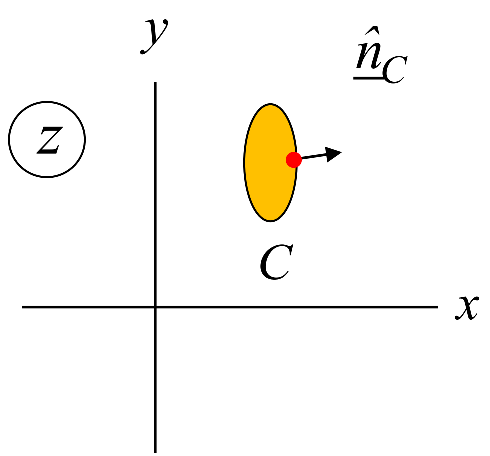
$$\rho_{sz} = \underline{D} \cdot \hat{\underline{n}}_C = \varepsilon \underline{E} \cdot \hat{\underline{n}}_C = -\varepsilon \nabla \phi \cdot \hat{\underline{n}}_C = -\varepsilon \frac{\partial \phi}{\partial n_C}$$

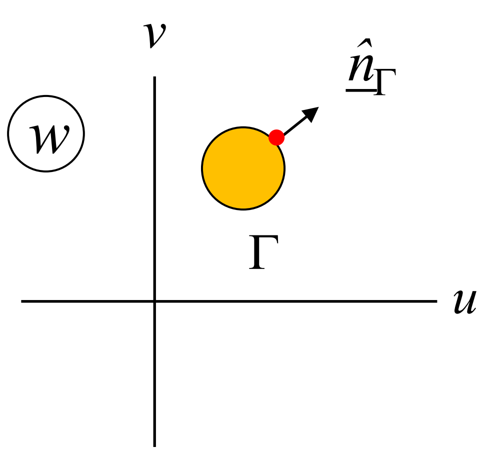
$$\rho_{sw} = \underline{D} \cdot \hat{\underline{n}}_\Gamma = \varepsilon \underline{E} \cdot \hat{\underline{n}}_\Gamma = -\varepsilon \nabla \psi \cdot \hat{\underline{n}}_\Gamma = -\varepsilon \frac{\partial \psi}{\partial n_\Gamma}$$

so

$$\frac{\rho_{sz}}{\rho_{sw}} = \frac{\partial n_\Gamma}{\partial n_C}$$

Note: $\partial \phi = \partial \psi$





Conformal Mapping (cont.)

From the last slide:

$$\frac{\rho_{sz}}{\rho_{sw}} = \frac{\partial n_\Gamma}{\partial n_C}$$

Note :

$$w = f(z)$$

$$dw = f'(z)\,dz$$

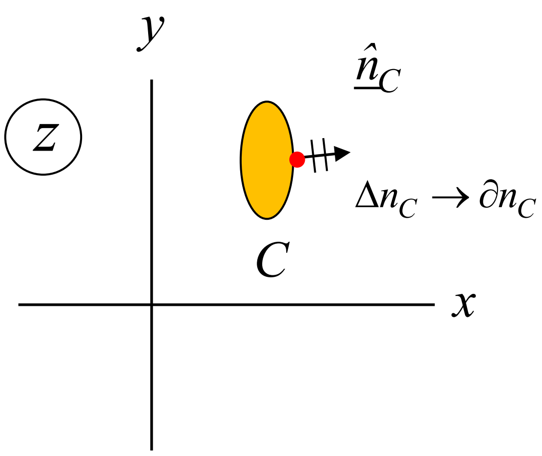
$$|dw| = |f'(z)|\,|dz|$$

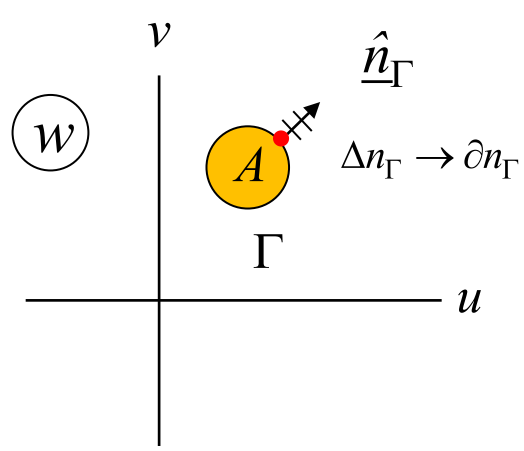
$$\Downarrow$$

$$\partial n_\Gamma = |f'(z)|\,\partial n_C$$

Hence, we have

$$\frac{\rho_{sz}}{\rho_{sw}} = |f'(z)| \qquad z \in C$$





Conformal Mapping (cont.)

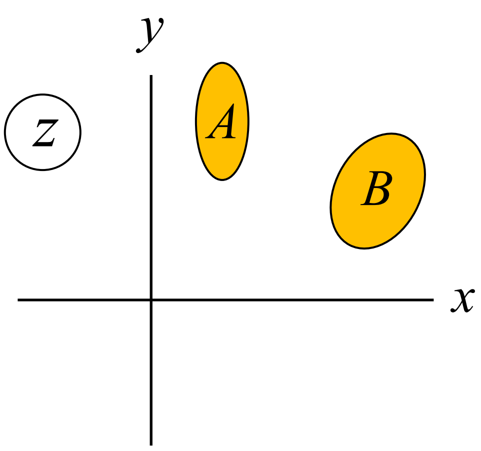
Relation Between Capacitance in the Two Planes

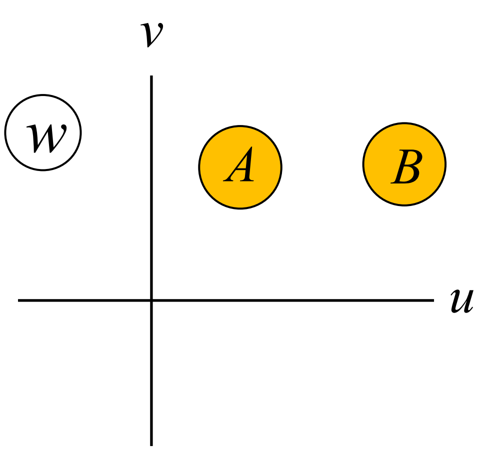
The capacitance (per unit length) between two conductive objects remains unchanged between the z and w planes.

Proof:

$$C_z \equiv \frac{Q_A^{(z)}}{V_{AB}} \qquad C_w \equiv \frac{Q_A^{(w)}}{V_{AB}}$$

$$V_{AB} \equiv \Phi_A - \Phi_B$$ (same voltage drop in both planes)





Conformal Mapping (cont.)

$$Q_A^{(z)} = \oint_{C_A} \rho_{sz} dl_C$$

$$Q_A^{(w)} = \int_{\Gamma_A} \rho_{sw}\, dl_\Gamma$$

Consider this ratio:

$$\frac{\rho_{sz}\, dl_C}{\rho_{sw}\, dl_\Gamma} = \left(\frac{\rho_{sz}}{\rho_{sw}}\right)\left(\frac{dl_C}{dl_\Gamma}\right) = \left|f'(z)\right| \frac{1}{\left|f'(z)\right|} = 1$$

Note:

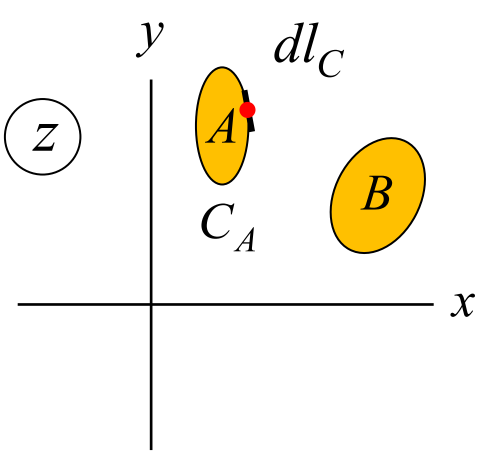
$$w = f(z)$$

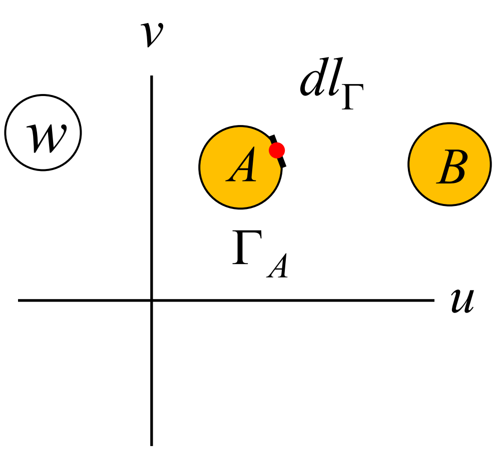
$$dw = f'(z)\, dz$$

$$|dw| = \left|f'(z)\right| |dz|$$

$$\Rightarrow \frac{dl_C}{dl_\Gamma} = \frac{1}{\left|f'(z)\right|}$$

Therefore, $Q_A^{(z)} = Q_A^{(w)}$ ⇨ $C_z = C_w$



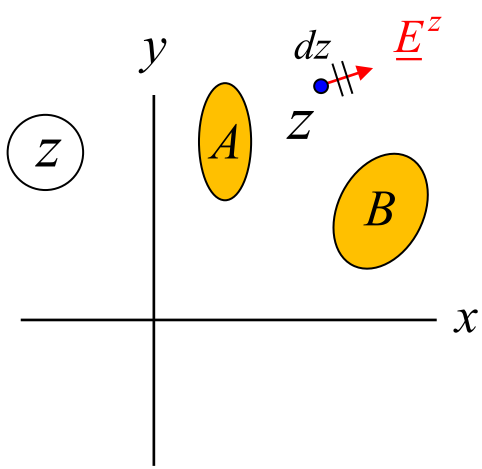


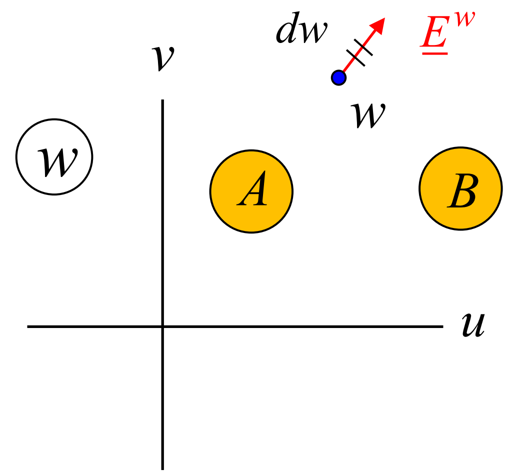
Conformal Mapping (cont.)

Relation Between Electric Field in the Two Planes

$$\underline{E}^{(z)}(x,y) = -\nabla \phi(x,y)$$

$$\underline{E}^{(w)}(u,v) = -\nabla \psi(u,v)$$





dz = small displacement along flux line

dw = corresponding small displacement along flux line

$$dw \approx f'(z)\,dz$$

Conformal Mapping (cont.)

Relation Between Electric Field in the Two Planes (cont.)

$$\underline{E}^{(z)}(x,y) = -\nabla\phi(x,y)$$

$$\left|\underline{E}^{(z)}(x,y)\right| = \left|\nabla\phi(x,y)\right| = \frac{|d\phi|}{|dz|}$$

when dz is in direction of $\nabla\phi$

(electric field in z plane)

$$\underline{E}^{(w)}(u,v) = -\nabla\psi(u,v)$$

$$\left|\underline{E}^{(w)}(u,v)\right| = \left|\nabla\psi(u,v)\right| = \frac{|d\psi|}{|dw|}$$

when dw is in direction of $\nabla\psi$

(electric field in w plane)

Note: The magnitude of the gradient gives us the rate of change of a function when we march in the direction of the gradient.

Hence

$$\frac{\left|\underline{E}^{(z)}(x,y)\right|}{\left|\underline{E}^{(w)}(u,v)\right|} = \frac{|d\phi|}{|dz|}\frac{|dw|}{|d\psi|} = \left|\frac{dw}{dz}\right|\left|\frac{d\phi}{d\psi}\right| = \left|\frac{dw}{dz}\right|$$

$$\Rightarrow \quad \frac{\left|\underline{E}^{(z)}(x,y)\right|}{\left|\underline{E}^{(w)}(u,v)\right|} = \left|f'(z)\right|$$

Note: The electric field vector in the z plane is also rotated from that in the w plane by $-\arg f'(z_0)$.

$$dz = dw\left(\frac{dz}{dw}\right) \Rightarrow \arg(dz) = \arg(dw) - \arg(f'(z))$$

Summary of Important Properties

Laplace Eq. ⇔ Laplace Eq.

