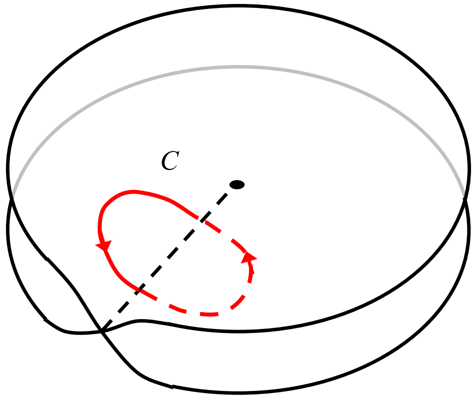


ECE 6382

Fall 2023

David R. Jackson

Notes 6

Branch Points and Branch Cuts

Notes are adapted from D. R. Wilton, Dept. of ECE

Preliminary

Consider: $f(z) = z^{1/2}$ $\qquad z = r\,e^{i\theta}$

$$z^{1/2} = \left(r\,e^{i\theta}\right)^{1/2} = \sqrt{r}\,e^{i\theta/2}$$

Choose: $|z| = r = 1$

$$\theta = 0: \qquad z^{1/2} = 1$$

$$\theta = 2\pi: \qquad z^{1/2} = -1$$

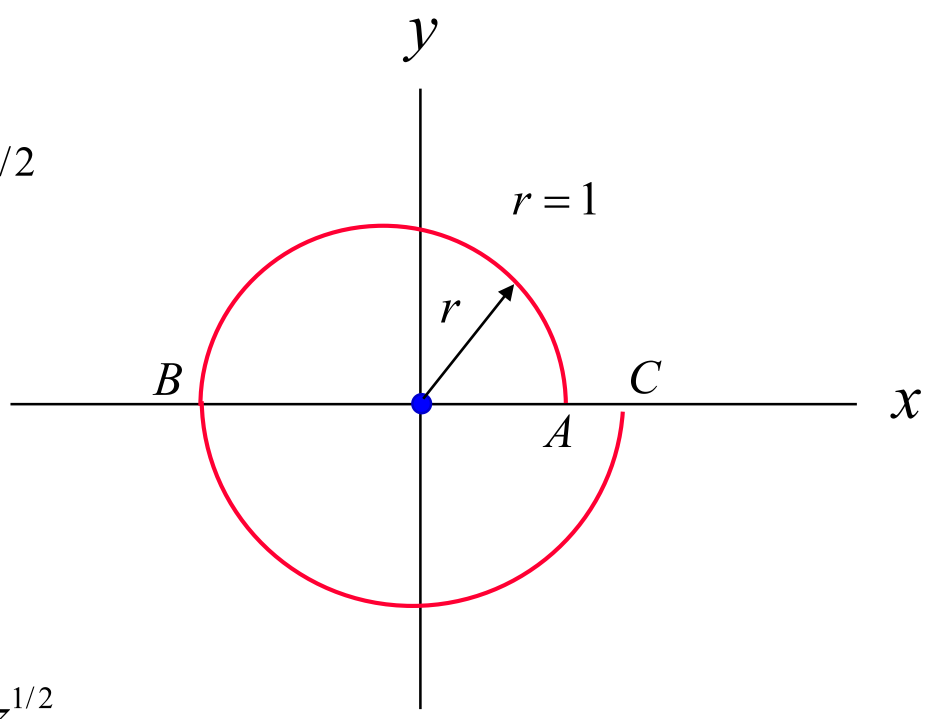
$$\theta = 4\pi: \qquad z^{1/2} = 1$$

There are two possible values.

Branch Cuts and Branch Points (cont.)

Consider what happens if we encircle the origin:

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2}$$



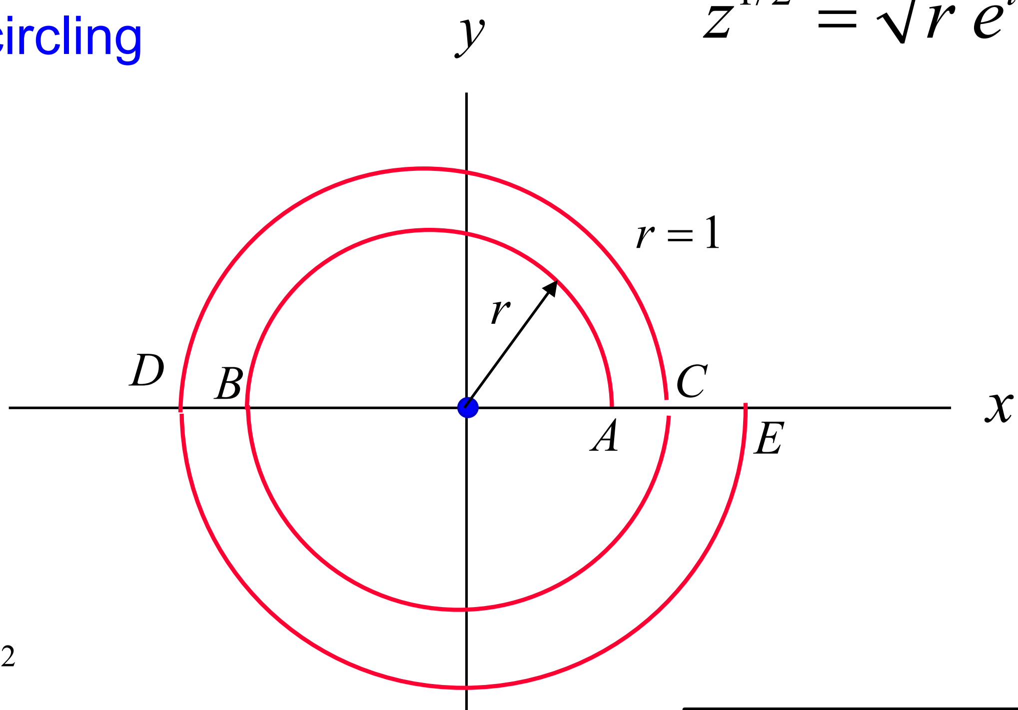
Point	θ	$z^{1/2}$
A	0	1
B	π	$+i$
C	2π	-1

We don't get back the same result!

Branch Cuts and Branch Points (cont.)

Now consider encircling the origin <u>twice</u>:

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2}$$



Point	θ	$z^{1/2}$
A	0	1
B	π	$+i$
C	2π	-1
D	3π	$-i$
E	4π	$+1$

Recall: $z^{p/q}$ has q distinct values (if p and q have no common factors).

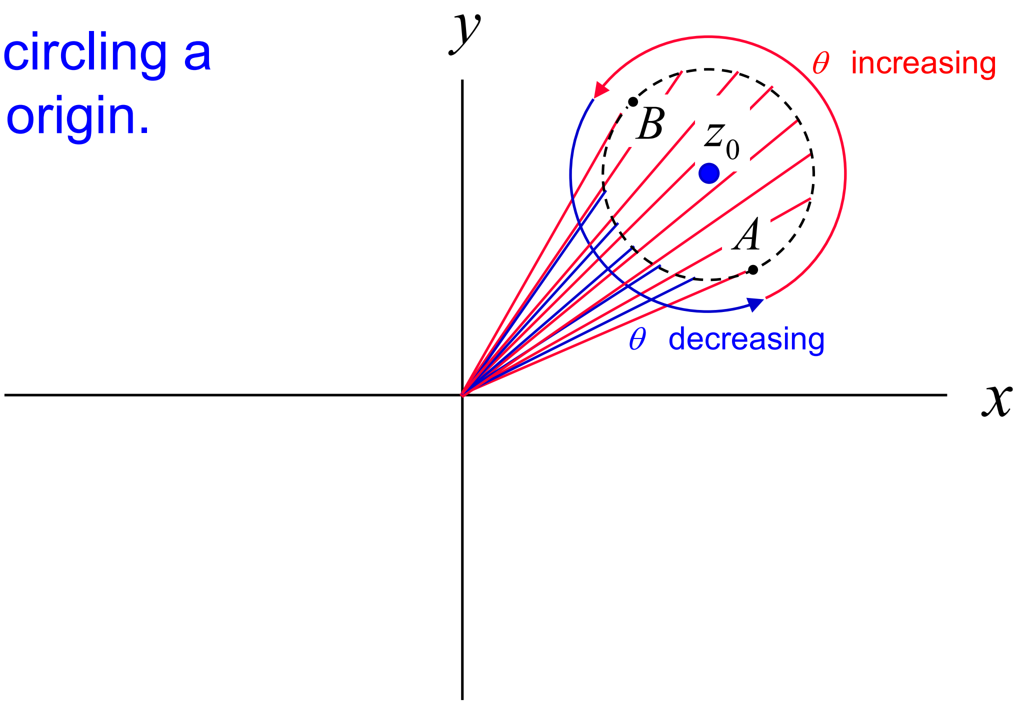
We now get back the same result!

Hence the square-root function is a double-valued function.

Branch Cuts and Branch Points (cont.)

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2}$$

Next, consider encircling a point z_0 not at the origin.

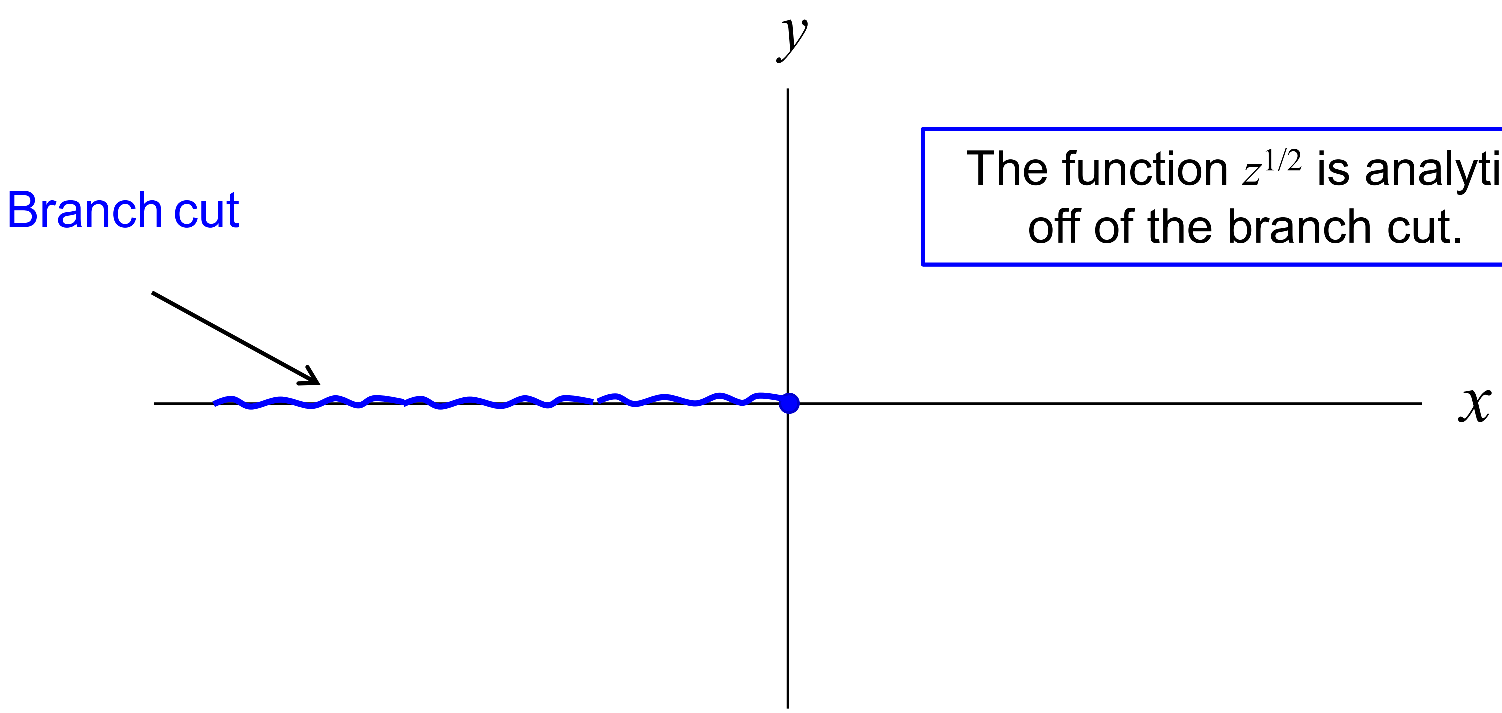


Unlike encircling the origin, now we return to the same result!

Branch Cuts and Branch Points (cont.)

The origin is called a branch point: we are not allowed to encircle it if we wish to make the square-root function single-valued.

In order to make the square-root function single-valued and analytic in the domain, we insert a “barrier” or “branch cut”.