Dirichlet ⇔ Dirichlet
Neumann ⇔ Neumann

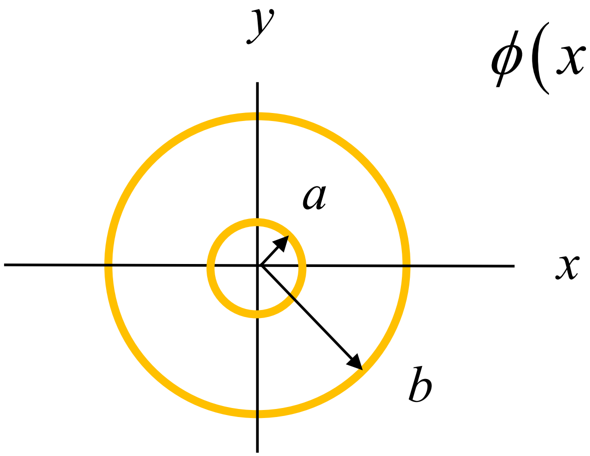
$$C_z = C_w$$

$$\frac{\rho_{sz}}{\rho_{sw}} = \left| f'(z) \right|$$

$$\frac{\left| \underline{E}^{(z)}(x,y) \right|}{\left| \underline{E}^{(w)}(u,v) \right|} = \left| f'(z) \right|$$

Example

Solve for the potential inside of a coax and the capacitance per unit length of a coax.



$$\phi(x,y)=\phi(\rho)$$

$$\phi(a)=1\ [\mathrm{V}]$$

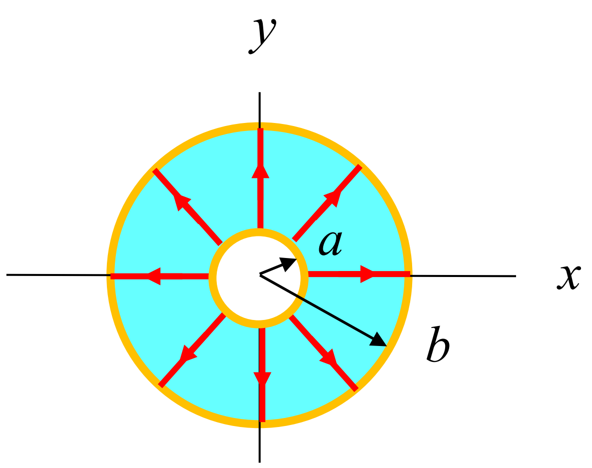
$$\phi(b)=0\ [\mathrm{V}]$$

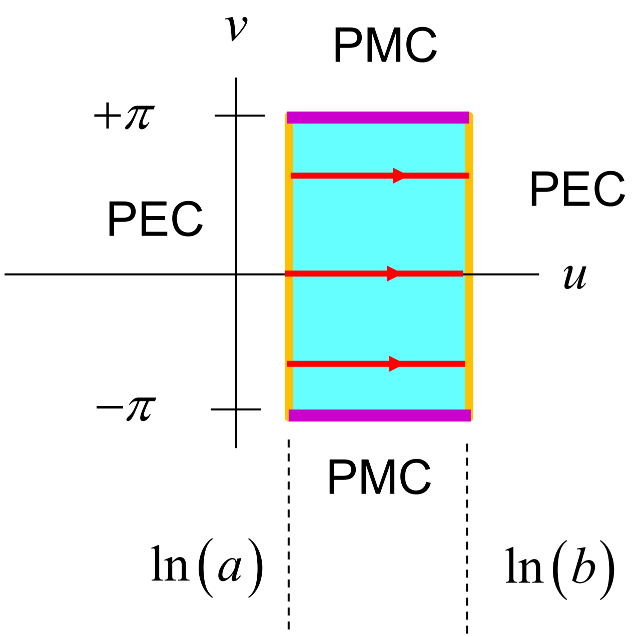
Example (cont.)

$$w = \ln(z)$$

$$w = u + iv = \ln\left(re^{i\theta}\right) = \ln(r) + i\theta$$

$$u = \ln(r)$$
$$v = \theta$$





Assume : $-\pi < \theta < \pi$

Example (cont.)

$$\nabla^2 \psi(u) = 0 \quad \Rightarrow \quad \psi = Au + B$$

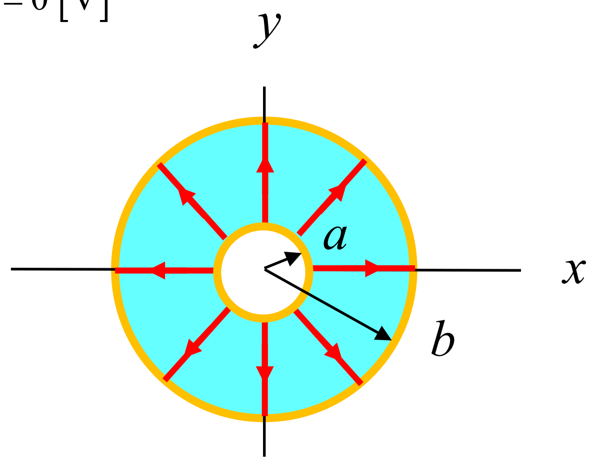
(no v dependence in ψ)

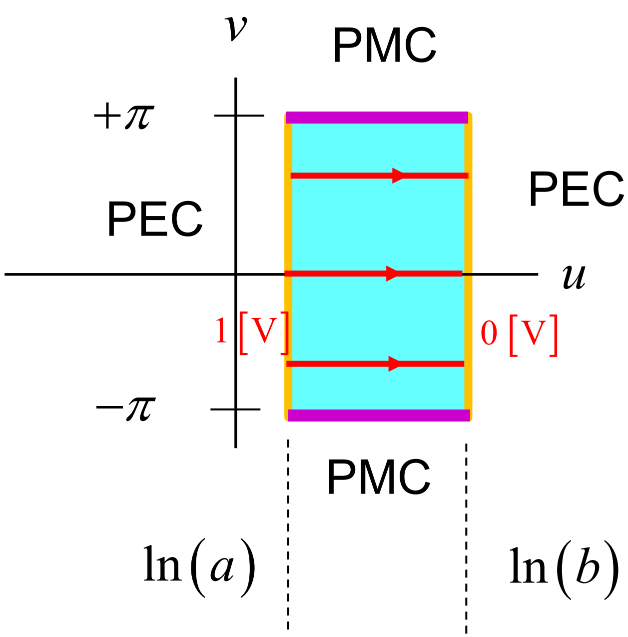
Boundary conditions (solve for A and B)

$$\psi = \left(\frac{1}{\ln a - \ln b}\right) u + \left(\frac{-\ln(b)}{\ln a - \ln b}\right) = \frac{\left(u - \ln(b)\right)}{\ln a - \ln b}$$

$\phi(a) = 1\ [\mathrm{V}]$

$\phi(b) = 0\ [\mathrm{V}]$





Assume : $-\pi < \theta < \pi$

Example (cont.)

$$\psi = \frac{\left(u - \ln(b)\right)}{\ln a - \ln b}$$

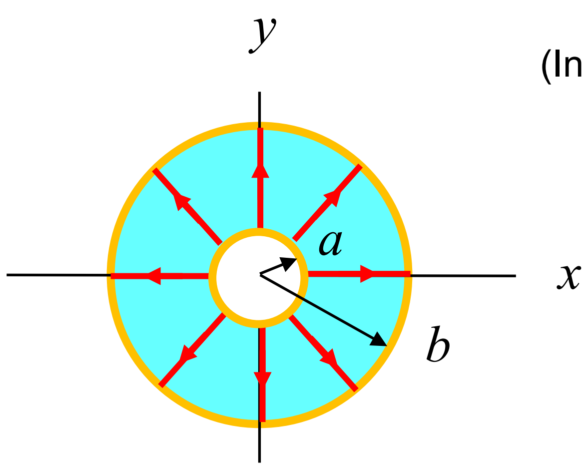
$$u = \ln(r)$$
$$v = \theta$$

so

$$\phi = \frac{\left(\ln(r) - \ln(b)\right)}{\ln a - \ln b}$$

or

$$\phi = \frac{\ln\left(\dfrac{b}{\rho}\right)}{\ln\left(\dfrac{b}{a}\right)}$$



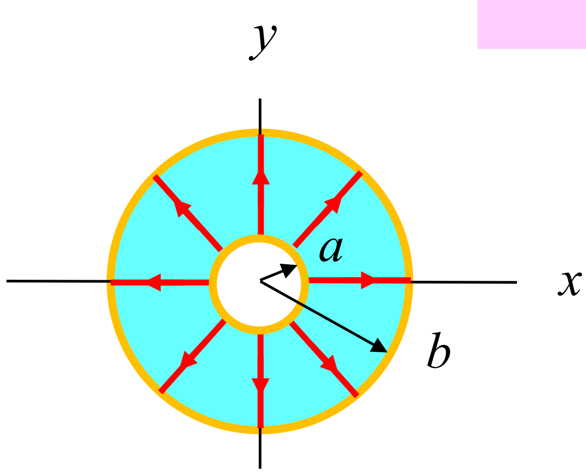
(In the final answer we use ρ instead of r.)

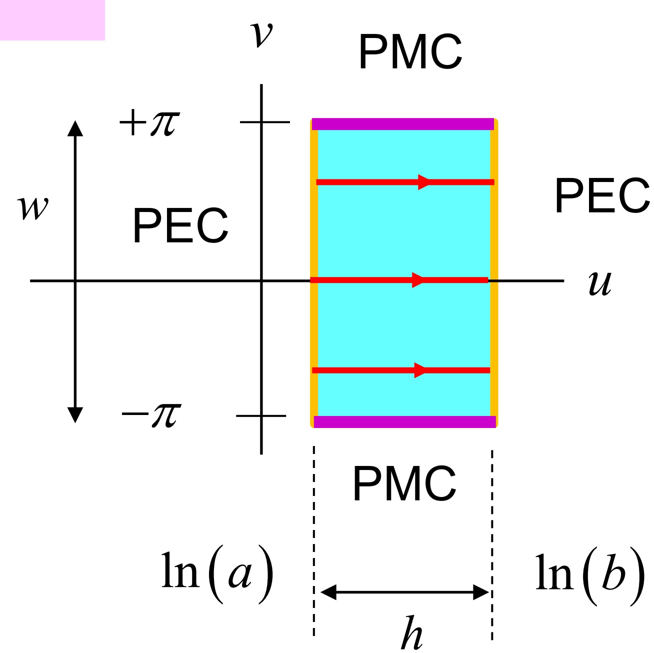
Example (cont.)

$$C_w = \varepsilon\left(\frac{w}{h}\right) = \varepsilon\left(\frac{2\pi}{\ln b - \ln a}\right) = \varepsilon\left(\frac{2\pi}{\ln\left(\frac{b}{a}\right)}\right)$$

so

$$C_z = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)} \quad [\mathrm{F/m}]$$



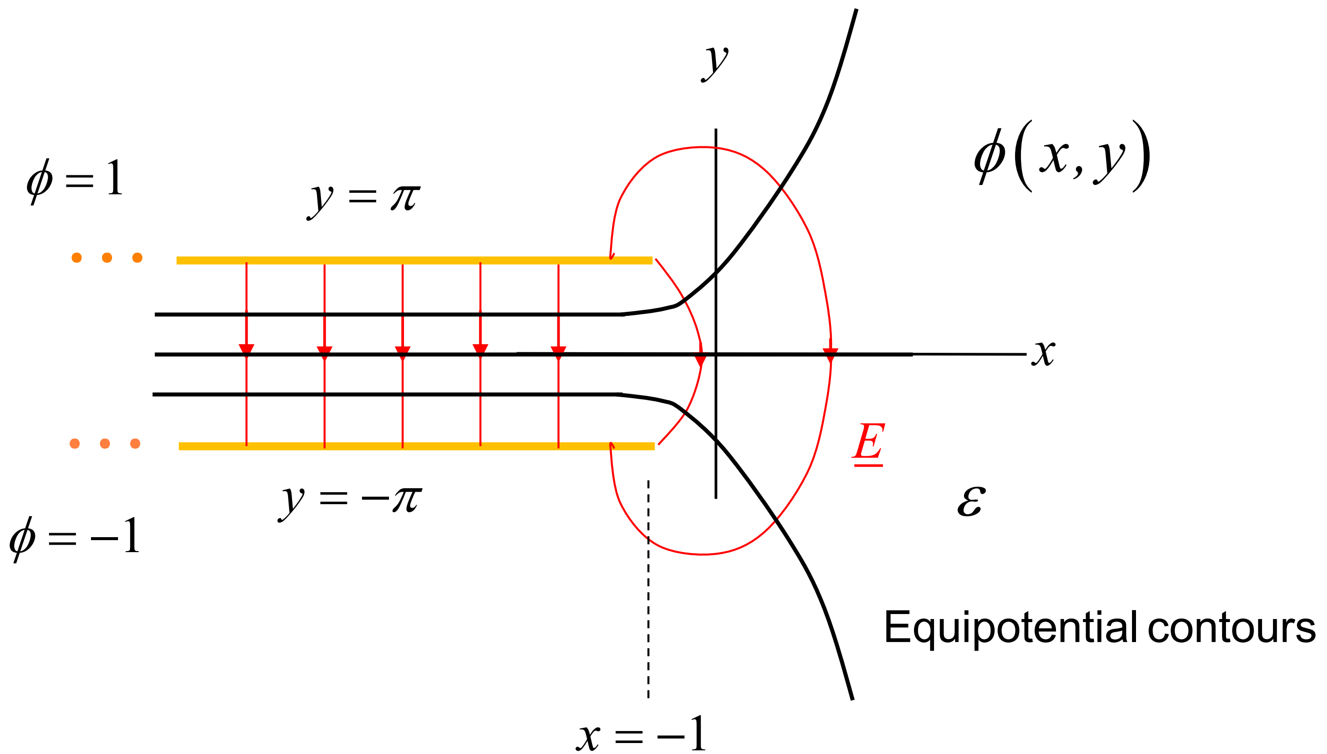


Assume : $-\pi < \theta < \pi$

Example

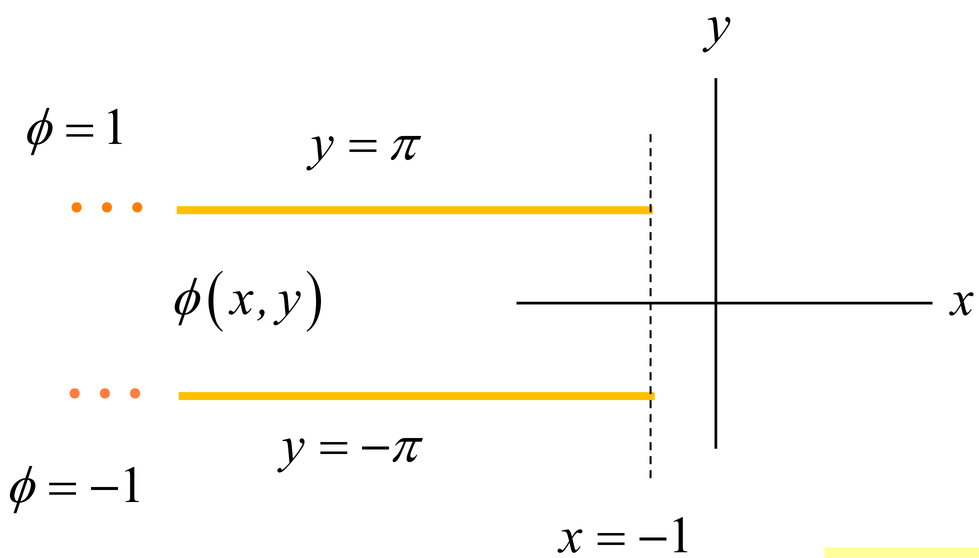
Solve for the potential inside and outside of a semi-infinite parallel-plate capacitor.

Find the surface charge density on the lower surface of the top plate.

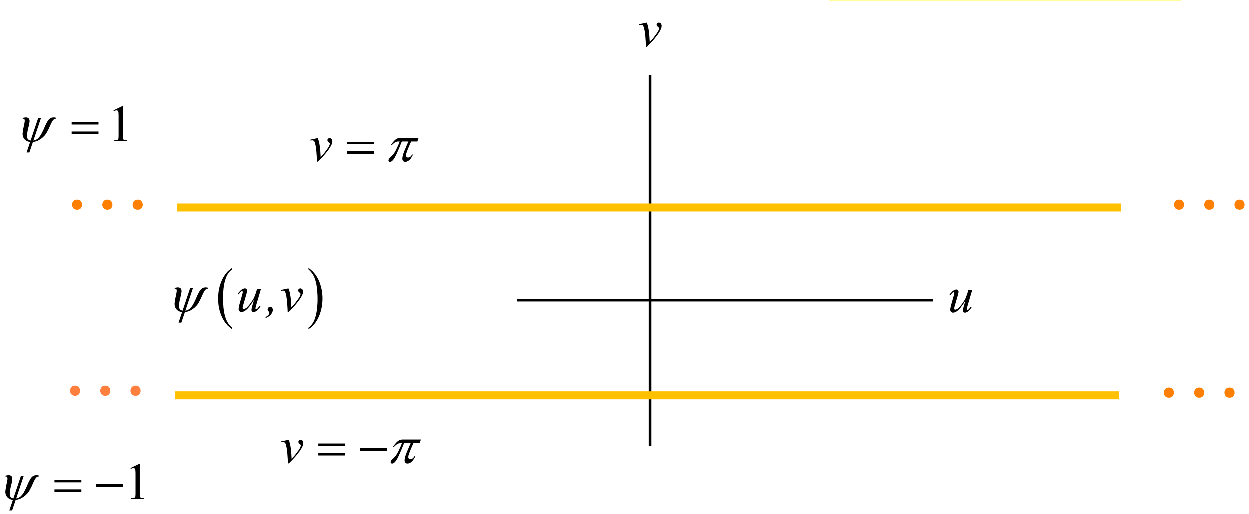


$$\underline{E} = -\nabla \phi(x,y)$$

Example (cont.)



$$z = e^{w} + w$$



$$z = e^{w} + w$$

$$\Downarrow$$

$$x + iy = e^{u+iv} + (u + iv)$$

$$\Downarrow$$

$$x = e^{u}\cos(v) + u$$

$$y = e^{u}\sin(v) + v$$

$$\Downarrow \quad v = \pm\pi$$

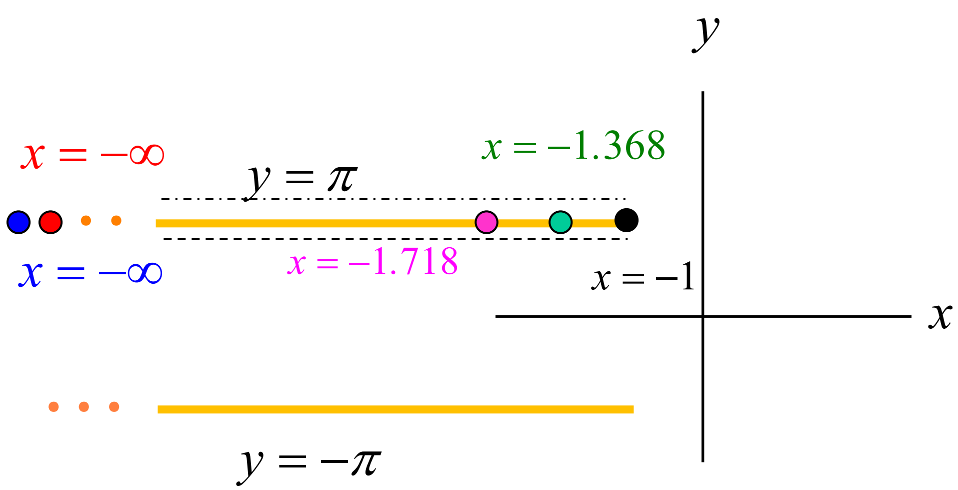
$$x = -e^{u} + u$$

$$y = v$$

$$\Downarrow$$

$$v = \pm\pi \quad \Rightarrow \quad y = \pm\pi$$

Example (cont.)

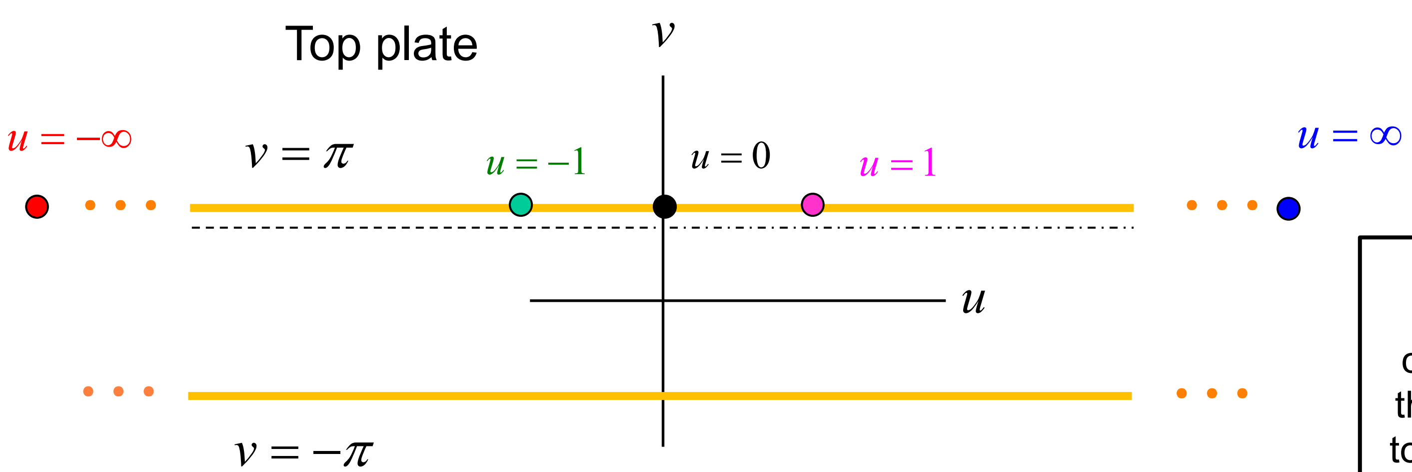


The corresponding colored dots show the mapping along the top plate.

$$x = -e^{u} + u$$

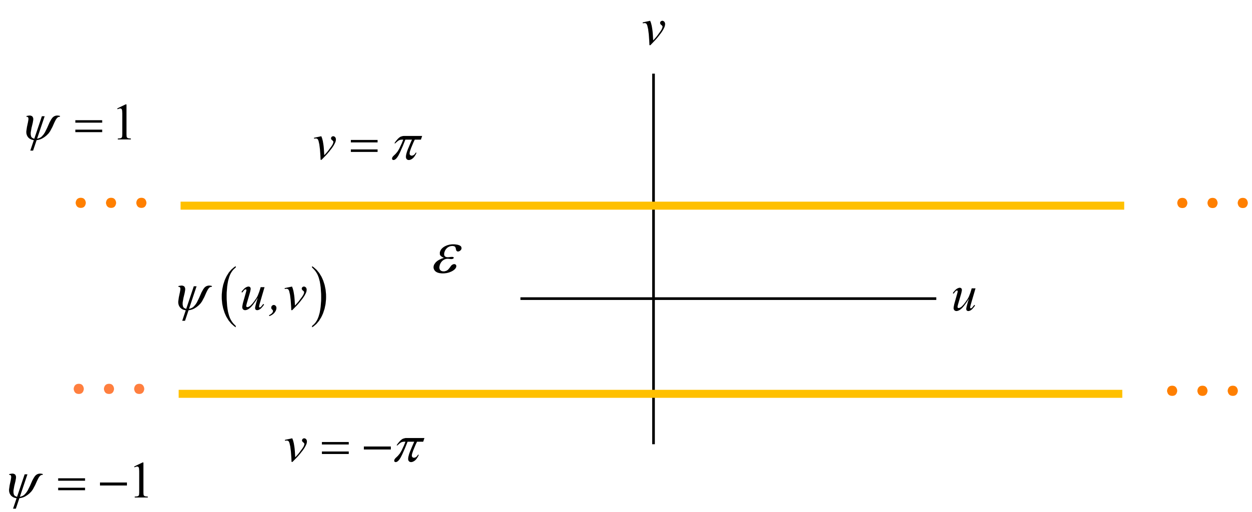
$$y = v$$

Note: x reaches a maximum (x = -1) at $u = 0$.



Note:
A negative value of u corresponds to being on the bottom surface of the top plate (see line styles).

Example (cont.)

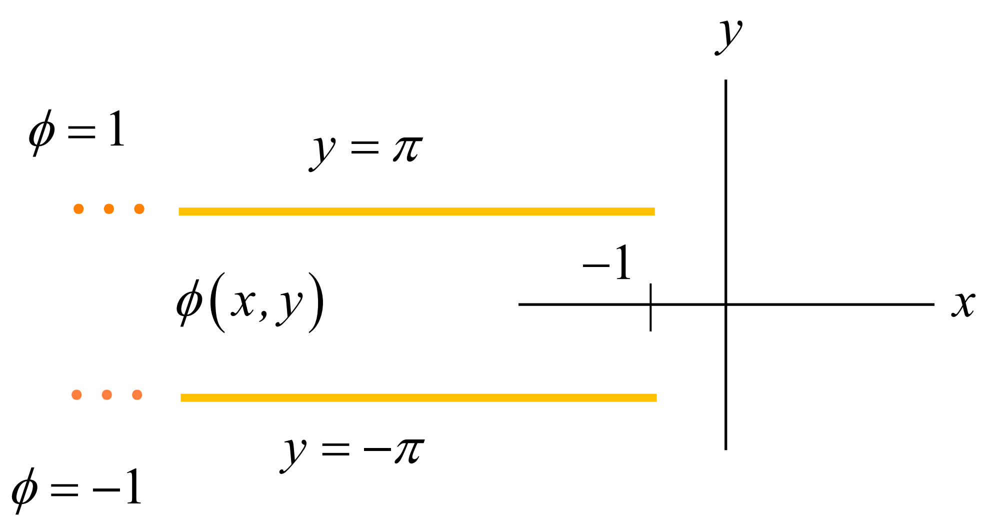


This is an ideal infinite parallel-plate capacitor, whose solution is simple:

$$\psi(u,v) = \frac{v}{\pi}, \quad -\pi < v < \pi$$

Note: The <u>inside</u> of the parallel-plate capacitor in the w plane gets mapped to the <u>entire</u> z plane.

Example (cont.)