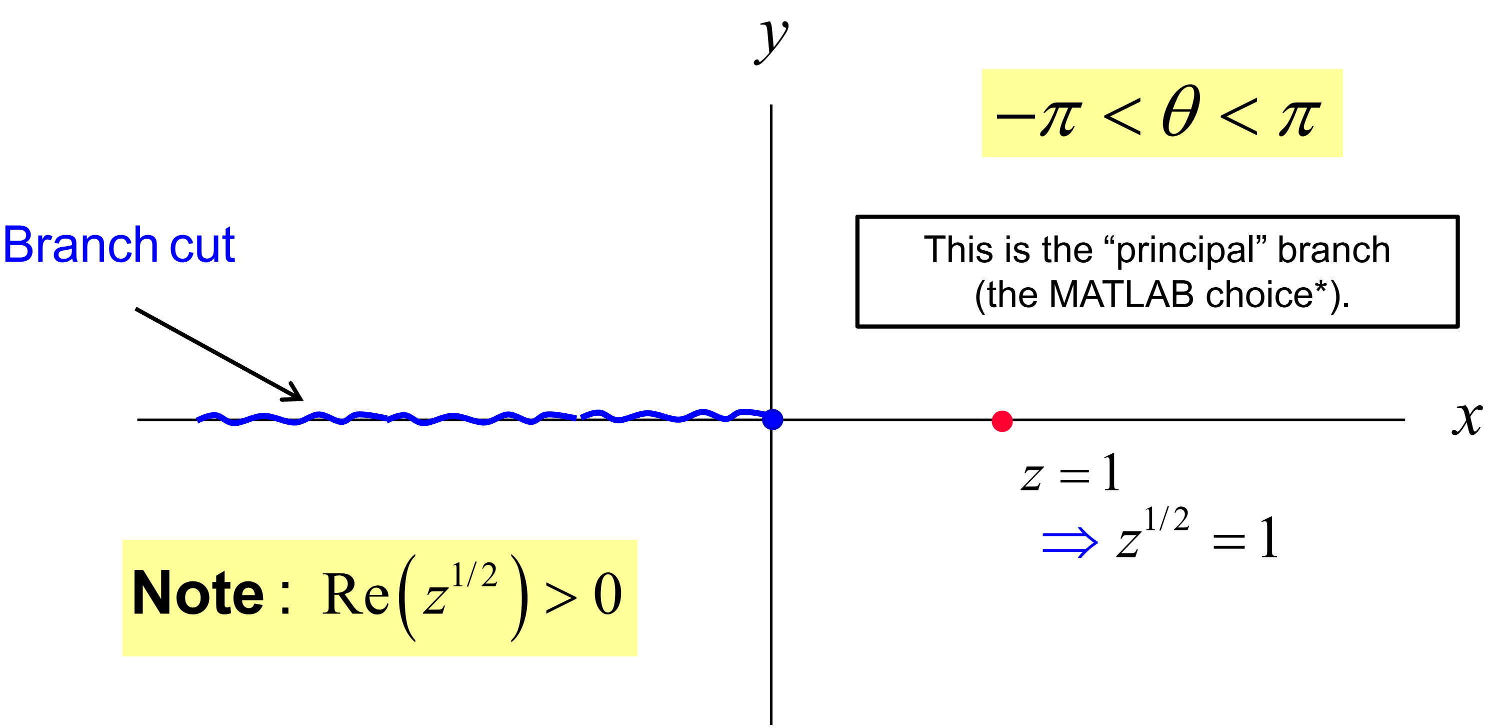


Here the branch cut is chosen to lie on the negative real axis (an arbitrary choice).

Branch Cuts and Branch Points (cont.)

We must now choose what "<u>branch</u>" of the function we want.

$$z = r\,e^{i\theta} \qquad z^{1/2} = \sqrt{r}\,e^{i\theta/2}$$

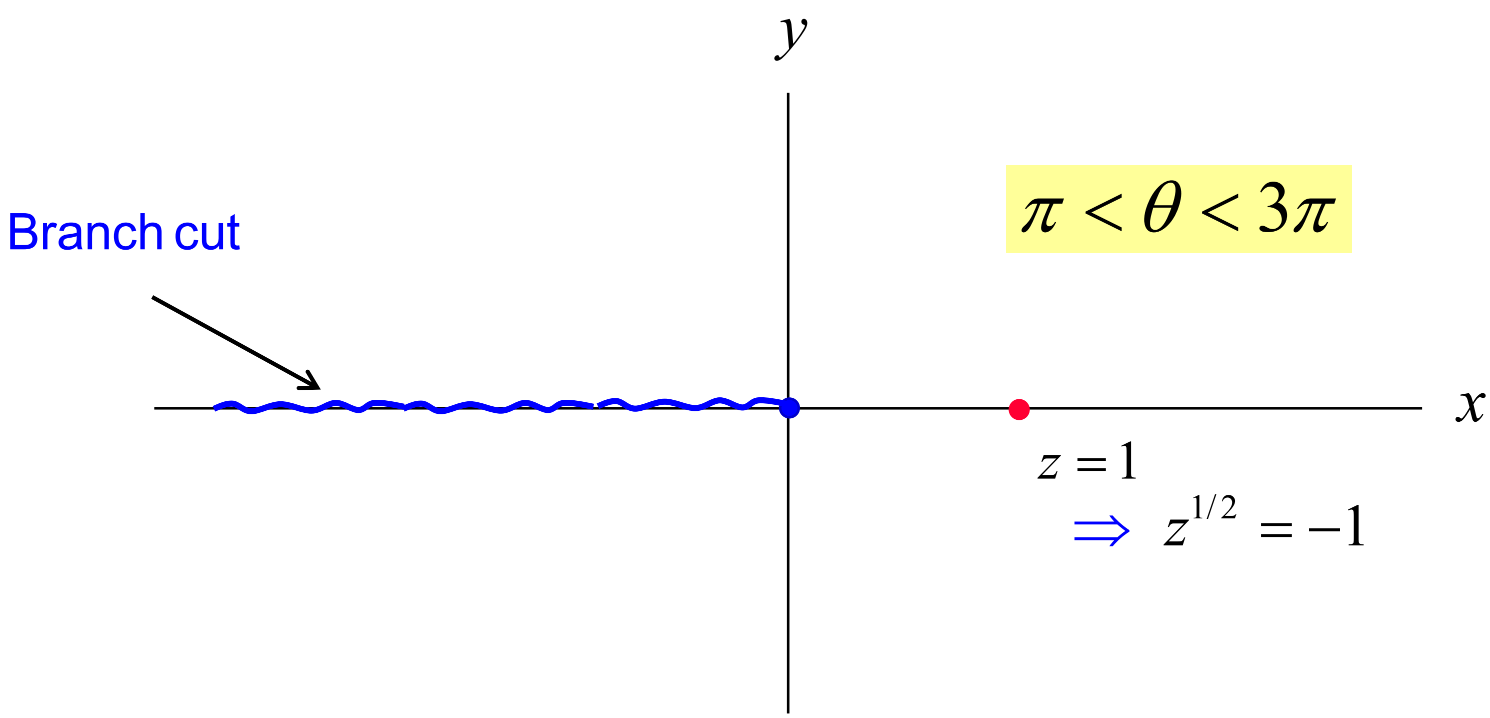


Note: MATLAB actually uses $-\pi < \theta \le \pi$.
The square-root function is then defined on the negative real axis (though it won't be analytic there): $\sqrt{-1} = i$

Branch Cuts and Branch Points (cont.)

Here is the other branch choice.

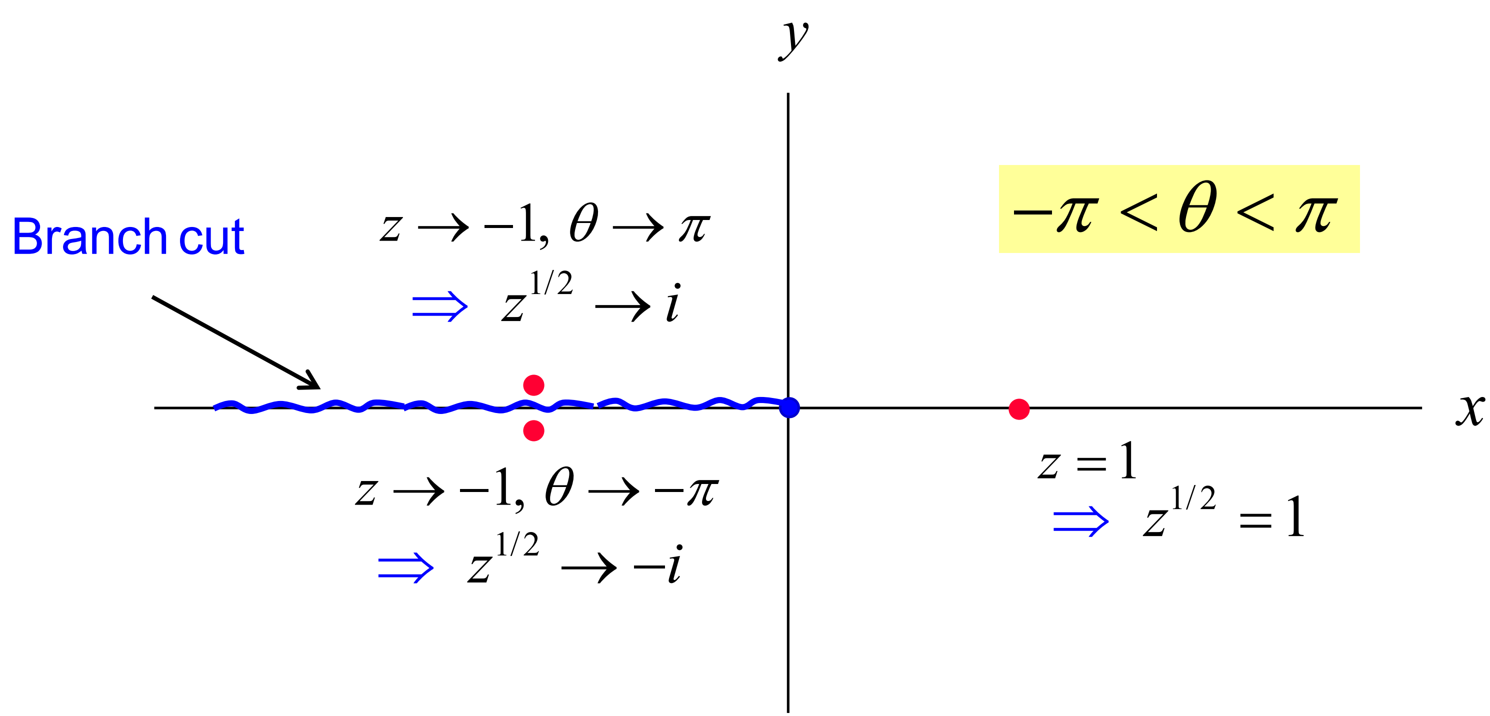
$$z = r\,e^{i\theta} \qquad z^{1/2} = \sqrt{r}\,e^{i\theta/2}$$



Branch Cuts and Branch Points (cont.)

Note that the function is discontinuous across the branch cut.

$$z = r\,e^{i\theta} \qquad z^{1/2} = \sqrt{r}\,e^{i\theta/2}$$

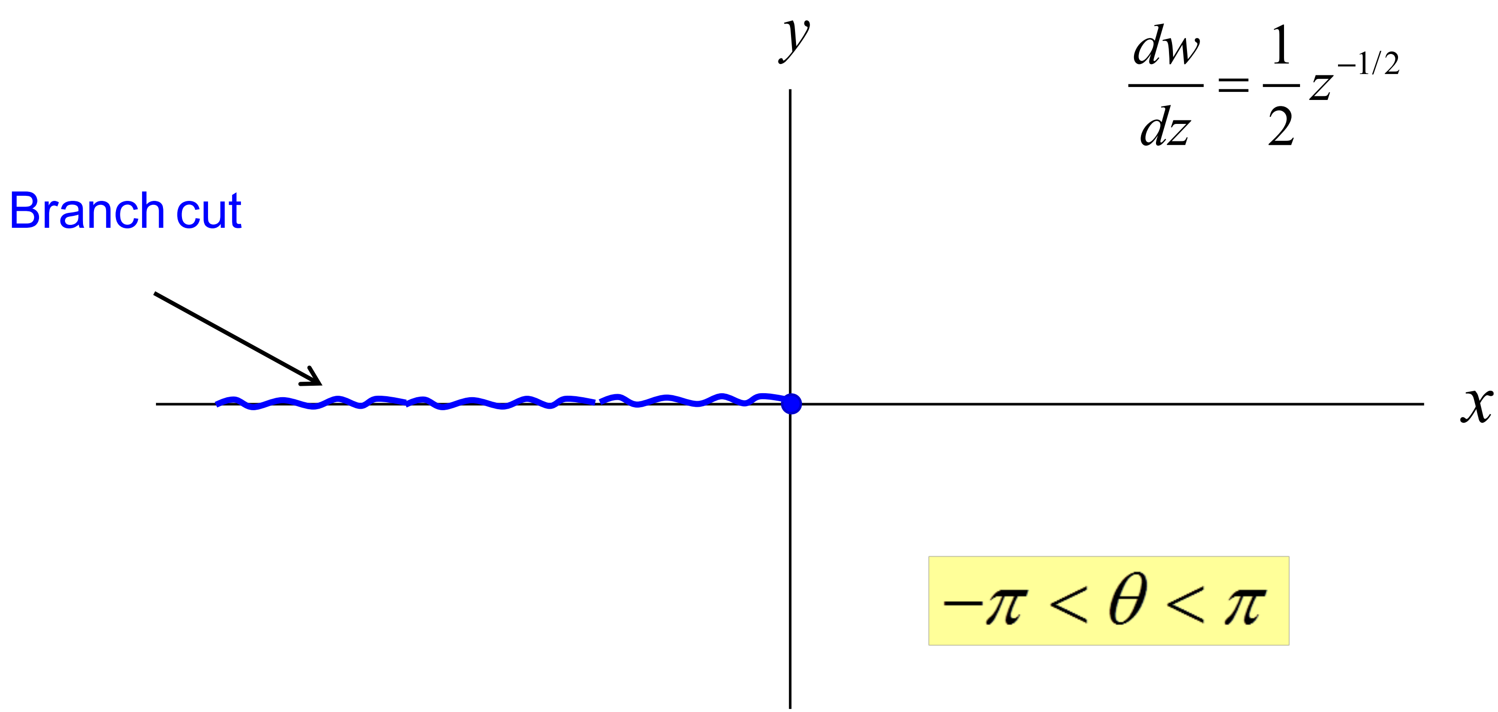


Branch Cuts and Branch Points (cont.)

The function $z^{1/2}$ is analytic <u>off</u> of the branch cut.

$$w = z^{1/2}$$

$$\frac{dw}{dz} = \frac{1}{2} z^{-1/2}$$



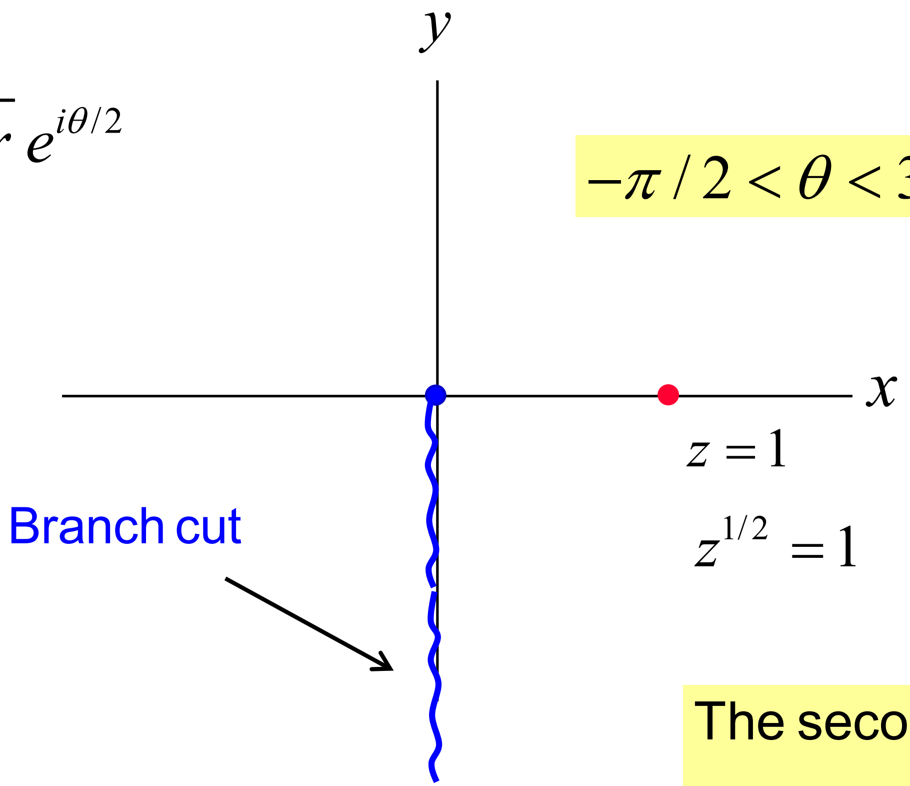
$$-\pi < \theta < \pi$$

Branch Cuts and Branch Points (cont.)

The <u>shape</u> of the branch cut is arbitrary.

$$z = r\, e^{i\theta}$$

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2}$$



$$-\pi / 2 < \theta < 3\pi / 2$$

$$z^{1/2} = 1$$

The second branch is

$$3\pi / 2 < \theta \le 7\pi / 2.$$

Branch Cuts and Branch Points (cont.)

The branch cut does not have to be a straight line.

In this case the branch is determined by requiring that the square-root function change <u>continuously</u> as we start from a specified value (e.g., $z = 1$).

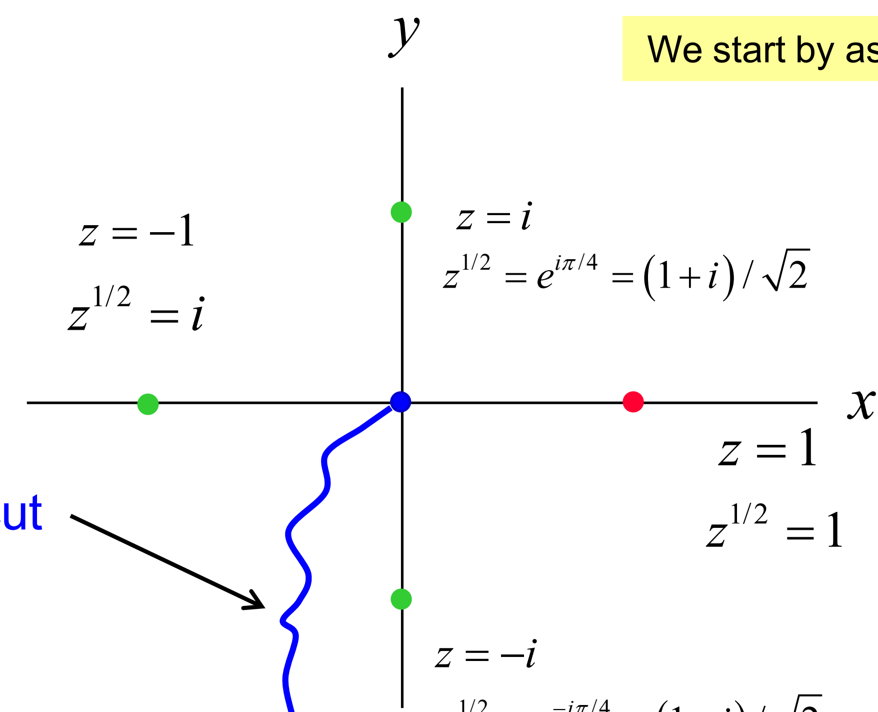
$$z = r\,e^{i\theta}$$

$$z^{1/2} = \sqrt{r}\,e^{i\theta/2}$$

(This means that the angle θ changes continuously.)

We start by assuming that $\theta = 0$ at $z = 1$.

Note:
For the second branch, we would choose $\theta = 2\pi$ at $z = 1$.



$z = -1$

$z^{1/2} = i$

$z = i$

$z^{1/2} = e^{i\pi/4} = (1+i)/\sqrt{2}$

$z = 1$

$z^{1/2} = 1 \quad (\Rightarrow \theta = 0)$

Branch cut

$z = -i$

$z^{1/2} = e^{-i\pi/4} = (1-i)/\sqrt{2}$

Branch Cuts and Branch Points (cont.)

Branch points usually appear in pairs; here one is at $z = 0$ and the other at $z = \infty$ as determined by using $\zeta = 1/z$ and then examining the function at $\zeta = 0$.

$$w = z^{1/2} = 1/\zeta^{1/2} = \frac{1}{\sqrt{r'}} e^{-i\theta'/2} \qquad \left(\zeta = r' e^{i\theta'}\right)$$

We get a different result when we encircle the origin in the ζ plane (θ' changes by 2π) , which means encircling the "point at infinity" in the z plane.

Hence the branch cut for the square-root function connects the origin and the "point at infinity".

Branch Cuts and Branch Points (cont.)

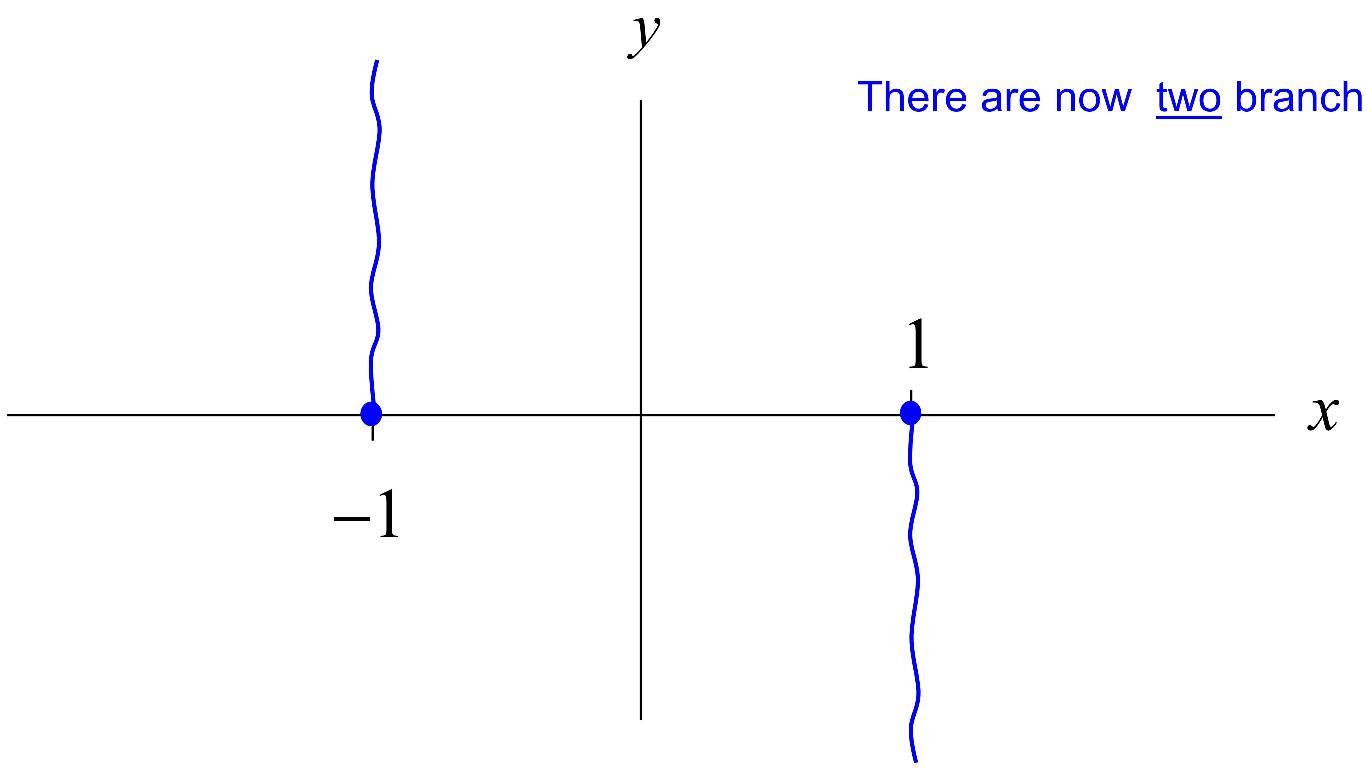
Consider this function:

$$f(z) = \left(z^2 - 1\right)^{1/2}$$

What do the branch points and branch cuts look like for this function?

Branch Cuts and Branch Points (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2} = \left(z-1\right)^{1/2}\left(z+1\right)^{1/2} = \left(z-1\right)^{1/2}\left(z-(-1)\right)^{1/2}$$



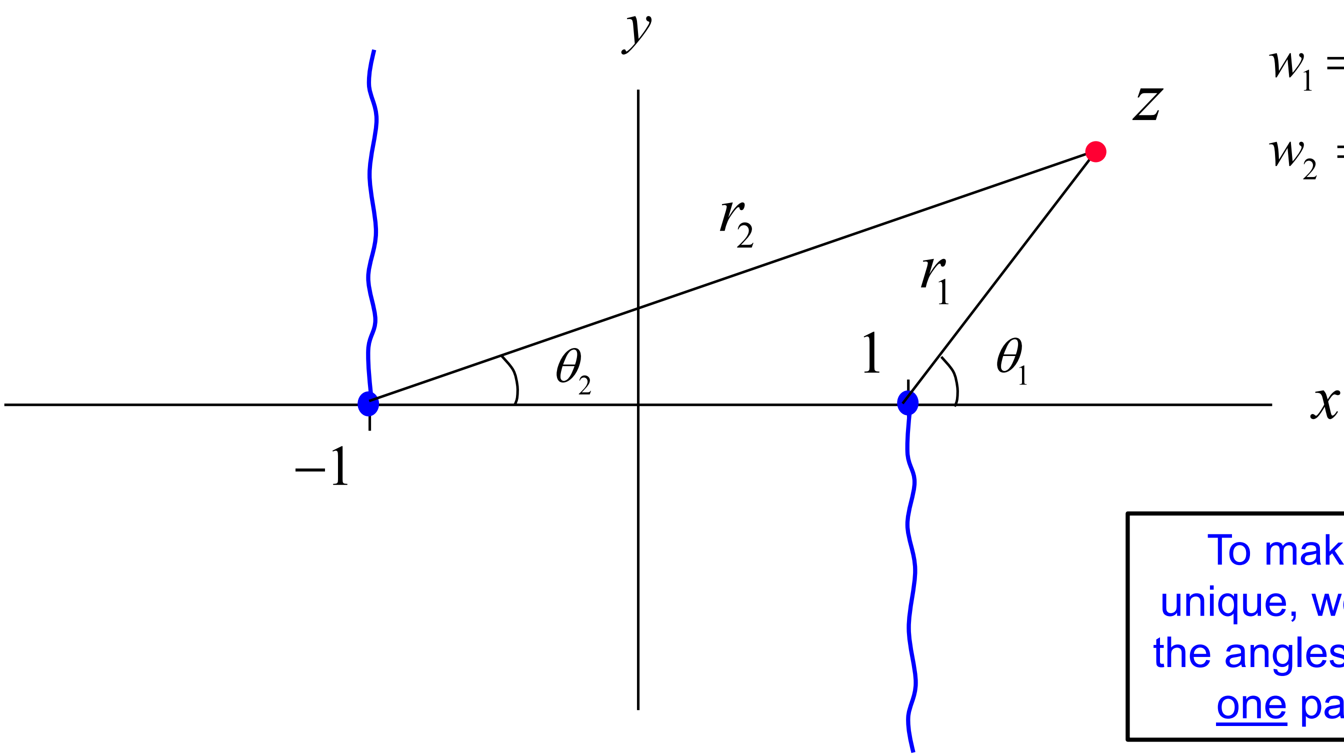
There are now two branch points.

There are two branch cuts: we are not allowed to encircle either branch point.

Branch Cuts and Branch Points (cont.)

Geometric interpretation

$$f(z) = (z-1)^{1/2}\left(z-(-1)\right)^{1/2} = w_1^{1/2}\, w_2^{1/2}$$



$$w_1 = z-1 = r_1\, e^{i\theta_1}$$

$$w_2 = z-(-1) = r_2\, e^{i\theta_2}$$

To make the function unique, we agree on what the angles θ_1 and θ_2 are at one particular point.