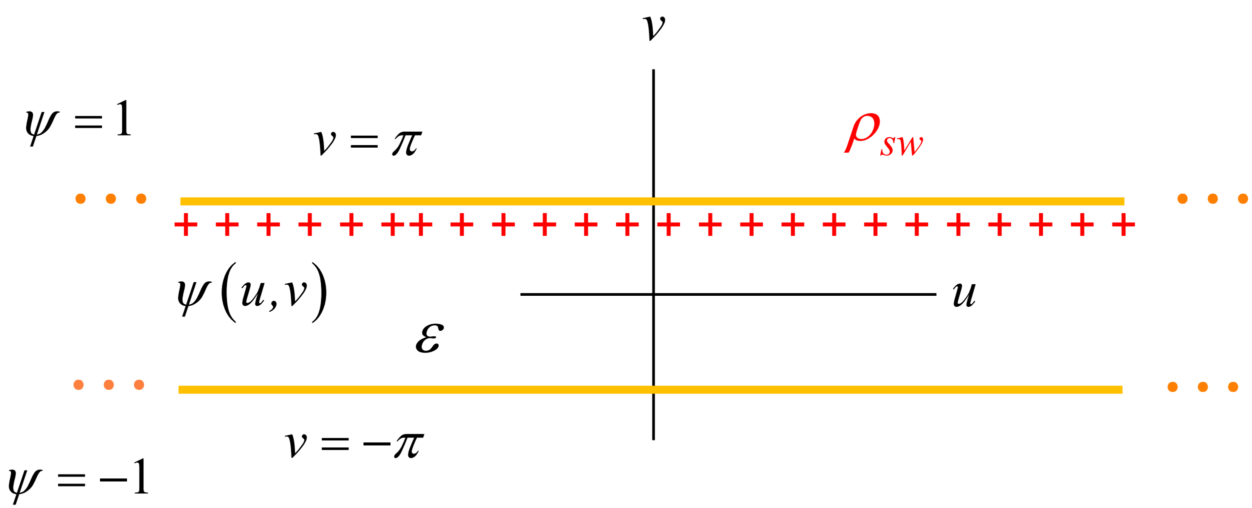


The solution is: $$\phi(x,y) = \frac{1}{\pi} v(x,y)$$

where
$$e^u \cos(v) + u = x$$
$$e^u \sin(v) + v = y$$

For any given (x,y), these two equations have to be solved numerically to find (u,v).

Example (cont.)



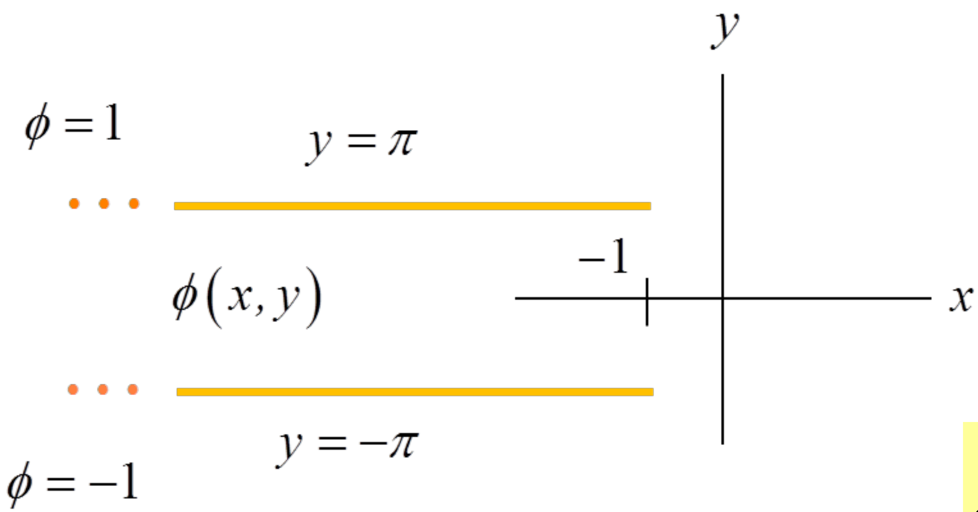
The charge density in the w plane on the lower surface of the top plate is:

$$\rho_{sw} = \underline{D} \cdot \underline{\hat{n}} = \varepsilon \underline{E} \cdot \underline{\hat{n}} = \varepsilon \underline{E} \cdot \left(-\underline{\hat{v}}\right) = \varepsilon\left(-E_v\right) = \varepsilon\left(\frac{V}{h}\right) = \varepsilon\left(\frac{2}{2\pi}\right) = \varepsilon\left(\frac{1}{\pi}\right)$$

The charge density in the z plane is:

$$\rho_{sz} = \rho_{sw}\left|f'(z)\right|$$

Example (cont.)



$$x = -e^{u} + u$$

$$z = e^{w} + w$$

$$\frac{dz}{dw} = e^{w} + 1$$

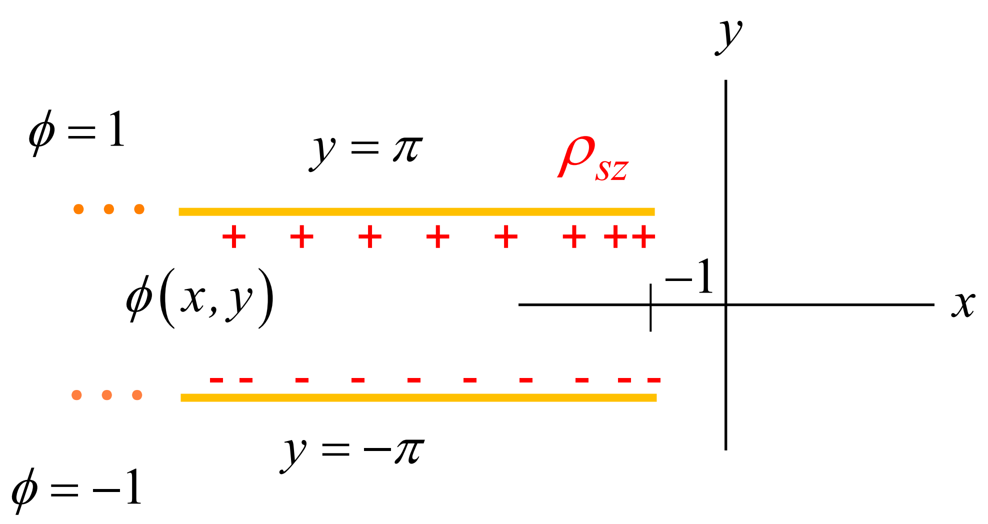
$$f'(z) = \frac{dw}{dz} = \frac{1}{e^{w} + 1}$$

Hence, we have:

$$\rho_{sz} = \varepsilon\left(\frac{1}{\pi}\right)\left|f'(z)\right|$$

$$\rho_{sz} = \varepsilon\left(\frac{1}{\pi}\right)\left|\frac{1}{e^{w}+1}\right| = \varepsilon\left(\frac{1}{\pi}\right)\left|\frac{1}{e^{(u+i\pi)}+1}\right| = \varepsilon\left(\frac{1}{\pi}\right)\left|\frac{1}{-e^{u}+1}\right|$$

Example (cont.)



Hence, we have on the bottom of the top plate:

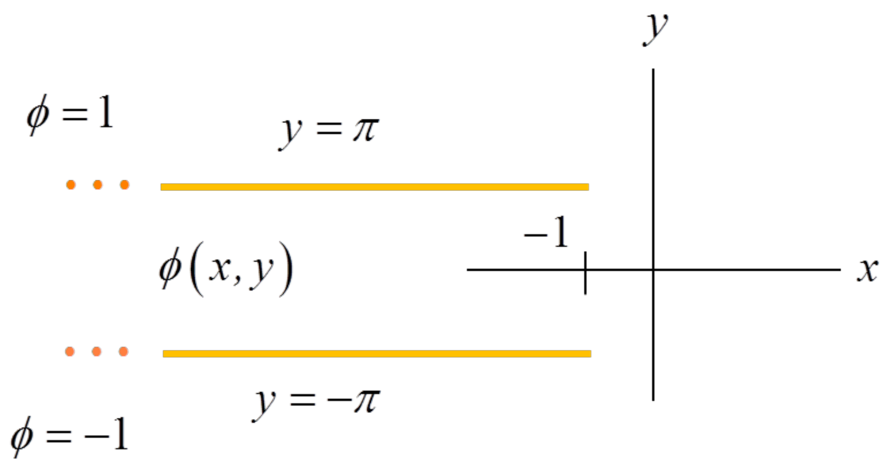
$$\rho_{sz} = \varepsilon \left(\frac{1}{\pi} \right) \left| \frac{1}{-e^u + 1} \right|$$

where

$$x = -e^u + u$$ (We must solve for u numerically, given a value of x.)

$$u < 0$$ (on bottom of top plate)

Example (cont.)



Rewriting in terms of x, we have:

$$\rho_{sz} = \varepsilon \left(\frac{1}{\pi} \right) \left| \frac{1}{1 - e^u} \right|$$

$$\Downarrow \quad x = -e^u + u$$

$$\rho_{sz} = \varepsilon \left(\frac{1}{\pi} \right) \left| \frac{1}{1 + x - u} \right|$$

$$\Downarrow \quad u \approx -\sqrt{-2(x+1)}$$

$$\rho_{sz} = \varepsilon \left(\frac{1}{\pi} \right) \left| \frac{1}{1 + x + \sqrt{-2(x+1)}} \right|$$

Choose negative sign
(a negative value of u
corresponds to the lower
surface of the top plate).
See slide 28.

→

Near Edge ($u \approx 0$)

$$x = -e^u + u$$

$$\Downarrow$$

$$x = -\left(1 + u + \frac{u^2}{2} + \ldots \right) + u$$

$$\approx -1 - \frac{u^2}{2} \quad (u \to 0)$$

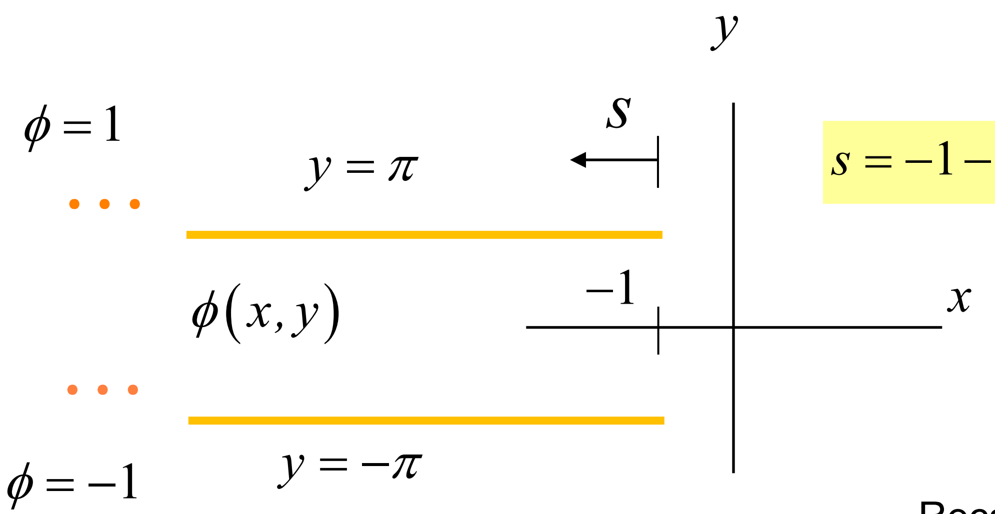
$$\Downarrow$$

$$u \approx \pm\sqrt{-2(x+1)}$$

$$\Downarrow$$

$$u \approx -\sqrt{-2(x+1)}$$

Example (cont.)



$$s = -1 - x = -(x+1)$$

Recall : $\rho_{sz} \approx \varepsilon \left(\frac{1}{\pi}\right) \left| \frac{1}{1 + x + \sqrt{-2(x+1)}} \right|$

Near Edge

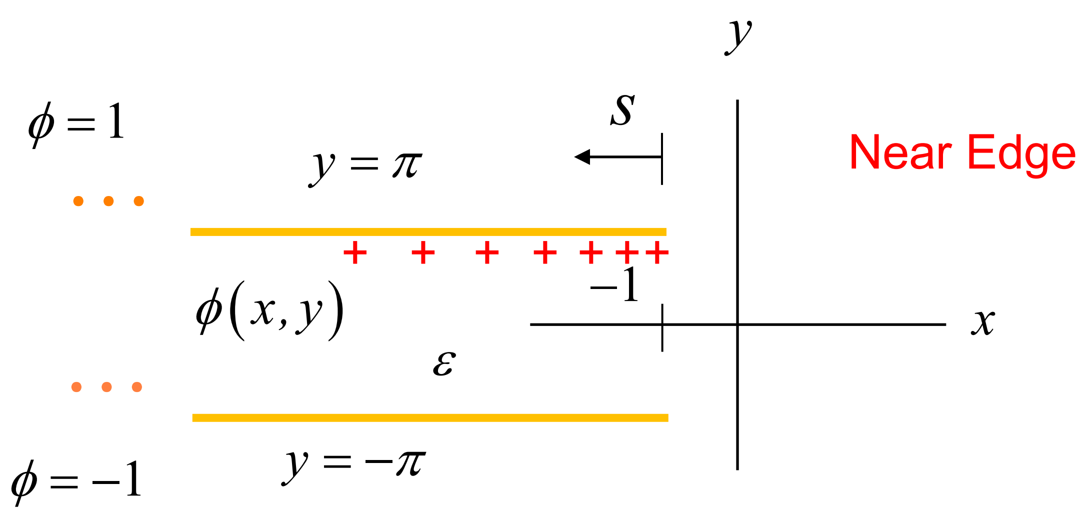
Hence, we have:

$$\rho_{sz} \approx \varepsilon \left(\frac{1}{\pi}\right) \left| \frac{1}{-s + \sqrt{2s}} \right|$$

or

$$\rho_{sz} \approx \varepsilon \left(\frac{1}{\pi}\right) \left| \frac{1}{\sqrt{2s} - s} \right|$$

Example (cont.)