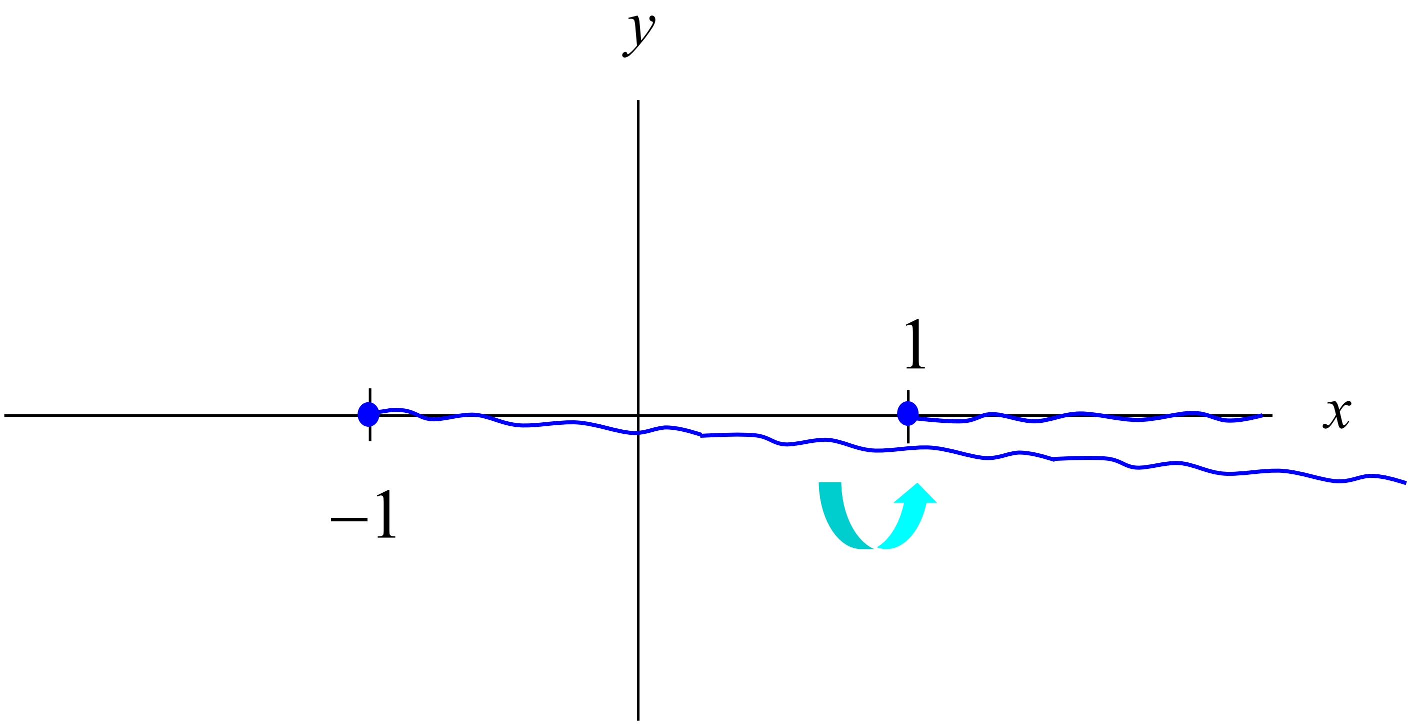
$$f(z) = \left(\sqrt{r_1}\, e^{i\theta_1/2}\right)\left(\sqrt{r_2}\, e^{i\theta_2/2}\right)$$

Example: For $z = 2$, choose $\theta_1 = \theta_2 = 0$.

$$\left(z^{1/2} = 3^{1/2} = +\sqrt{3}\right)$$

Branch Cuts and Branch Points (cont.)

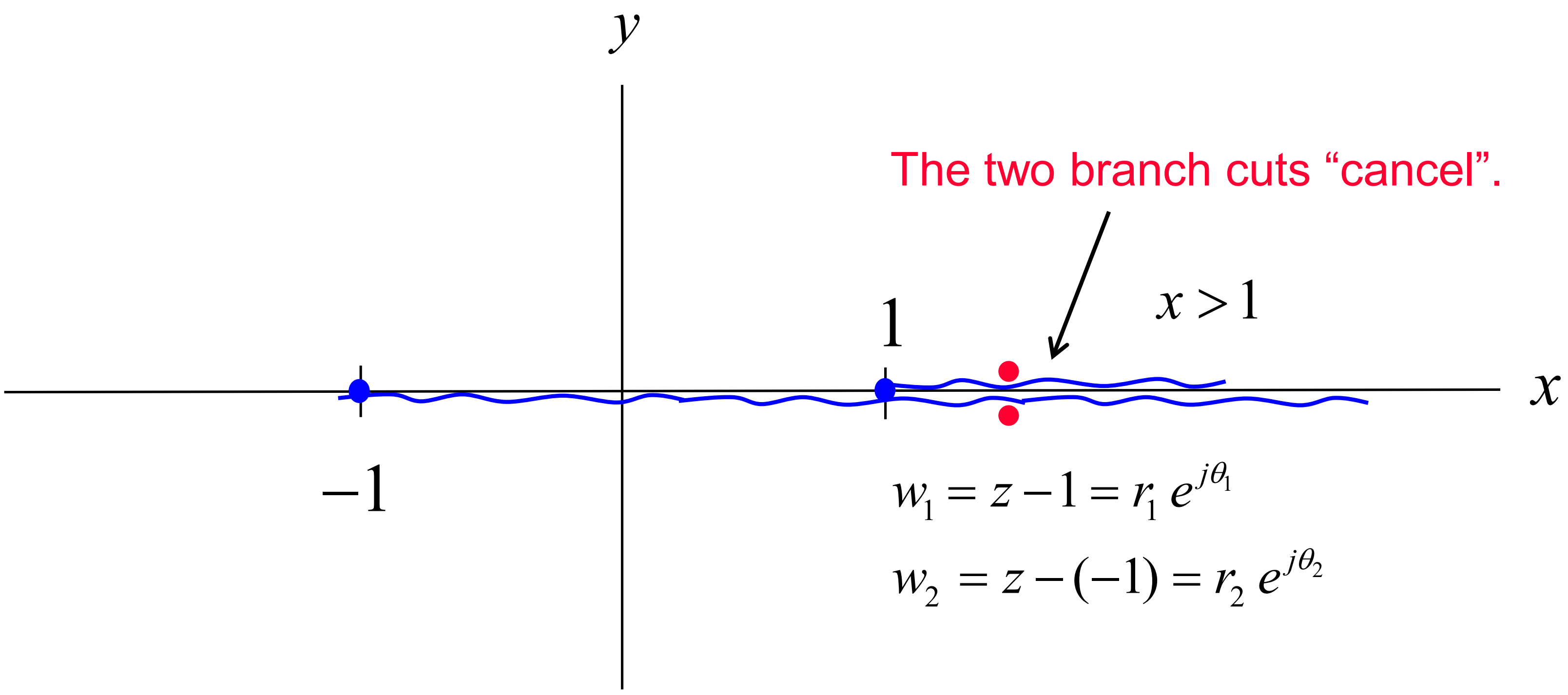
$$f(z) = \left(z^2 - 1\right)^{1/2} = \left(z - 1\right)^{1/2}\left(z - (-1)\right)^{1/2}$$



We can rotate both branch cuts to the real axis.

Branch Cuts and Branch Points (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2} = \left(z - 1\right)^{1/2} \left(z - (-1)\right)^{1/2}$$

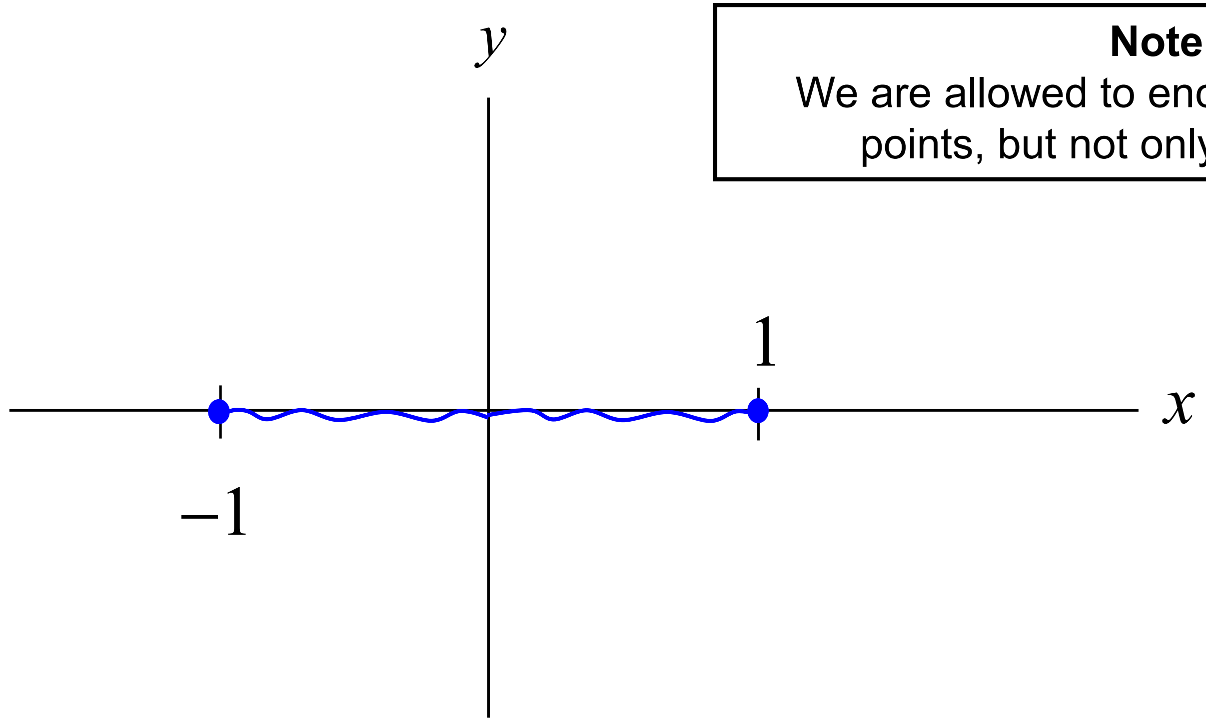


Both θ_1 and θ_2 have changed by 2π if we encircle both branch points.

Note that the function is the same at the two points shown.

Branch Cuts and Branch Points (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2} = \left(z - 1\right)^{1/2}\left(z - (-1)\right)^{1/2}$$



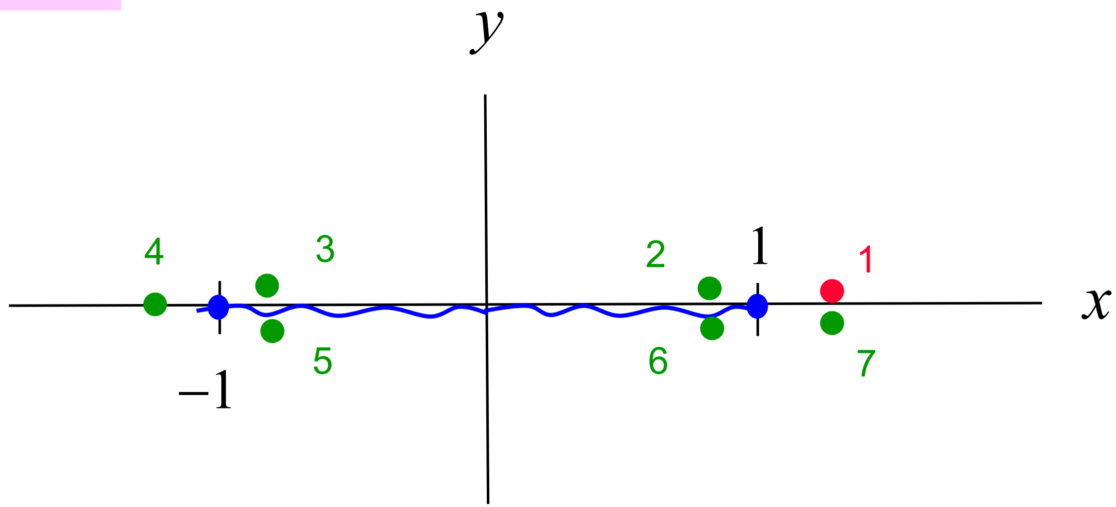
Note:
We are allowed to encircle both branch points, but not only one of them!

An <u>alternative</u> branch cut.

Branch Cuts and Branch Points (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2}$$

Example:

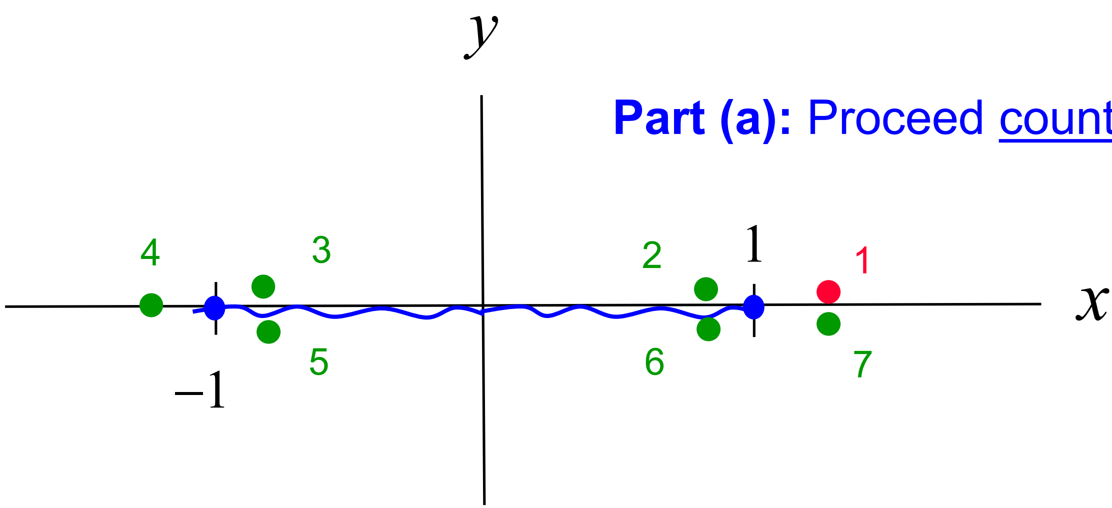


Suppose we agree that at the point #1, $\theta_1 = \theta_2 = 0$. This should uniquely determine the value (branch) of the function everywhere in the complex plane.

Find the angles θ_1 and θ_2 at the other points labeled.

Branch Cuts and Branch Points (cont.)

$$f(z)=\left(z^2-1\right)^{1/2}=\left(z-1\right)^{1/2}\left(z-(-1)\right)^{1/2}=\left(\sqrt{r_1}\,e^{i\theta_1/2}\right)\left(\sqrt{r_2}\,e^{i\theta_2/2}\right)$$



Part (a): Proceed counterclockwise from point 1.