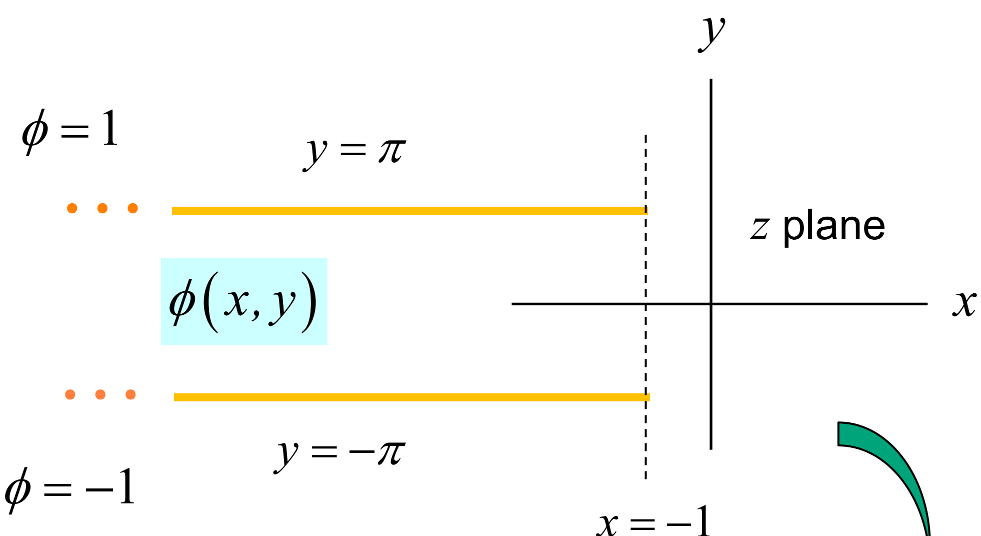
Near the upper edge (on the lower surface) we then have:

$$\rho_{sz} \approx \varepsilon \left(\frac{1}{\sqrt{2}\pi} \right) \frac{1}{\sqrt{s}} \left(\frac{1}{1 - \sqrt{s/2}} \right), \quad s \to 0$$

Note the square-root singularity at the edge!

Example (cont.)

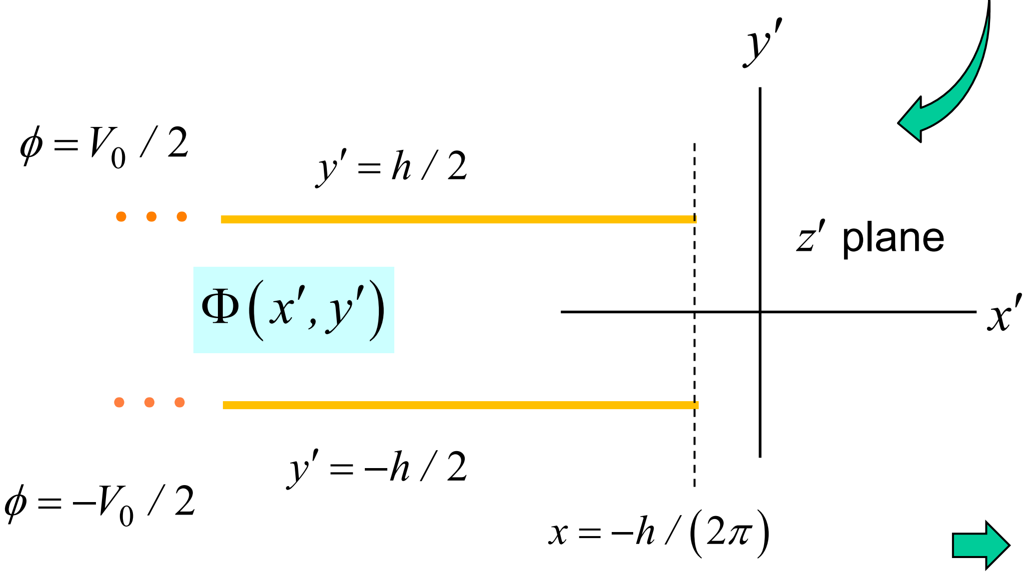
Scaling the Answer



$$\phi(x, y) = \text{previous solution}$$

$$z = z'\left(\frac{2\pi}{h}\right) \quad \begin{cases} x = x'\left(\dfrac{2\pi}{h}\right) \\ y = y'\left(\dfrac{2\pi}{h}\right) \end{cases}$$

(This does not affect the Laplace equation.)

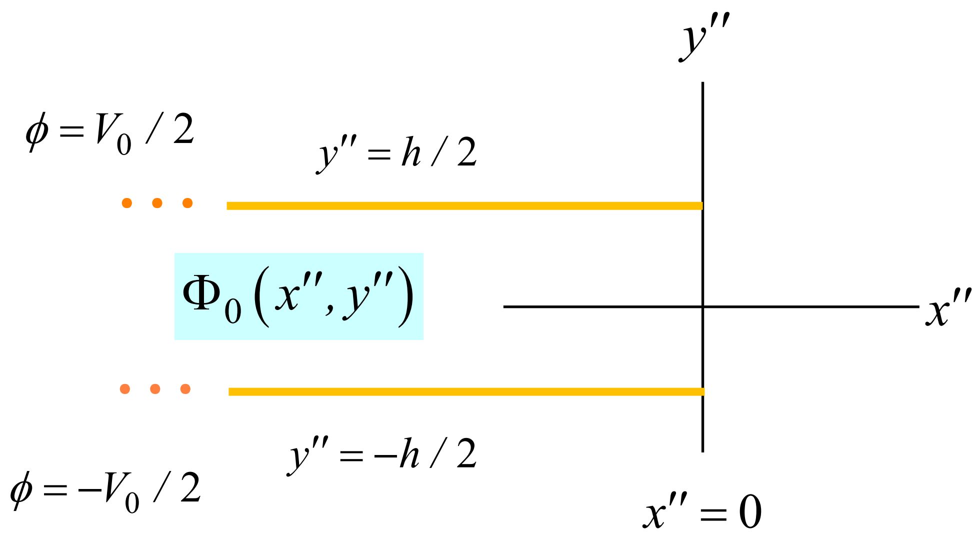


$$\Phi(x', y') = \left(\frac{V_0}{2}\right)\phi(x, y)$$

$$\Rightarrow \quad \Phi(x', y') = \left(\frac{V_0}{2}\right)\phi\left(x'\left(\frac{2\pi}{h}\right), y'\left(\frac{2\pi}{h}\right)\right)$$

Example (cont.)

Shifting the Answer



$$z' = z'' - \frac{2\pi}{h} \qquad \begin{cases} x' = x'' - \dfrac{2\pi}{h} \\ y' = y'' \end{cases}$$

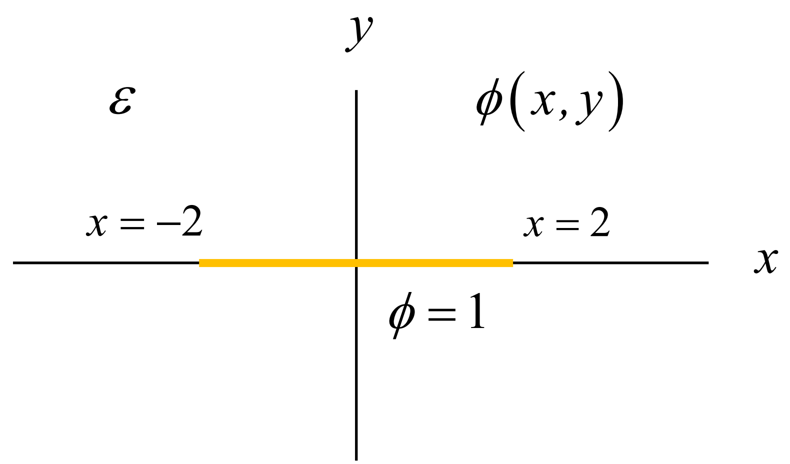
$$\Phi_0\left(x'', y''\right) = \Phi\left(x', y'\right) = \Phi\left(x'' - \frac{2\pi}{h}, y''\right)$$

where

$$\Phi\left(x', y'\right) = \left(\frac{V_0}{2}\right)\phi\left(x'\left(\frac{2\pi}{h}\right), y'\left(\frac{2\pi}{h}\right)\right)$$

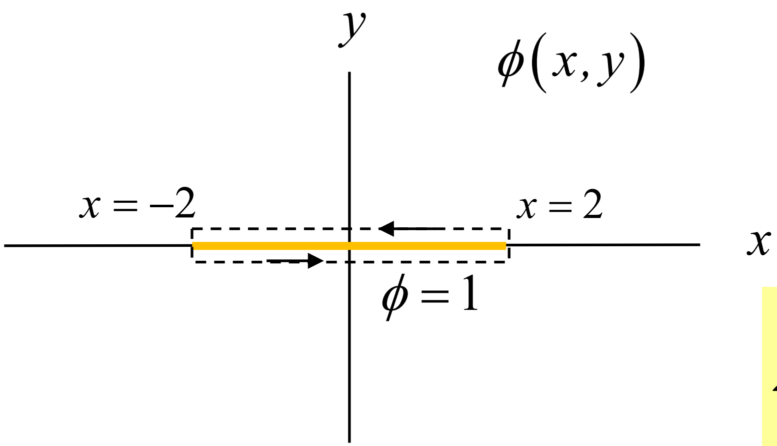
Example

Solve for the potential surrounding a metal strip, and the surface charge density on the strip.



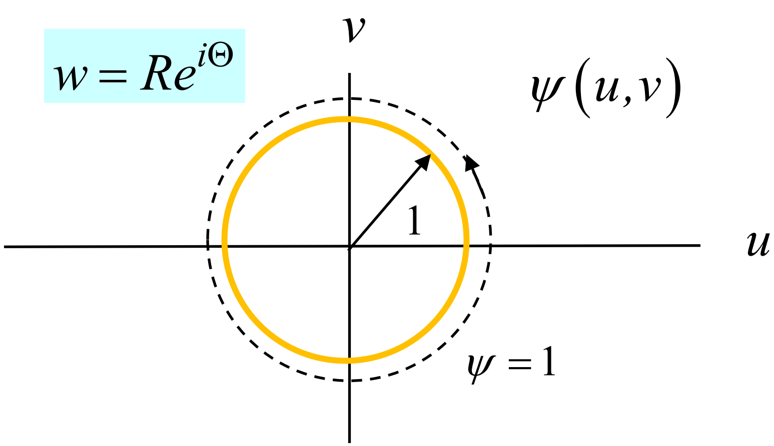
Note: The potential goes to -∞ as $\rho \to \infty$.

Example (cont.)



$$z = w + \frac{1}{w}$$

$$w = Re^{i\Theta}$$



$$z = w + \frac{1}{w}$$

$R=1$

$$z = e^{i\Theta} + e^{-i\Theta}$$

$$z = 2\cos\Theta$$

$$x = 2\cos\Theta$$

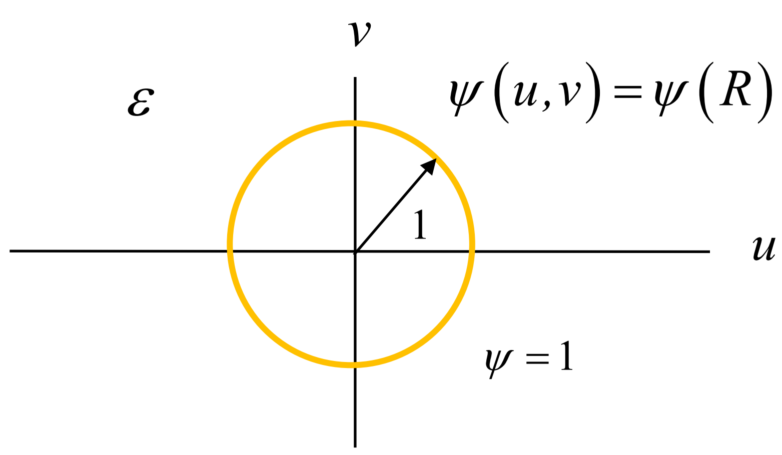
$$y = 0$$

The <u>outside</u> of the circle gets mapped into the <u>entire</u> z plane.

The top of the strip corresponds to $0 < \Theta < \pi$.

Note: The inside of the cylinder also gets mapped to the entire z plane. However, inside the cylinder we have $\psi \equiv 1$, so this gives us a trivial solution with no electric field and no charge on the strip.

Example (cont.)



ρ_l = equivalent line charge density on cylinder

ρ_s = surface charge density on outside of cylinder

$$\rho_l = (2\pi R)_{R=1}\, \rho_s$$

Outside the circle, we have (from electrostatic theory):

$$\psi(u,v) = -\ln(R) + 1$$

$$R^2 = u^2 + v^2$$

Note : $A_1 = 1 \;\Rightarrow\; \rho_l = 2\pi\varepsilon$ (see below)

From electrostatics (with no ϕ variation):

$$\underbrace{\underline{E} = \underline{\hat{R}}\left(\frac{\rho_l}{2\pi\varepsilon R}\right)}_{\text{Gauss's law}} = \underbrace{-\nabla\psi = \underline{\hat{R}}\left(\frac{A_1}{R}\right)}_{\text{Gradient calculation}} \;\Rightarrow\; A_1 \equiv \frac{\rho_l}{2\pi\varepsilon}$$

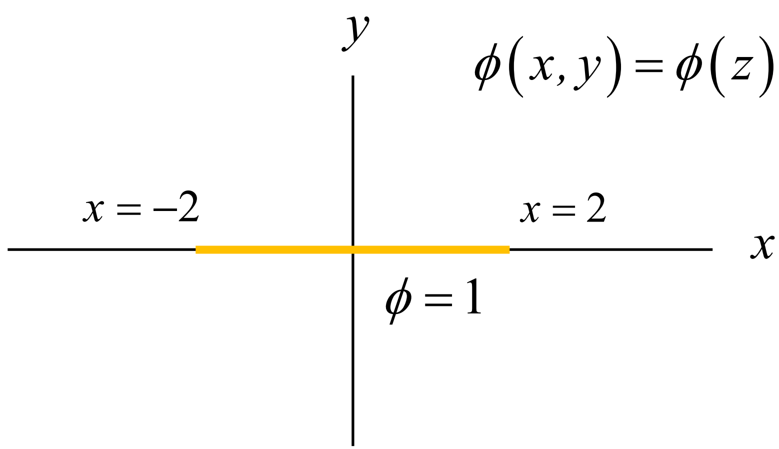
Note:
To be more general, we could use

$$\psi(u,v) = -A_1 \ln(R) + A_2$$

Changing the constant A_1 changes the total charge on the strip.