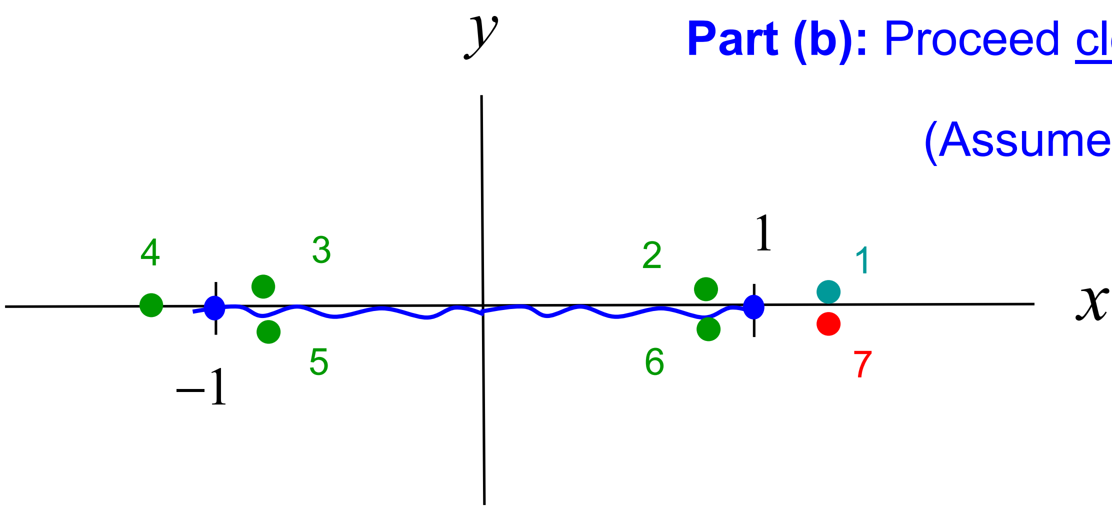
Point	θ_1	θ_2
1	0	0
2	π	0
3	π	0
4	π	π
5	π	2π
6	π	2π
7	2π	2π

For example, at point 6:

$$\begin{aligned}f(z)&=\left(z-1\right)^{1/2}\left(z-(-1)\right)^{1/2}\\&=\left(\sqrt{|z-1|}e^{i(\pi/2)}\right)\left(\sqrt{|z+1|}e^{i(2\pi/2)}\right)\\&=\left((+i)\sqrt{|z-1|}\right)\left((-1)\sqrt{|z+1|}\right)\\&=-i\sqrt{|z-1|}\sqrt{|z+1|}\end{aligned}$$

Branch Cuts and Branch Points (cont.)

$$f(z)=\left(z^2-1\right)^{1/2}=\left(z-1\right)^{1/2}\left(z-(-1)\right)^{1/2}=\left(\sqrt{r_1}\,e^{i\theta_1/2}\right)\left(\sqrt{r_2}\,e^{i\theta_2/2}\right)$$



Part (b): Proceed clockwise from point 7.

(Assume that $\theta_1=\theta_2=0$ at point 7)

Point	θ_1	θ_2
1	-2π	-2π
2	$-\pi$	-2π
3	$-\pi$	-2π
4	$-\pi$	$-\pi$
5	$-\pi$	0
6	$-\pi$	0
7	0	0

For example, at point 6:

$$\begin{aligned}f(z)&=\left(z-1\right)^{1/2}\left(z-(-1)\right)^{1/2}\\&=\left(\sqrt{|z-1|}e^{i(-\pi/2)}\right)\left(\sqrt{|z+1|}e^{i(0/2)}\right)\\&=\left((-i)\sqrt{|z-1|}\right)\left((+1)\sqrt{|z+1|}\right)\\&=-i\sqrt{|z-1|}\sqrt{|z+1|}\end{aligned}$$

Sommerfeld Branch Cuts



Arnold Sommerfeld (1868-1951)

Sommerfeld branch cuts are the most common choice in dealing with radiation types of problems, where there is a square-root wavenumber function.

Sommerfeld Branch Cuts (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2}$$

With this choice of branch cuts we have:

First branch: $\operatorname{Re} f(z) \geq 0$

Second branch: $\operatorname{Re} f(z) \leq 0$

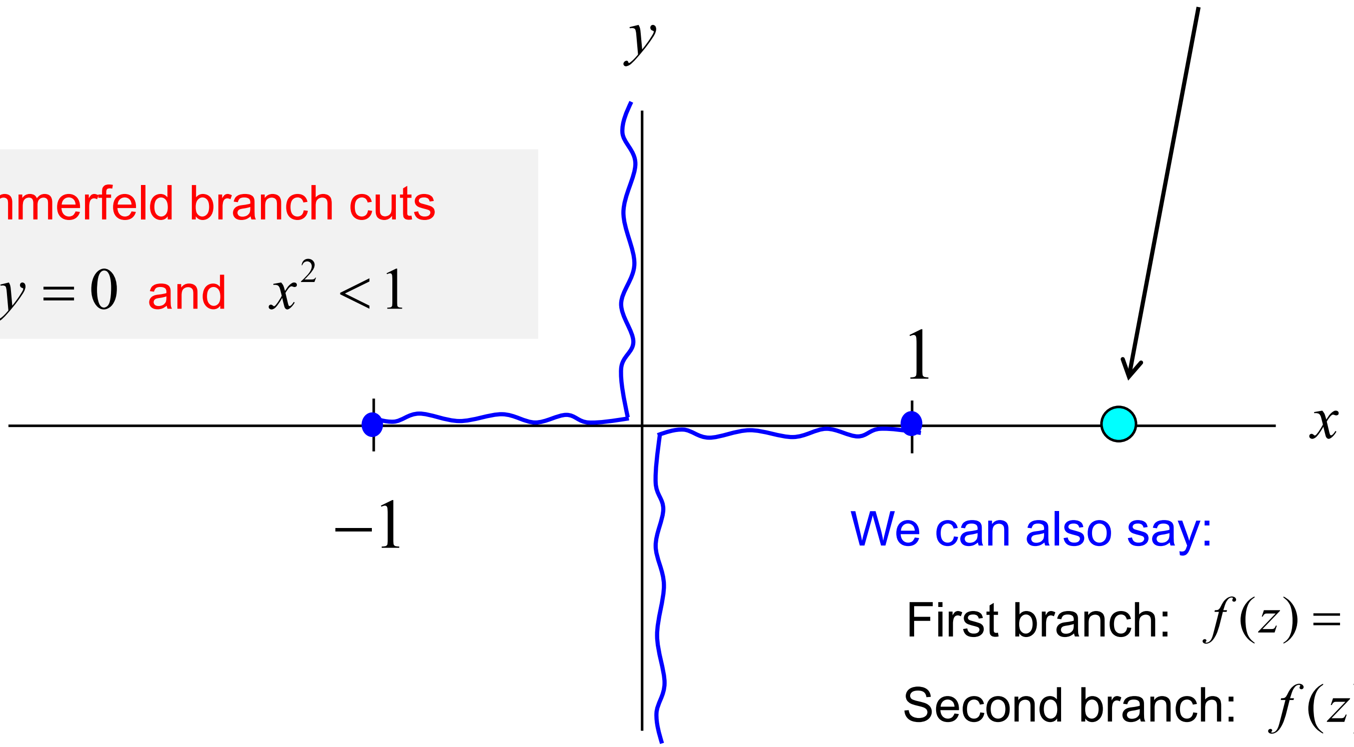
The first branch is defined by:

$$\left(x^2 - 1\right)^{1/2} = \sqrt{x^2 - 1}, \quad x > 1$$

E.g., $f(2) = +\sqrt{3}$

Sommerfeld branch cuts

$xy = 0$ and $x^2 < 1$



We can also say:

First branch: $f(z) = \sqrt{z^2 - 1}$

Second branch: $f(z) = -\sqrt{z^2 - 1}$

Sommerfeld Branch Cuts (cont.)

Proof:

We first solve for the "boundary curve" where $\mathrm{Re}(f(z)) = 0$.

Set $\mathrm{Re}\, f(z) = 0$

$$\Rightarrow \; f(z) = \text{imaginary}$$

$$\Rightarrow \; f^2(z) = \text{real} < 0$$

$$\Rightarrow \left(z^2 - 1\right) = \text{real} < 0$$

$$\Rightarrow \left((x+iy)^2 - 1\right) = \text{real} < 0$$

$$\Rightarrow \left(x^2 - y^2 - 1\right) + i\left(2xy\right) = \text{real} < 0$$

$$\Rightarrow xy = 0 \;\text{ and }\; x^2 - y^2 < 1 \;\Rightarrow\; x^2 < 1 \;\; (\text{for } y = 0)$$

First branch: $\mathrm{Re}\, f(z) \geq 0$

Second branch: $\mathrm{Re}\, f(z) \leq 0$

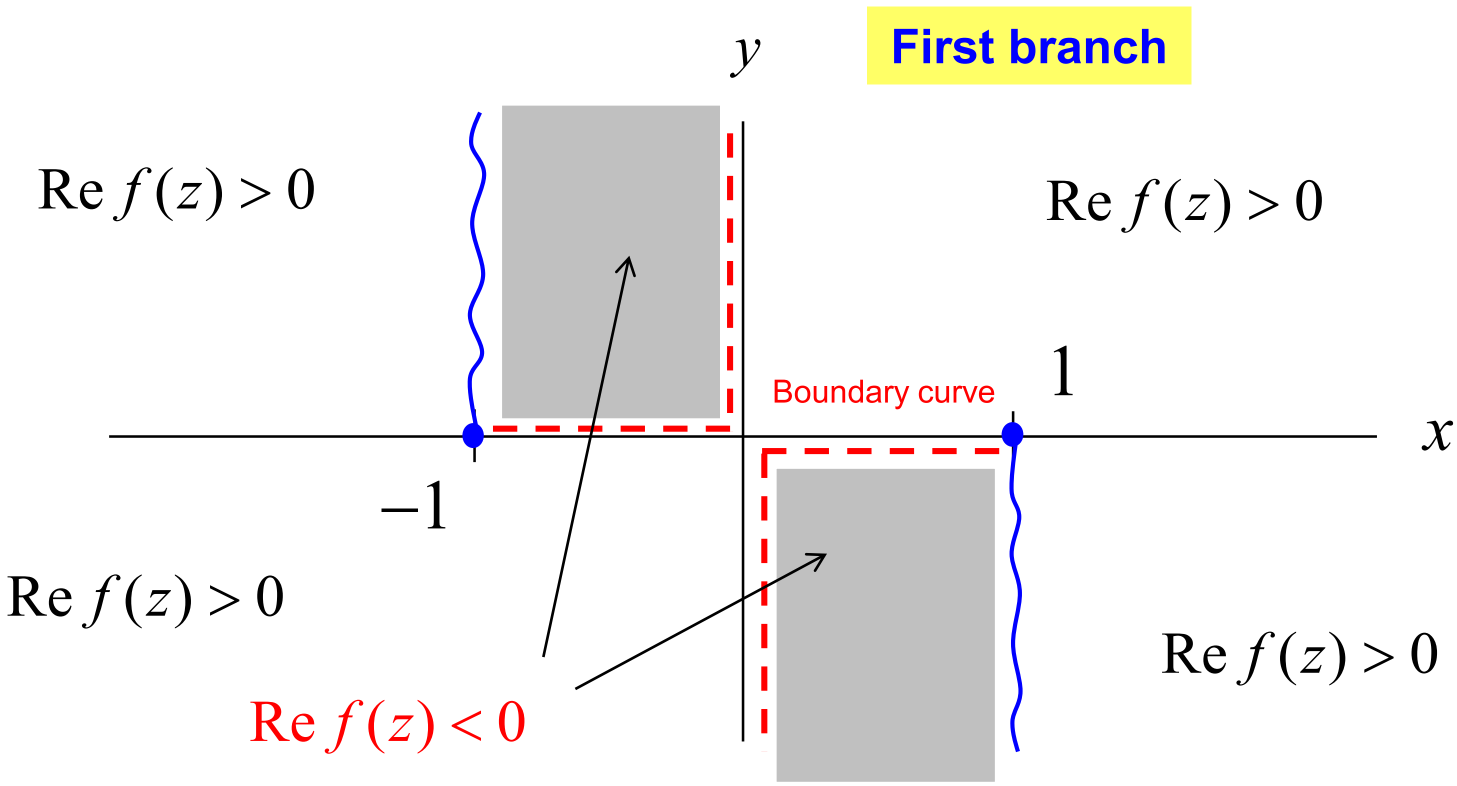
$$f(z) = \left(z^2 - 1\right)^{1/2} = \left(z - 1\right)^{1/2}\left(z - (-1)\right)^{1/2}$$

As long as we do not cross this hyperbolic contour, the real part of f does not change. Hence, the entire complex plane must have a real part that is either positive or negative (depending on which branch we are choosing) if the branch cuts are chosen to lie along this contour (i.e., the Sommerfeld branch cuts).

Sommerfeld Branch Cuts (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2}$$

If the branch cuts are deformed to the boundary curve, the gray area disappears.



Sommerfeld Branch Cuts (cont.)

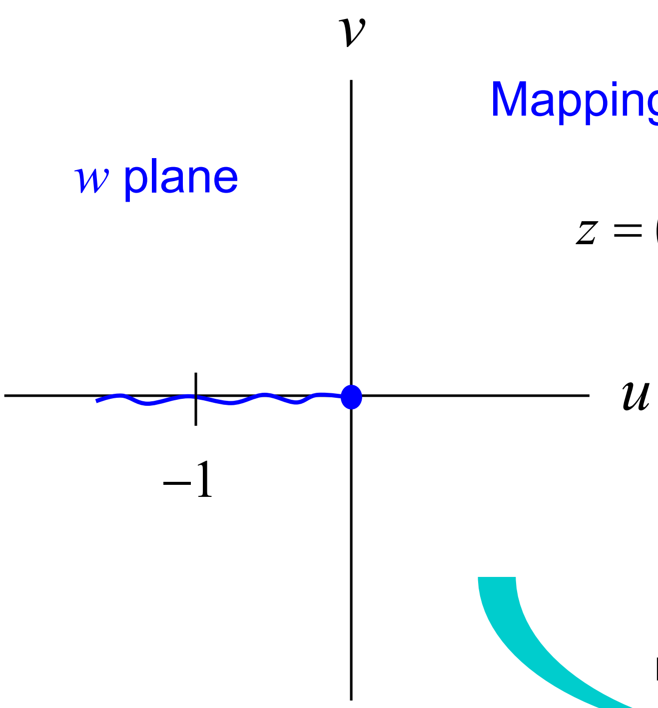
$$f(z) = \left(z^2 - 1\right)^{1/2}$$

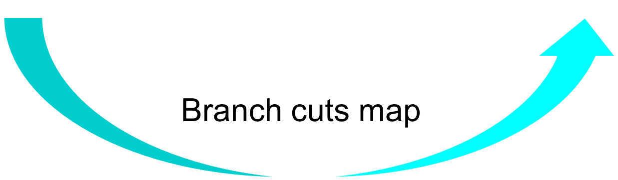
Another point of view: Let $w = z^2 - 1$ ⇨ $f(z) = w^{1/2}$

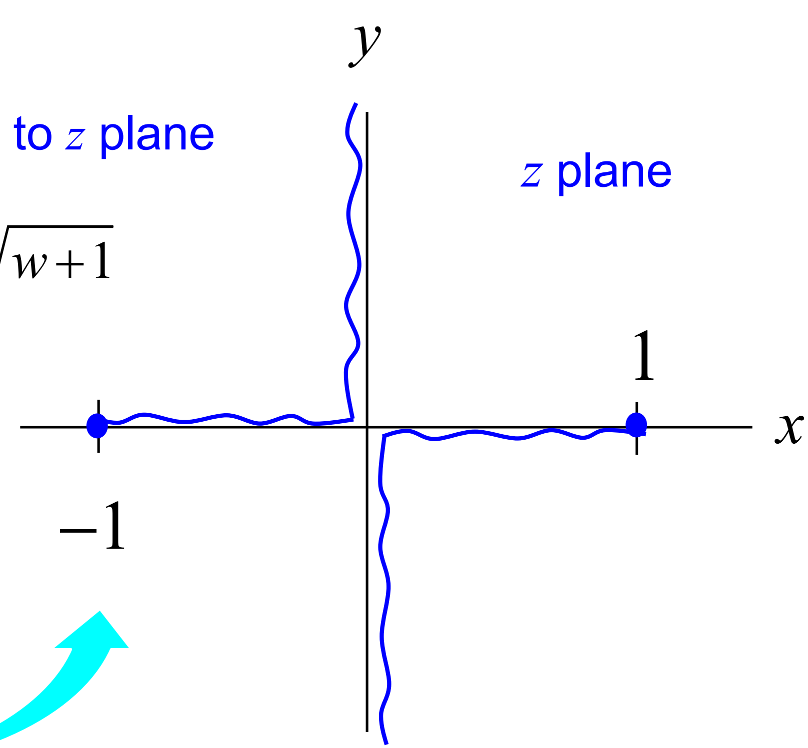
Principal branch: $f(z) = \sqrt{w}$ $\left(\operatorname{Re} f(z) \geq 0\right)$ The branch point is on the negative real axis.

Mapping from w plane to z plane

$$z = \left(w+1\right)^{1/2} = \pm\sqrt{w+1}$$







Sommerfeld Branch Cuts (cont.)

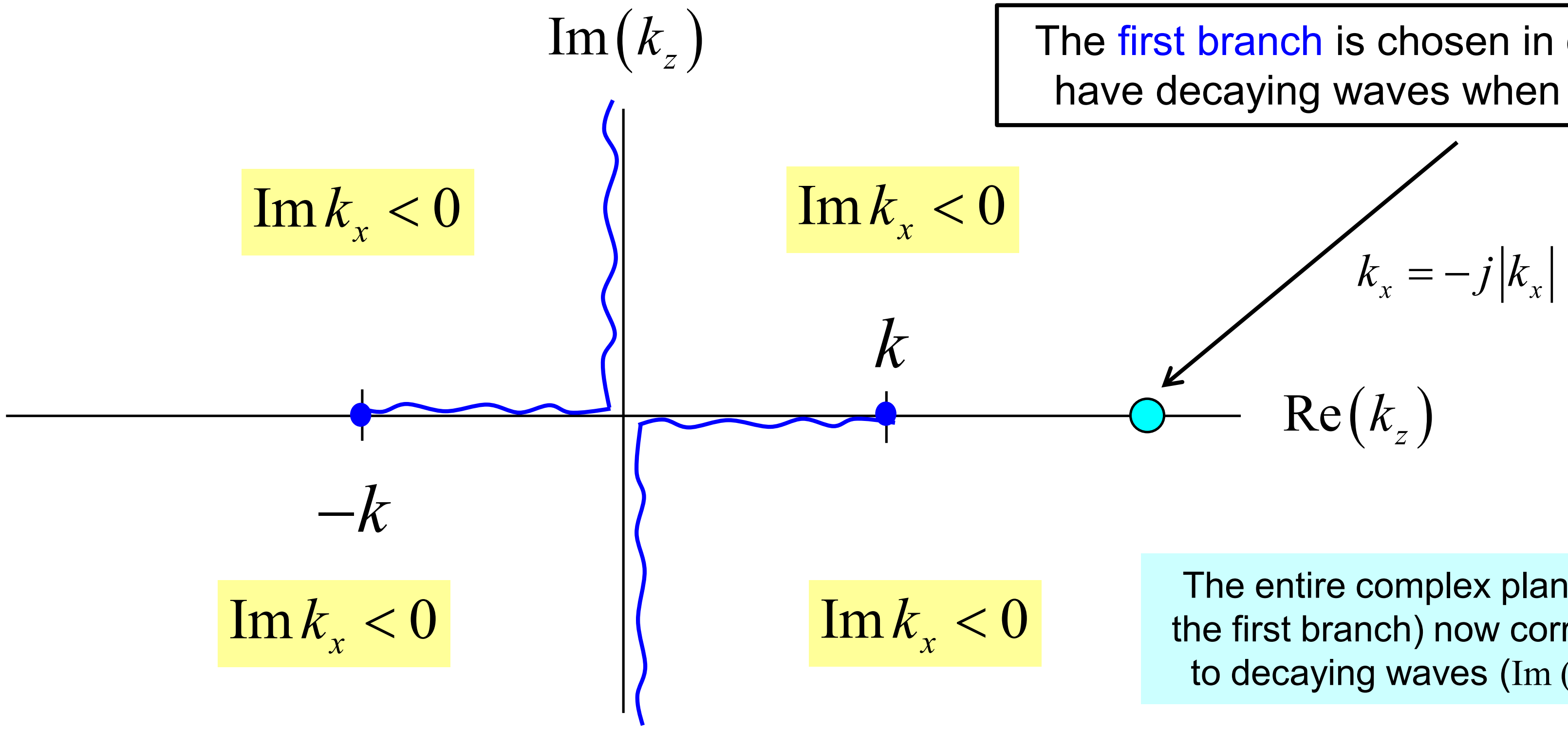
Application: electromagnetic (and other) problems involving a wavenumber.

$$k_x = \left(k_0^2 - k_z^2\right)^{1/2}$$

or $$k_x = -j\left(k_z^2 - k_0^2\right)^{1/2}$$

Note: j is used here instead of i.

(The – sign in front is an arbitrary choice here.)



The first branch is chosen in order to have decaying waves when $k_z > k_0$.

The entire complex plane (using the first branch) now corresponds to decaying waves ($\mathrm{Im}\,(k_x) < 0$).

Riemann Surface

A Riemann surface is a surface that combines the different sheets of a multi-valued function.

It is useful since it displays all possible values of the function at one time.

(his signature)

Georg Friedrich Bernhard Riemann (1826-1866)

Riemann Surface (cont.)

The concept of the Riemann surface is first illustrated for

$$f(z) = z^{1/2} \qquad z = r\,e^{i\theta}$$

The Riemann surface is really <u>two</u> complex planes connected together.

The function $z^{1/2}$ is analytic everywhere on this surface (there are <u>no branch cuts</u>), except at the origin. It also assumes all possible values on the surface.

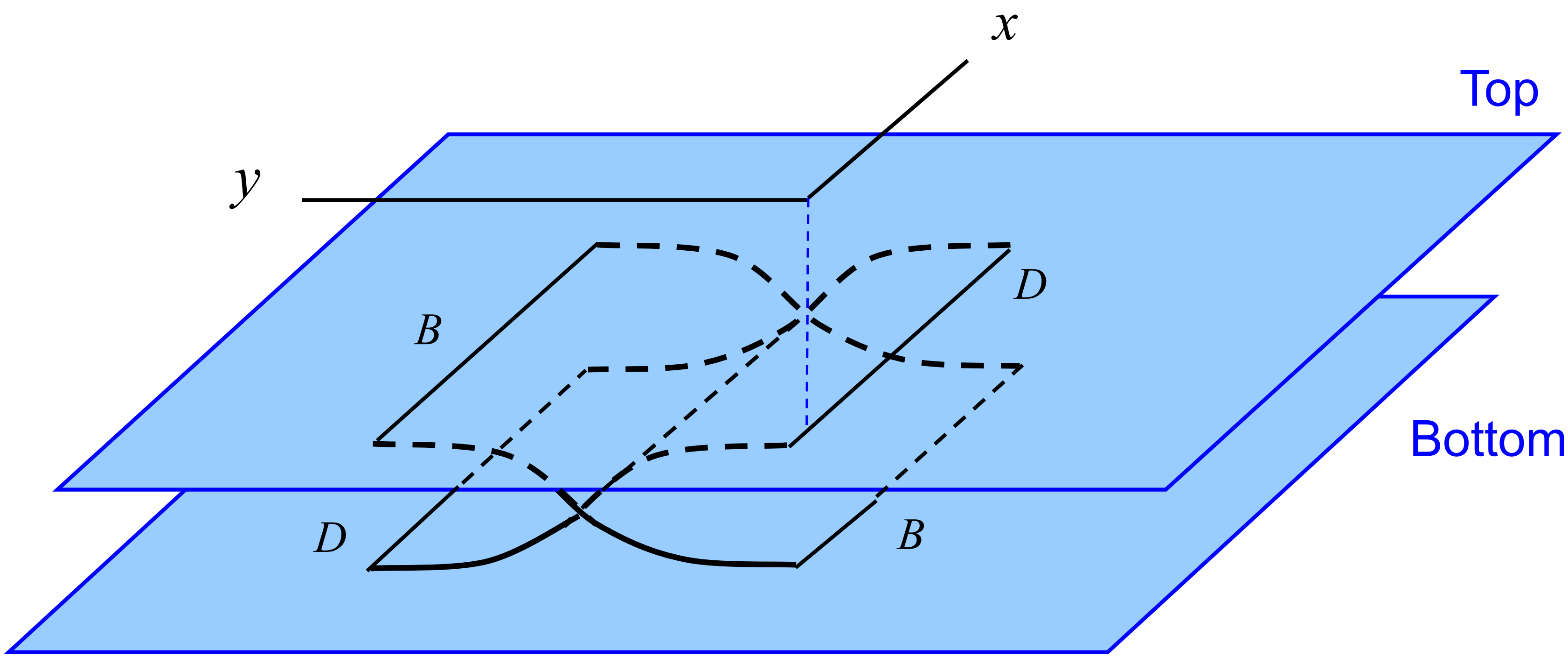
Consider this choice:

Top sheet: $-\pi < \theta < \pi \quad (1^{1/2} = 1)$

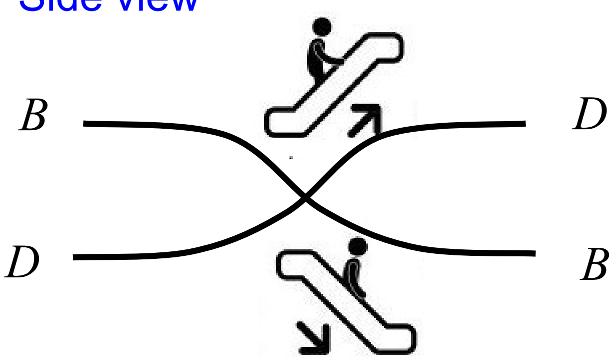
Bottom sheet: $\pi < \theta < 3\pi \quad (1^{1/2} = -1)$

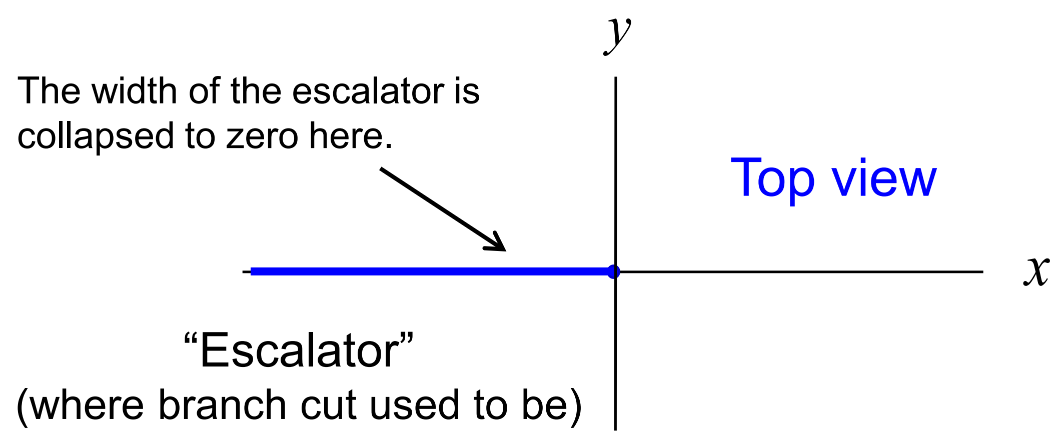
Riemann Surface (cont.)

The angle θ changes smoothly on the surface!