Changing the constant A_2 changes the voltage on the strip.

Example (cont.)



Hence, we have:

$$\phi(x,y)=1-\ln\big(R(x,y)\big)$$

$$z=w+\frac{1}{w}$$

$$x+iy=Re^{i\Theta}+\frac{1}{R}e^{-i\Theta}$$

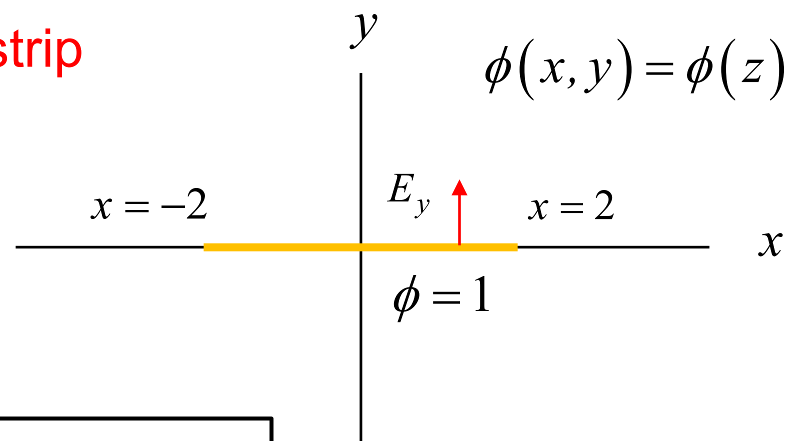
For any given (x,y), these two equations have to be solved numerically to find (R,Θ).

$$R\cos\Theta+\frac{1}{R}\cos\Theta=x$$

$$R\sin\Theta-\frac{1}{R}\sin\Theta=y$$

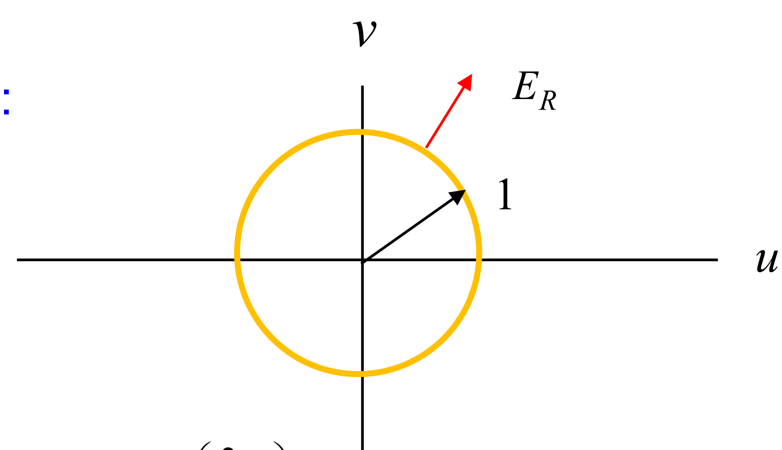
Example (cont.)

Charge density on strip



Note:
Going normal to the strip in the z plane means going normal to the circle in the w plane.

On the upper surface of the strip, we have:



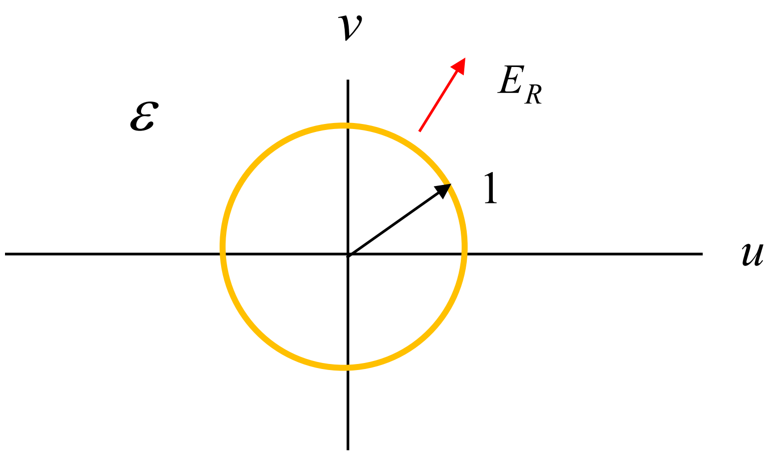
$$\rho_{sz} = \rho_{sw} \left| f'(z) \right|$$

$$\rho_{sw} = \varepsilon E_R = -\varepsilon \frac{\partial \psi}{\partial R}$$

Note : $\underline{E} = -\nabla \psi = -\underline{\hat{R}} \left(\frac{\partial \psi}{\partial R} \right)$

Example (cont.)

$$\rho_{sw} = \varepsilon E_R = -\varepsilon \frac{\partial \psi}{\partial R}$$



$$\psi(R,\Theta) = 1 - \ln(R)$$

Hence, on the circle : $\dfrac{\partial \psi}{\partial R} = -\dfrac{1}{R}\Big|_{R=1} = -1$

$\Rightarrow$ $\rho_{sw} = \varepsilon$ $\Rightarrow$

Total charge on cylinder (per meter):

$$\rho_l = \rho_{sw}\left(2\pi R\right)_{R=1} = 2\pi\varepsilon$$

We then have

$$\rho_{sz} = \rho_{sw}\left|f'(z)\right| = \varepsilon\left|f'(z)\right|$$

Example (cont.)

$$\rho_s = \varepsilon \left| f'(z) \right|$$

Next, use

$$z = w + \frac{1}{w}$$

so

$$\frac{dz}{dw} = 1 - \frac{1}{w^2}$$

so

$$f'(z) = \left(1 - \frac{1}{w^2} \right)^{-1}$$

or

$$f'(z) = \left(1 - \frac{1}{R^2 e^{i2\Theta}} \right)^{-1} = \left(1 - e^{-i2\Theta} \right)^{-1} \qquad (R = 1)$$

Example (cont.)

On the circle:

$$f'(z) = \left(1 - e^{-i2\Theta}\right)^{-1}$$
$$= \left(1 - \cos(2\Theta) + i\sin(2\Theta)\right)^{-1}$$
$$= \left(\left(1 - \cos^2\Theta + \sin^2\Theta\right) + i\left(2\sin\Theta\cos\Theta\right)\right)^{-1}$$
$$= \left(2\sin^2\Theta + i\left(2\sin\Theta\cos\Theta\right)\right)^{-1}$$

so

$$\left|f'(z)\right| = \left|\left(2\sin^2\Theta + i\left(2\sin\Theta\cos\Theta\right)\right)^{-1}\right|$$
$$= \left|2\sin^2\Theta + i\left(2\sin\Theta\cos\Theta\right)\right|^{-1}$$
$$= \frac{1}{2}\left(\sqrt{\sin^4\Theta + \sin^2\Theta\cos^2\Theta}\right)^{-1}$$
$$= \frac{1}{2}\left(\sqrt{\sin^2\Theta\left(\sin^2\Theta + \cos^2\Theta\right)}\right)^{-1}$$
$$= \frac{1}{2}\left|\frac{1}{\sin\Theta}\right|$$

Example (cont.)

$$\left|f'(z)\right| = \frac{1}{2}\left|\frac{1}{\sin\Theta}\right|$$

On the strip:

$$x = 2\cos\Theta$$

$$\Rightarrow \ \sin\Theta = \pm\sqrt{1-\left(\frac{x}{2}\right)^2}$$

so

$$\left|f'(z)\right| = \frac{1/2}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

Hence, we have

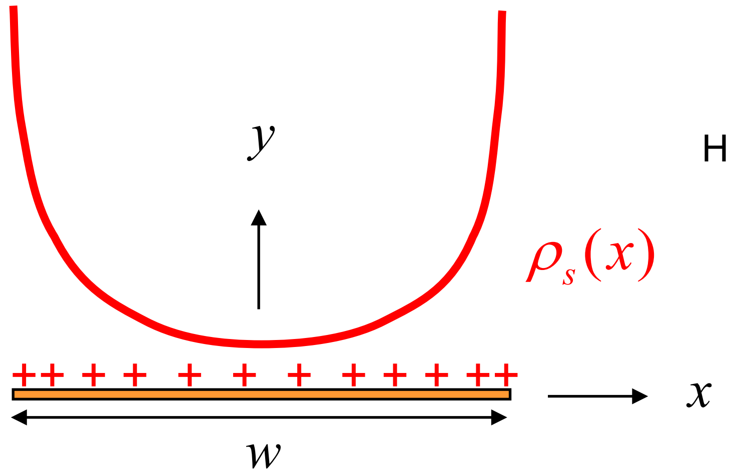
$$\rho_s = \varepsilon\frac{1/2}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

Note:
The surface charge density goes to infinity as we approach the edges.

This result was first derived by Maxwell!

Example (cont.)

Strip of width w



Knife-edge singularity

$$\rho_s \propto \frac{1}{\sqrt{s}}$$

Here s is the distance from the edge.

The strip now has a width of w.

Previous solution (relabeled with x')

$$\rho_s = \varepsilon \frac{1/2}{\sqrt{1-\left(\frac{x'}{2}\right)^2}}$$

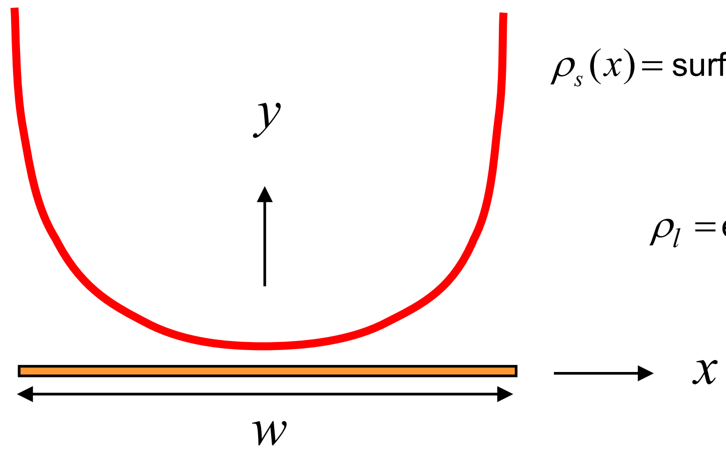
$$x' = x\left(\frac{4}{w}\right)$$

Note: The total charge gets changed by a factor of $w/4$.

New solution

$$\rho_s = \varepsilon \frac{1/2}{\sqrt{1-\left(\frac{2x}{w}\right)^2}}$$

Example (cont.)



$\rho_s(x) =$ surface charge density on top of strip

$\rho_l =$ equivalent line charge density on strip

The total line charge density is now assumed to be ρ_l [C/m].

$$\rho_s^{\text{tot}}(x) = \rho_l \left(\frac{2}{w}\right) \frac{1/\pi}{\sqrt{1-\left(\dfrac{2x}{w}\right)^2}}$$

Note:
The normalization of $1/\pi$ corresponds to a unity total line charge density:

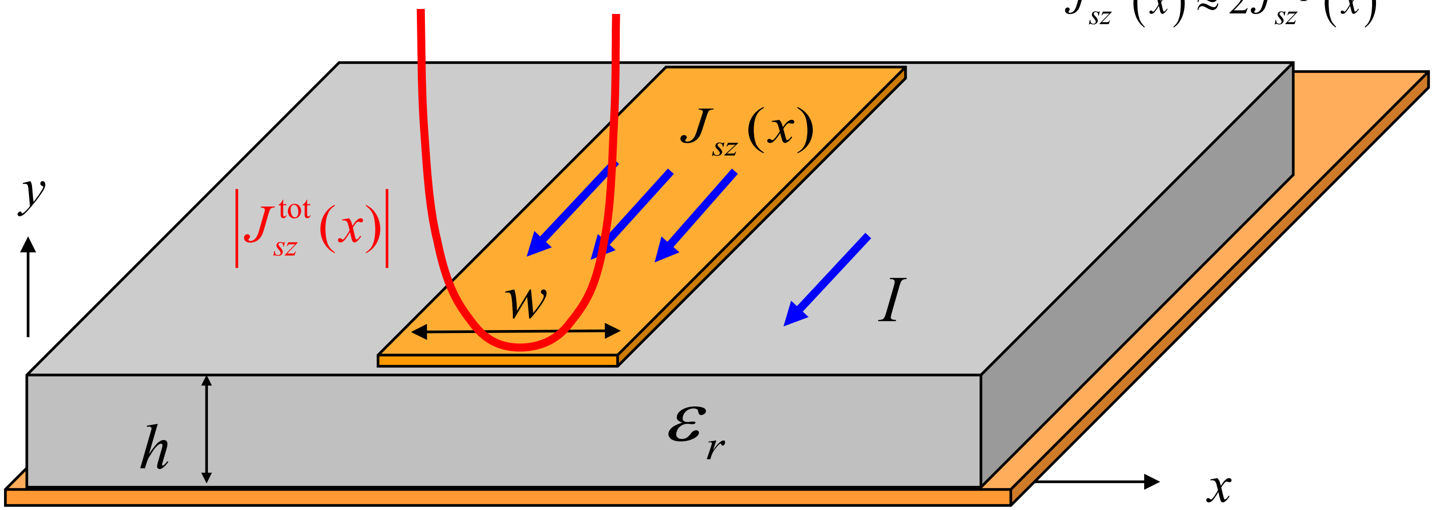
$$\int_{-w/2}^{w/2} \frac{1/\pi}{\sqrt{1-\left(\dfrac{2x}{w}\right)^2}}\, dx = w/2$$

$$\rho_l = \int_{-w/2}^{w/2} \rho_s^{\text{tot}}(x)\, dx$$

$$\left(\rho_s^{\text{tot}}(x) = \rho_s^{\text{top}}(x) + \rho_s^{\text{bot}}(x) = 2\rho_s^{\text{top}}(x) = 2\rho_s(x)\right)$$

Example (cont.)