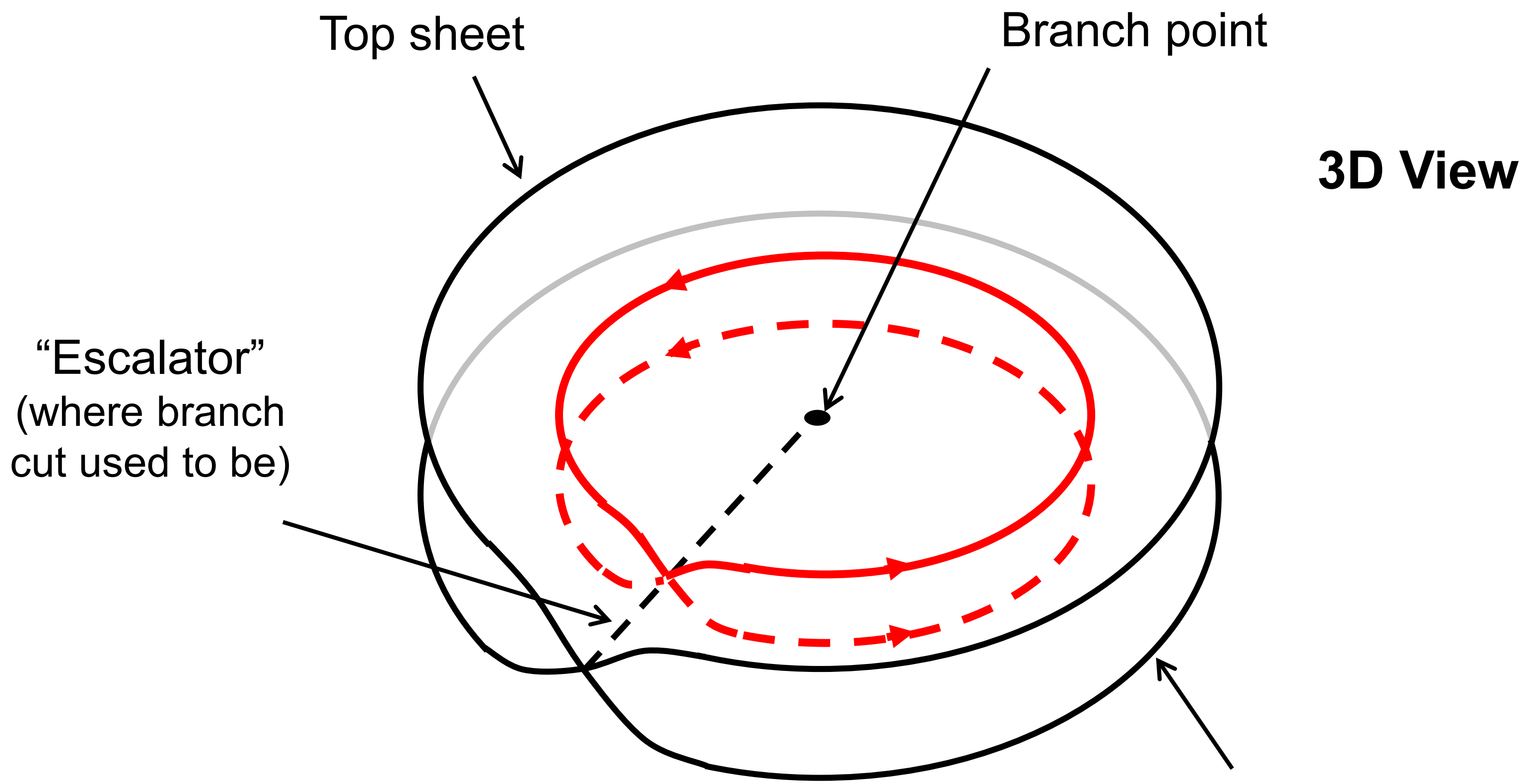
Side view





Riemann Surface (cont.)

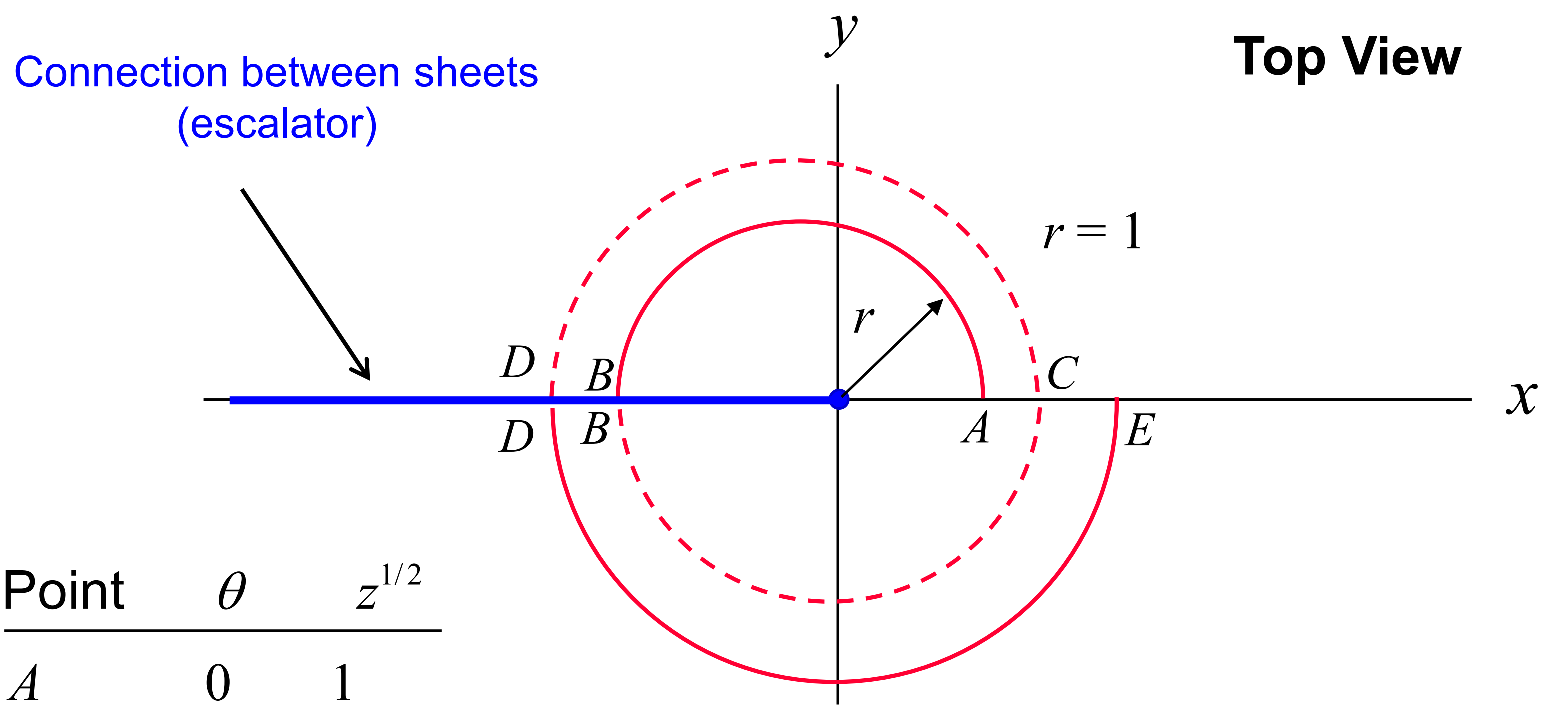
This figure shows going around a branch point twice.



Note:
We are not allowed to jump from
one escalator to the other!

Riemann Surface (cont.)

This figure shows going around a branch point twice.



Point	θ	$z^{1/2}$
A	0	1
B	π	$+i$
C	2π	-1
D	3π	$-i$
E	4π	$+1$

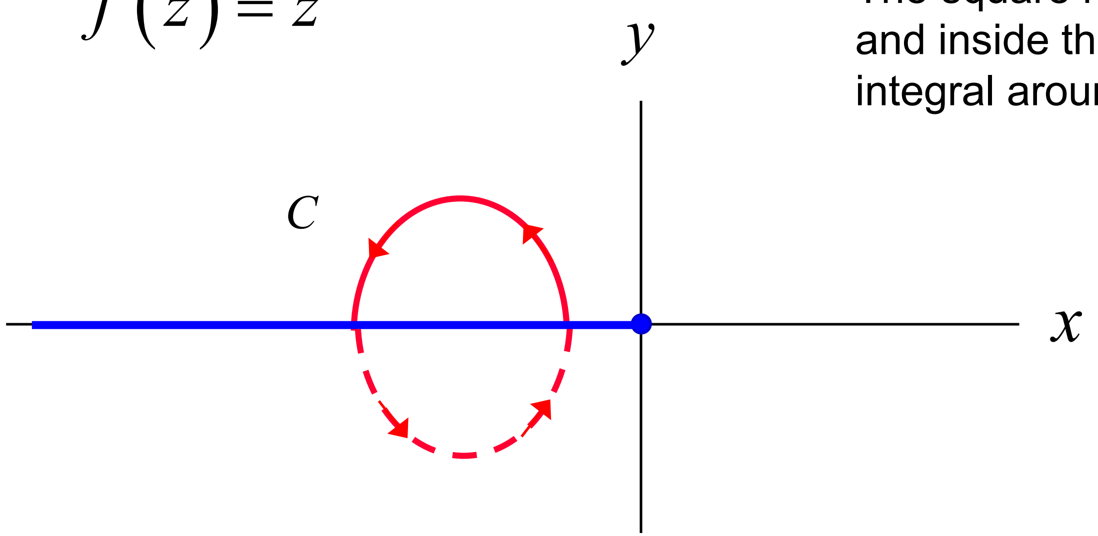
$$z = re^{i\theta}$$

$$f(z) = z^{1/2} = (\sqrt{1})\, e^{i\theta/2} = e^{i\theta/2}$$

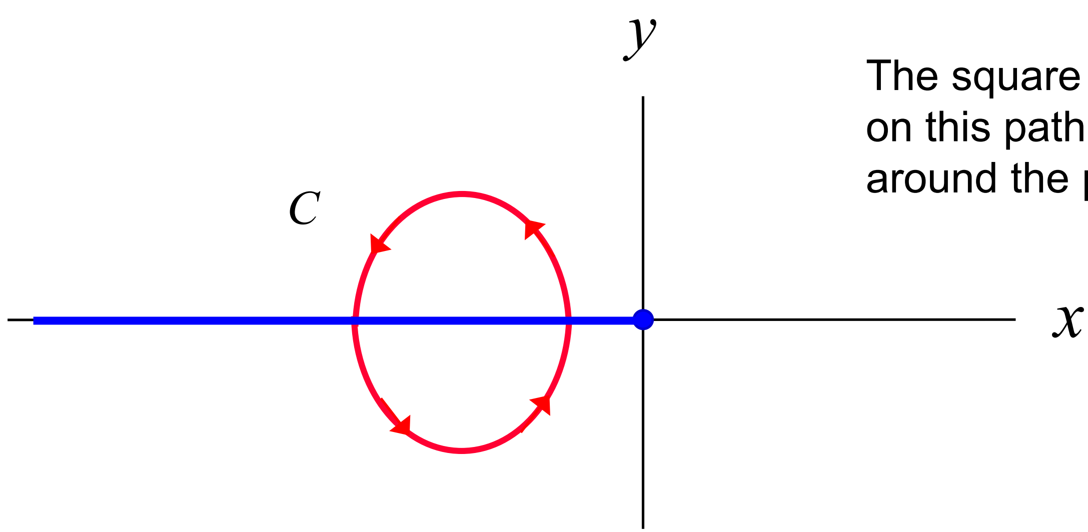
Note: 4π is the same as 0 for this function.

Riemann Surface (cont.)

$$f(z) = z^{1/2}$$



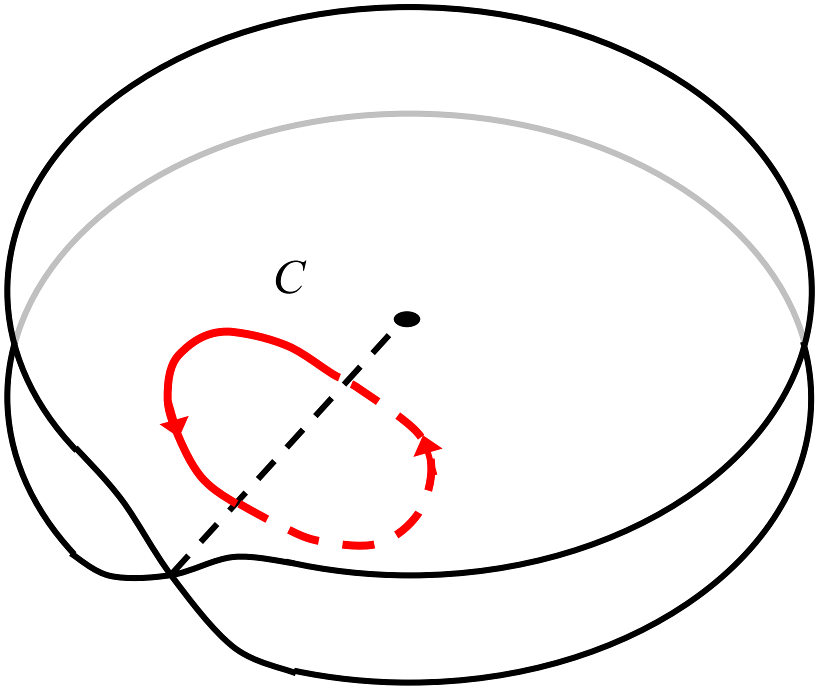
The square root function is <u>analytic</u> on and inside this path, and the closed line integral around the path is thus zero.



The square root function is <u>discontinuous</u> on this path, and the closed line integral around the path is not zero.

Riemann Surface (cont.)

$$f(z) = z^{1/2}$$



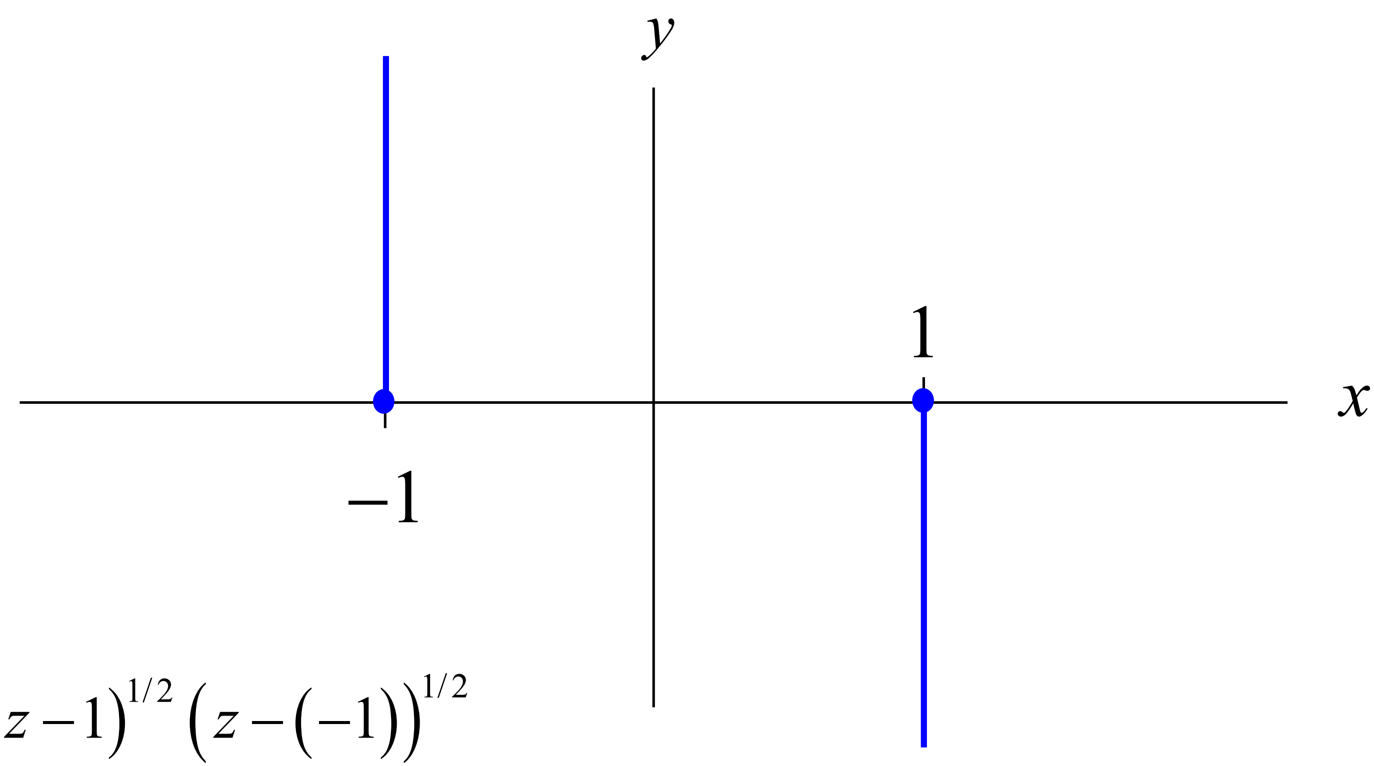
The 3D perspective makes it more clear that the integral around this closed path on the Riemann surface is zero. (The path can be shrunk continuously to zero.)