Microstrip line



Narrow strip:

$$J_{sz}^{\text{tot}}(x) \approx 2 J_{sz}^{\text{top}}(x)$$

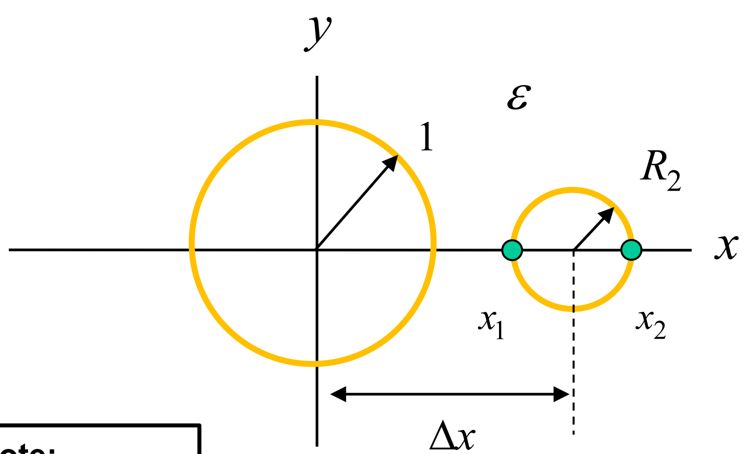
$$J_{sz}^{\text{tot}} \approx I\left(\frac{2}{w}\right)\frac{1/\pi}{\sqrt{1-\left(\frac{2x}{w}\right)^2}}$$

Note:
The increased current density near the edges causes increased conductor loss and susceptibility to dielectric breakdown.

(This ignores the effects of the ground plane and the substrate – accurate for narrow strips.)

Example

Find the capacitance between two wires (tubes).



$$w = \frac{z-a}{az-1}$$

$$a = \frac{1 + x_1 x_2 + \sqrt{\left(x_1^2 - 1\right)\left(x_2^2 - 1\right)}}{x_1 + x_2}$$

$$R_0 = \frac{x_1 x_2 - 1 - \sqrt{\left(x_1^2 - 1\right)\left(x_2^2 - 1\right)}}{x_2 - x_1}$$

(from Churchill book)

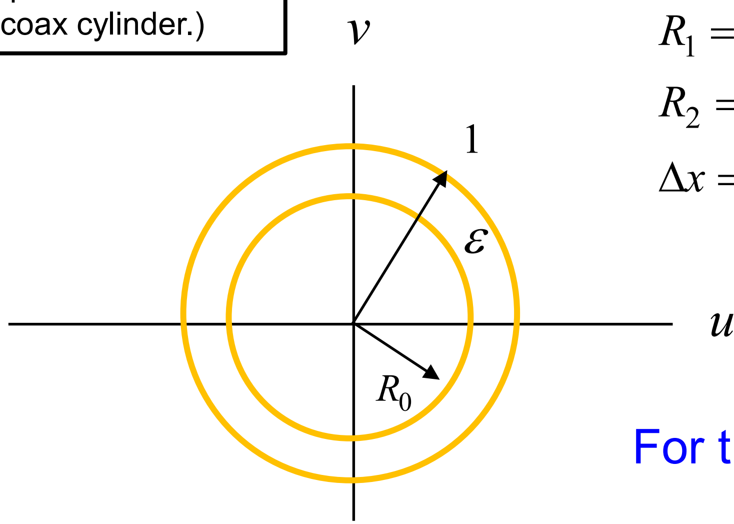
Note:
The left cylinder gets mapped into the outer coax cylinder.)

Radii and offset:

$$R_1 = 1$$

$$R_2 = \left(x_2 - x_1\right)/2$$

$$\Delta x = \left(x_1 + x_2\right)/2$$

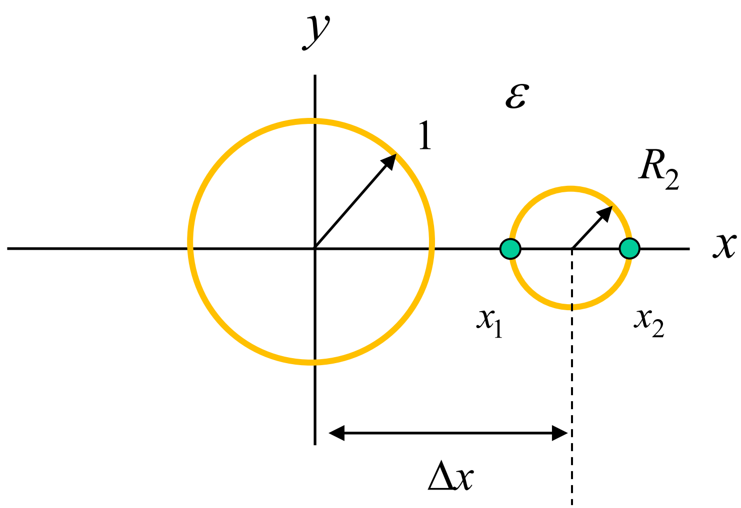


For the coax we have:

$$C_w = \frac{2\pi\varepsilon}{\ln\left(\dfrac{1}{R_0}\right)} \quad [\mathrm{F/m}]$$

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 9th Ed., McGraw-Hill, 2013.

Example (cont.)



We therefore have ($C_z = C_w$):

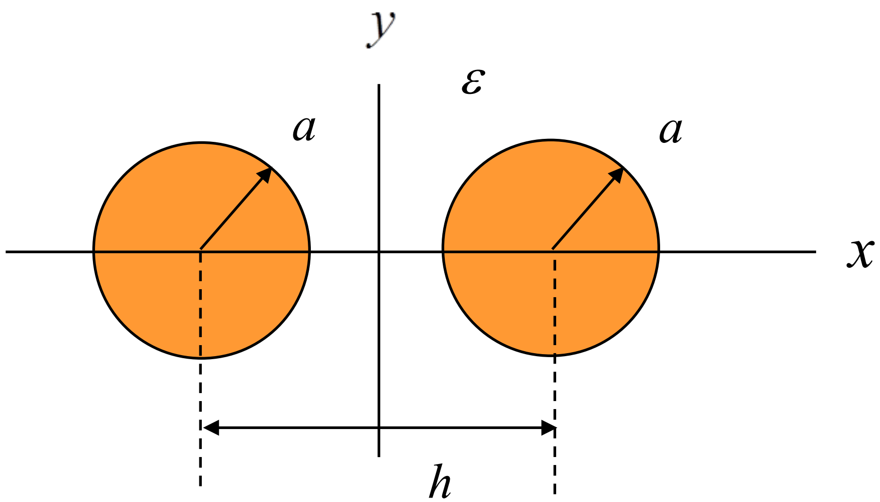
$$C_z = \frac{2\pi\varepsilon}{\ln\left(\dfrac{x_2 - x_1}{x_1 x_2 - 1 - \sqrt{\left(x_1^2 - 1\right)\left(x_2^2 - 1\right)}}\right)} \quad \text{[F/m]}$$

where

$$x_1 = \Delta x - R_2$$
$$x_2 = \Delta x + R_2$$

Example (cont.)

Symmetrical "twin lead" Transmission Line

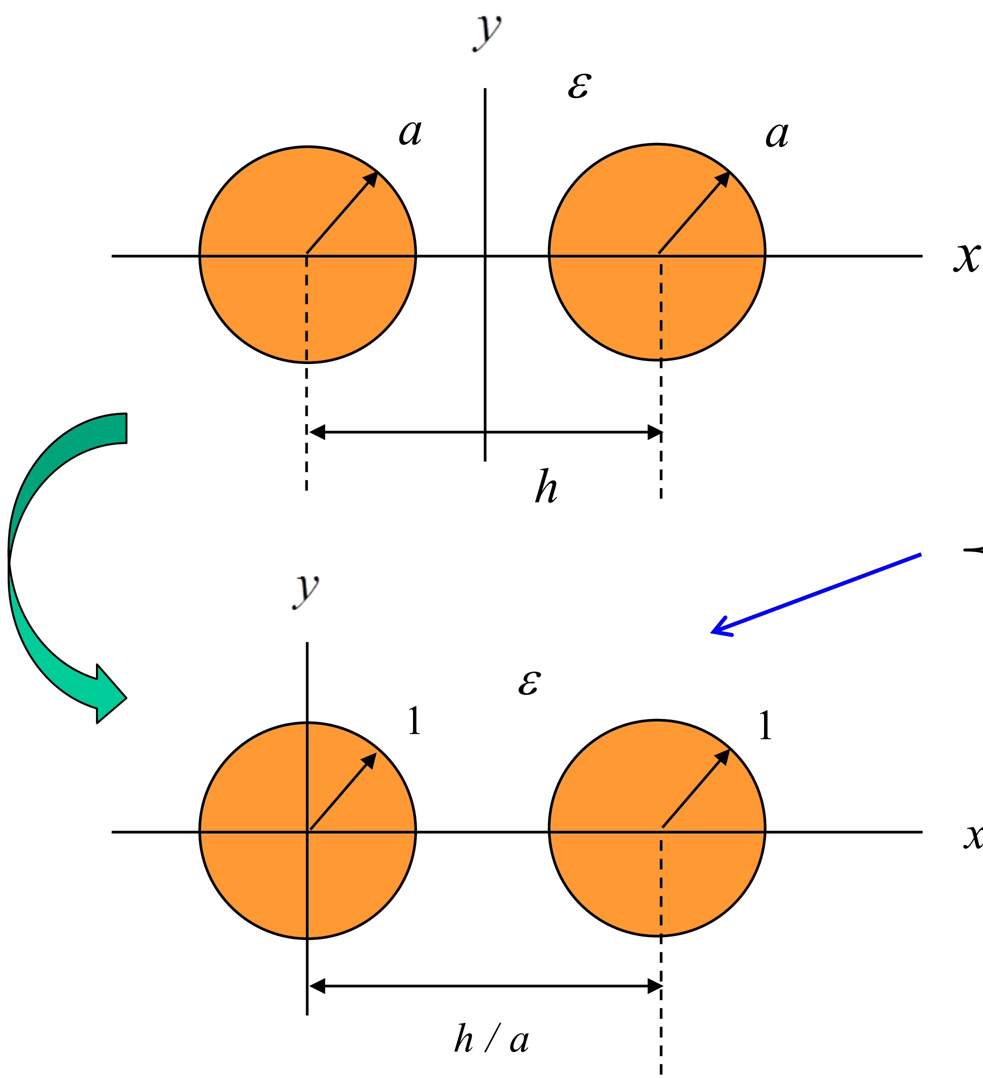


Scale this geometry by $1/a$.

(This does not change the capacitance.)

Example (cont.)

Symmetrical "twin lead" Transmission Line

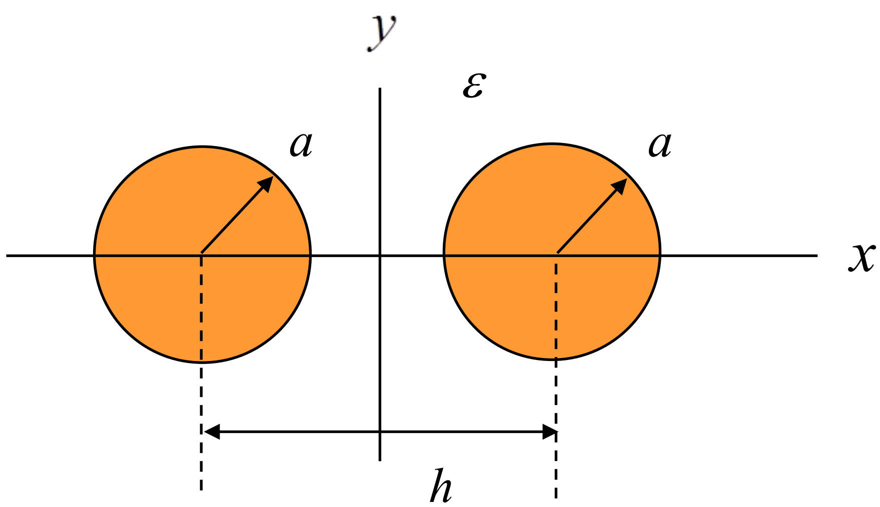


Scale by $1/a$
and shift

$$
\begin{cases}
R_1 = R_2 = 1 \\
\Delta x = h / a \\
x_1 = h / a - 1 \\
x_2 = h / a + 1 \\
x_2 - x_1 = 2R_2 = 2
\end{cases}
$$

Example (cont.)

Symmetrical “twin lead” transmission line



$$C = \frac{2\pi\varepsilon}{\ln\left(\dfrac{2}{\left(\left(\dfrac{h}{a}\right)^2 - 1\right) - 1 - \sqrt{\left(\left(\dfrac{h}{a} - 1\right)^2 - 1\right)\left(\left(\dfrac{h}{a} + 1\right)^2 - 1\right)}}\right)} \quad \text{[F/m]}$$

Example (cont.)

Define: $x \equiv \dfrac{h}{2a} \quad \left(\dfrac{h}{a} = 2x\right)$

$$C = \frac{2\pi\varepsilon}{\ln\left(\dfrac{2}{\left(4x^2-1\right)-1-\sqrt{\left((2x-1)^2-1\right)\left((2x+1)^2-1\right)}}\right)} \quad \text{[F/m]}$$

$$C = \frac{2\pi\varepsilon}{\ln\left(\dfrac{2}{4x^2-2-\sqrt{\left(4x^2-4x\right)\left(4x^2+4x\right)}}\right)} \quad \text{[F/m]}$$

Example (cont.)

$$C = \frac{2\pi\varepsilon}{\ln\left(\dfrac{2}{4x^2 - 2 - \sqrt{\left(4x^2 - 4x\right)\left(4x^2 + 4x\right)}}\right)} \quad [\text{F/m}]$$

$$C = \frac{2\pi\varepsilon}{\ln\left(\dfrac{2}{4x^2 - 2 - \sqrt{16x^4 - 16x^2}}\right)} \quad [\text{F/m}]$$

$$C = \frac{2\pi\varepsilon}{\ln\left(\dfrac{1}{2x^2 - 1 - 2\sqrt{x^4 - x^2}}\right)} \quad [\text{F/m}]$$

Example (cont.)

$$C = \frac{2\pi\varepsilon}{\ln\left(\dfrac{1}{2x^2 - 1 - 2\sqrt{x^4 - x^2}}\right)} \quad \text{[F/m]}$$

$$C = \frac{2\pi\varepsilon}{\ln\left(\dfrac{1}{2x^2 - 1 - 2x\sqrt{x^2 - 1}}\right)} \quad \text{[F/m]}$$

Note: $\dfrac{1}{2x^2 - 1 - 2x\sqrt{x^2 - 1}} = 2x^2 - 1 + 2x\sqrt{x^2 - 1}$

$$C = \frac{2\pi\varepsilon}{\ln\left(2x^2 - 1 + 2x\sqrt{x^2 - 1}\right)} \quad \text{[F/m]}$$

$$C = \frac{\pi\varepsilon}{\ln\left(\sqrt{2x^2 - 1 + 2x\sqrt{x^2 - 1}}\right)} \quad \text{[F/m]}$$

Example (cont.)

$$C = \frac{\pi\varepsilon}{\ln\left(\sqrt{2x^2 - 1 + 2x\sqrt{x^2 - 1}}\right)} \quad \text{[F/m]}$$

Note : $2x^2 - 1 + 2x\sqrt{x^2 - 1} = \left(x + \sqrt{x^2 - 1}\right)^2$

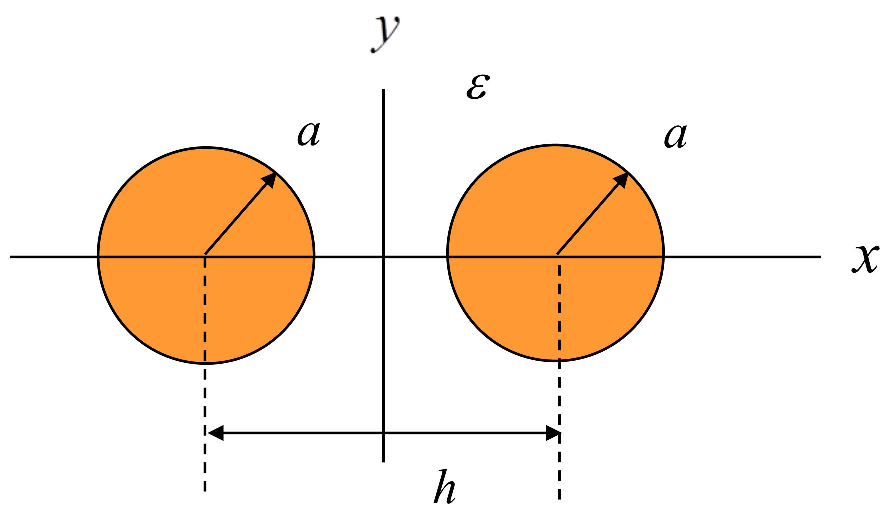
$$C = \frac{\pi\varepsilon}{\ln\left(x + \sqrt{x^2 - 1}\right)} \quad \text{[F/m]}$$

Note : $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

$$C = \frac{\pi\varepsilon}{\cosh^{-1}(x)} \quad \text{[F/m]}$$

Example (cont.)

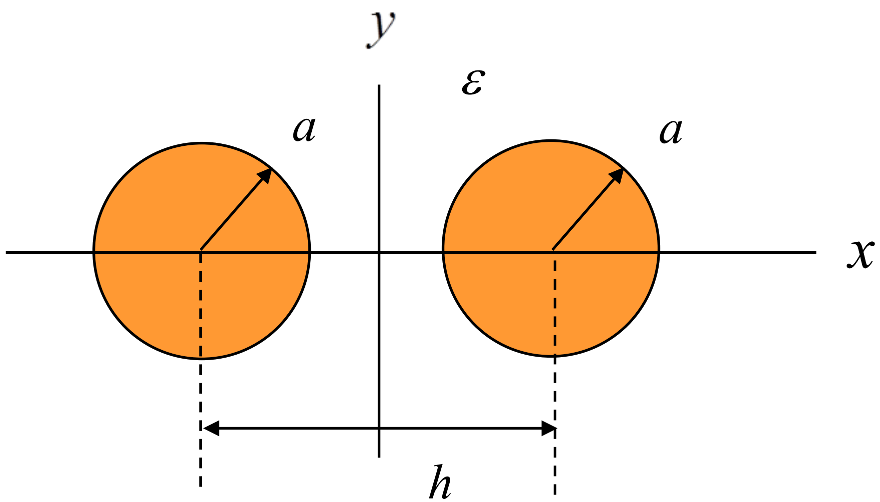
Final Result



$$C = \frac{\pi \varepsilon}{\cosh^{-1}\left(\frac{h}{2a}\right)} \; [\mathrm{F/m}]$$

Example (cont.)

Final Result



Note: $Z_0 = \sqrt{\dfrac{L}{C}} = \dfrac{\sqrt{LC}}{C} = \dfrac{\sqrt{\mu\varepsilon}}{C} = \dfrac{\sqrt{\mu_0\varepsilon_0}}{C}\sqrt{\varepsilon_r} \quad (\mu = \mu_0)$

$$Z_0 = \frac{\eta_0}{\pi}\sqrt{\frac{1}{\varepsilon_r}}\cosh^{-1}\left(\frac{h}{2a}\right) \; [\Omega]$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$\eta_0 \doteq 376.7303 \; [\Omega]$$

Example

Conductor attenuation on stripline

An Exact TEM Calculation of Loss in a Stripline of Arbitrary Dimensions

Sujata Rawal and David R. Jackson, *Member, IEEE*

Abstract **—An exact expression for the quasi-static conductive attenuation in a symmetrical stripline is derived. The formulation is based on a TEM assumption, which assumes that the skin depth is much smaller than the strip thickness. The conductive attenuation is related to the charge density on the conductive surfaces, which is determined by a conformal mapping originally proposed by Bates. An analytic extraction of a charge singularity term is used to obtain a numerically efficient calculation, in which no singular integrations appear.**

I. Introduction

SYMMETRICAL stripline, shown in Fig. 1, is one of the most common transmission lines in use at microwave frequencies. The dominant mode on this structure is approximately TEM, provided the conductor losses are small. Because of this, a quasi-static analysis may be used to determine both the characteristic impedance Z_0 and the attenuation constant α [1]. An exact calculation of Z_0 using a conformal mapping method was originally given by Bates [2]. The total attenuation constant α is in general the sum of a conductive attenuation constant α_c and a dielectric attenuation constant α_d, with α_d given (for low loss) by the simple expression [1]

$$\alpha_d = k_0 \frac{\sqrt{\epsilon_r}}{2} \tan \delta_d \tag{1}$$

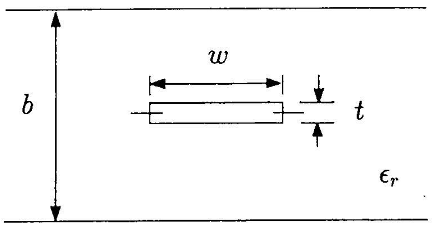


Fig. 1. Geometry of a symmetrical stripline.

the well-known quasi-static formula

$$\alpha_c = \frac{R_s}{2Z_0} \int_{\Gamma_T} \frac{\rho_s^2 \, d\ell}{Q^2} \tag{2}$$

where ρ_s is the surface charge density on all parts of the conductive system, Q is the total charge on the center conductor, and R_s is the surface resistance of the metal. The total contour Γ_T includes both the center conductor and the metal ground planes.

In order to determine the charge density ρ_s, the conformal mapping of Bates is used [2]. In this transformation, the

Example (cont.)

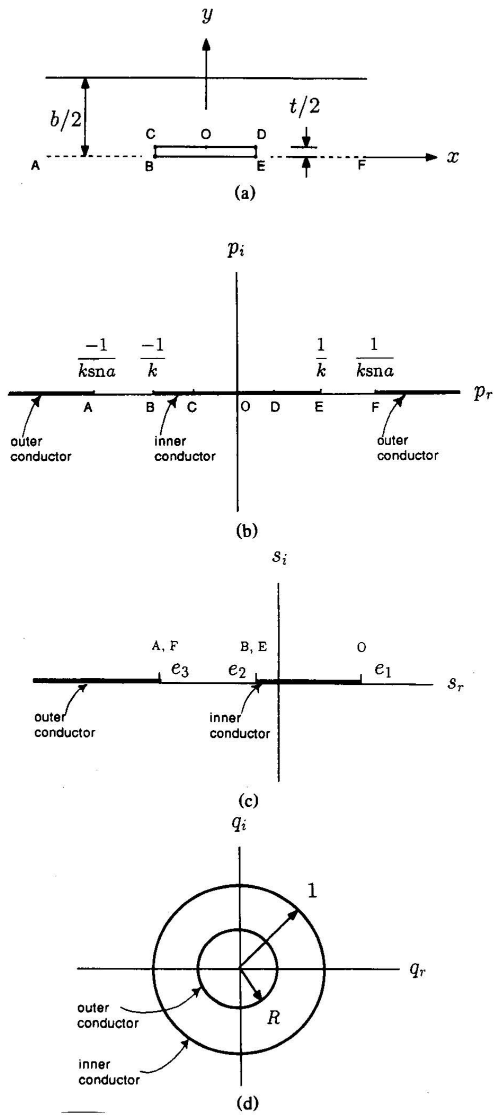


Fig. 2. The conformal mappings used to transform the upper half of the original stripline into a pair of coaxial cylinders. The mappings of the inner conductor (strip) and outer conductor (top ground plane) are indicated in each plane. (a) The z plane. (b) The p plane. (c) The s plane. (d) The q plane.

Conformal mapping of Bates:

$$\frac{dz}{dp} = \frac{\left(1-p^2\right)^{1/2}}{\left(1-k^2p^2\right)^{1/2}\left(1-k^2p^2\text{sn}^2 a\right)}$$

$$s = -p^2 C_1 + C_0$$

$$s = P\left(\ln\left(\frac{q}{R}\right)\right)$$

sn = Jacobi elliptic function

P = Weierstrass elliptic function

R. H. T. Bates, “The characteristic impedance of the shielded slab line,” *IRE Trans. Microwave Theory and Techniques*, vol. MTT-4, pp. 28-33, Jan. 1956.

Example (cont.)

TABLE I

COMPARISON OF NORMALIZED ATTENUATION CONSTANTS FOR THE CASE $t/b = 0.001$

w/b	Exact Method	Narrow Strip Cohn Formula	% Error	Wide Strip Cohn Formula	% Error
		$\frac{\alpha_c b}{R_s\sqrt{\epsilon_r}}$ $\left(\frac{t}{b} = 0.001\right)$			
0.124	2.4137 E-02	2.4215 E-02	0.32	1.7145 E-02	29.0
0.175	2.0178 E-02	2.0356 E-02	0.88	1.6174 E-02	19.8
0.247	1.7087 E-02	1.7432 E-02	2.02	1.5044 E-02	12.0
0.350	1.4658 E-02	1.5304 E-02	4.40	1.3781 E-02	5.98
0.501	1.2713 E-02	1.3927 E-02	9.55	1.2425 E-02	2.27
0.604	1.1868 E-02	1.3557 E-02	14.2	1.1725 E-02	1.20
0.733	1.1081 E-02	1.3468 E-02	21.5	1.1018 E-02	0.57
0.901	1.0332 E-02	1.3802 E-02	33.6	1.0307 E-02	0.24
1.125	9.6037 E-03	1.4918 E-02	55.3	9.5933 E-03	0.11
1.438	8.8855 E-03	1.7971 E-02	102.2	8.8791 E-03	0.079
1.909	8.1705 E-03	2.9618 E-02	262.5	8.1641 E-03	0.076