Riemann Surface (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2}$$

Top sheet:

Defined by $\theta_1 = \theta_2 = 0$ on real axis for $x > 1$

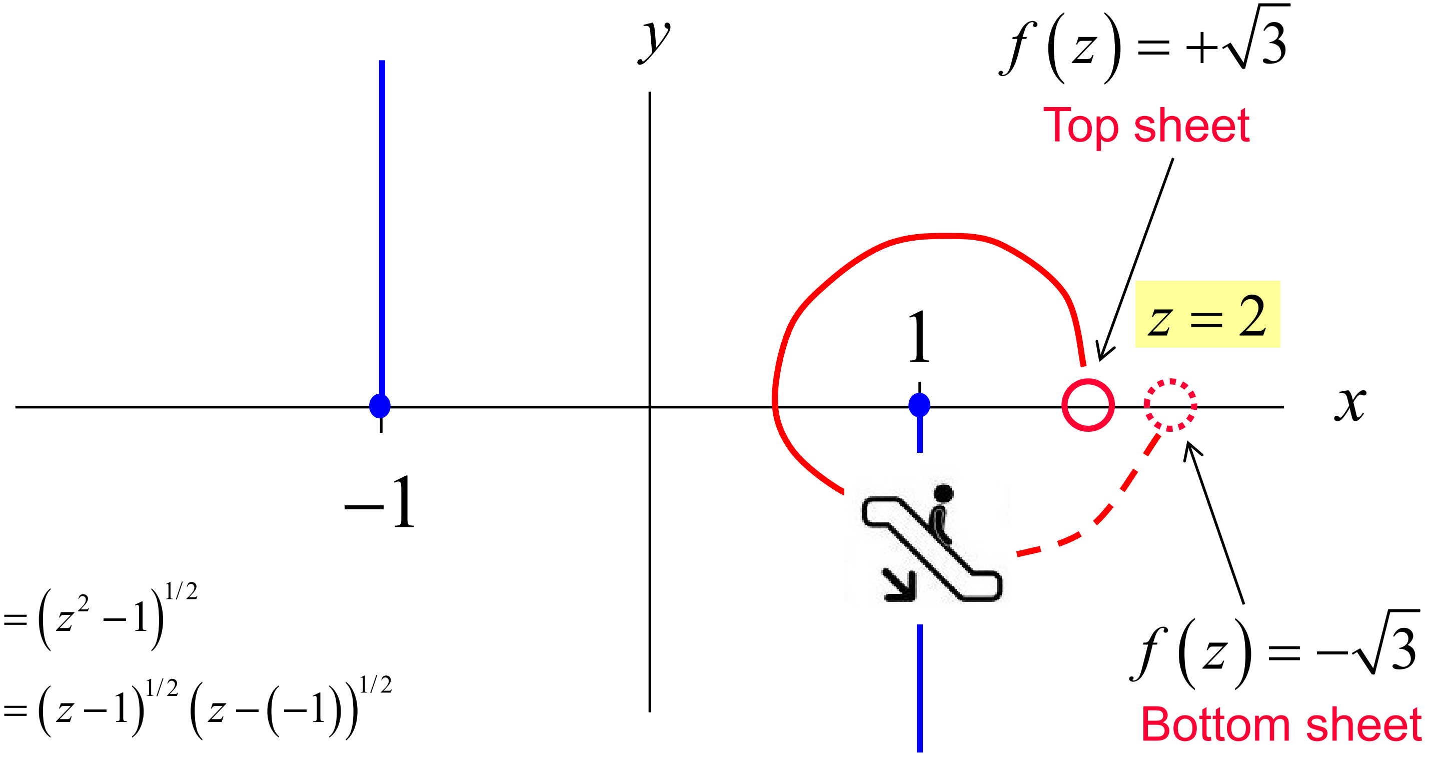


$$f(z) = \left(z-1\right)^{1/2}\left(z-\left(-1\right)\right)^{1/2}$$

There are two sets of up and down "escalators" that now connect the top and bottom sheets of the surface.

Riemann Surface (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2}$$



$$\begin{aligned} f(z) &= \left(z^2 - 1\right)^{1/2} \\ &= (z-1)^{1/2}\left(z-(-1)\right)^{1/2} \\ &= \left(\sqrt{r_1}\, e^{i\theta_1/2}\right)\left(\sqrt{r_2}\, e^{i\theta_2/2}\right) \end{aligned}$$

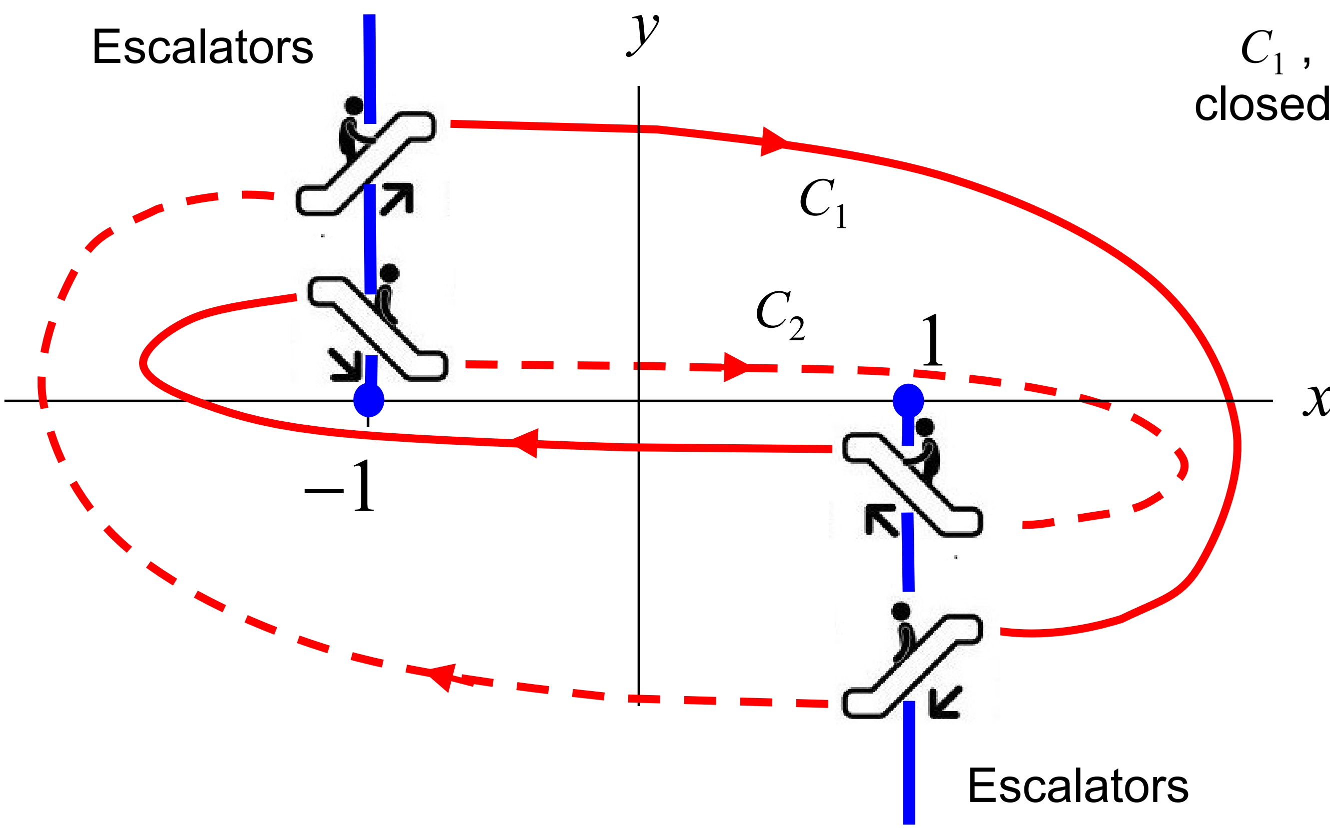
The angle θ_1 has changed by 2π as we go back to the point $z = 2$.

Riemann Surface (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2}$$

C_1 , C_2 are closed curves on the Riemann surface.

The integral around them is not zero!
(The paths cannot be shrunk to zero.)



C_1 , C_2 are closed curves.

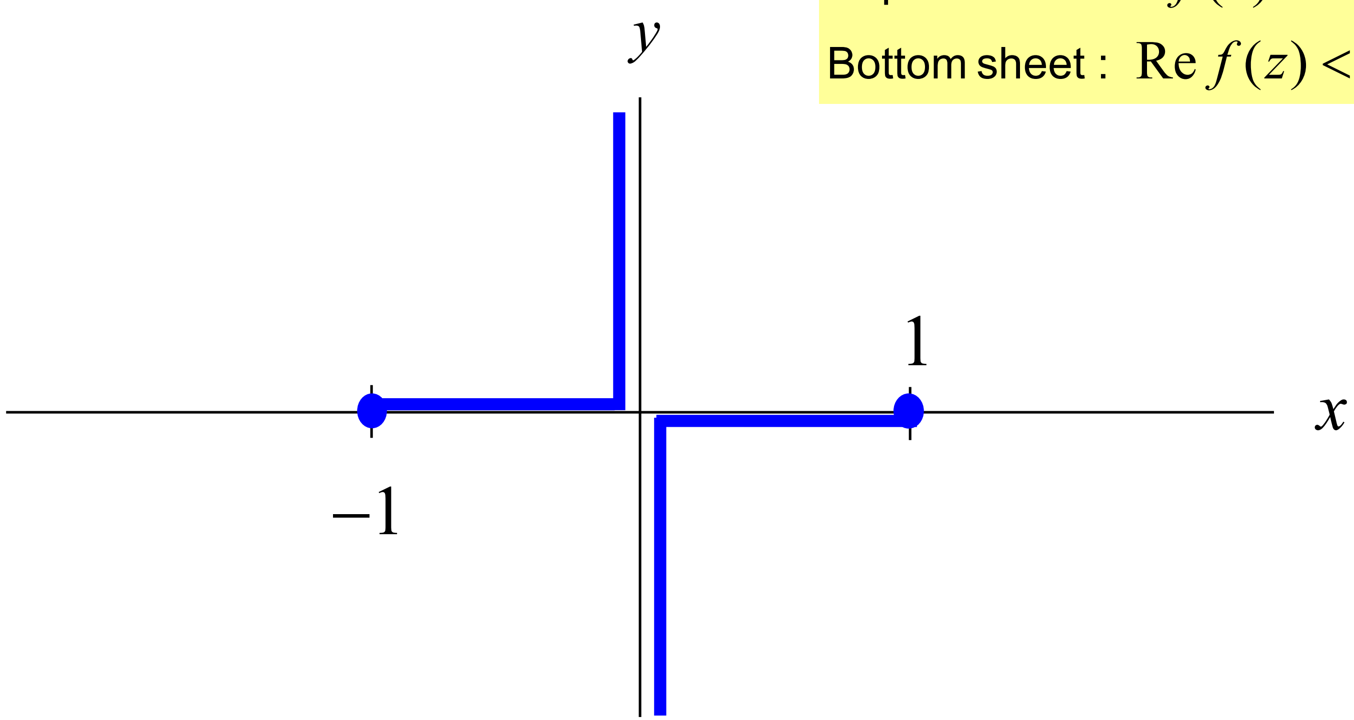
Riemann Surface (cont.)

Sommerfeld (hyperbolic) shape of escalators

$$f(z) = \left(z^2 - 1\right)^{1/2}$$

Top sheet : $\operatorname{Re} f(z) > 0$

Bottom sheet : $\operatorname{Re} f(z) < 0$



Riemann Surface (cont.)

Application to leaky modes:

$$\psi(x,z) = A\,e^{-jk_z z}e^{-jk_x x}$$

Leaky modes (radiating modes) are improper (exponentially increasing away from the structure).

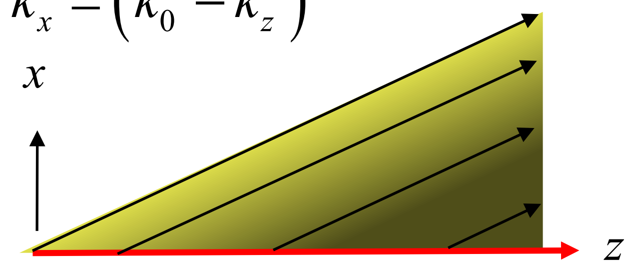
Note: j is used here instead of i.

k_z = complex propagation wavenumber of mode

$$k_z = \beta - j\alpha, \qquad \beta > 0, \quad \alpha > 0 \quad \text{(forward wave)}$$

In the air region: $k_x^2 + k_y^2 + k_z^2 = k_0^2 \Rightarrow k_x = \left(k_0^2 - k_z^2\right)^{1/2}$

(assume $k_y = 0$)



Leaky mode radiating in air

$$(\beta_x - j\alpha_x)^2 + (\beta - j\alpha)^2 = k_0^2$$

Imaginary part: $\beta_x \alpha_x = -\beta\alpha$

Assume: $\beta_x > 0$ ⟹ Conclusion: $\alpha_x < 0$ (improper!)

Riemann Surface (cont.)

$$\psi(x,z) = A e^{-jk_x x} e^{-jk_z z}$$

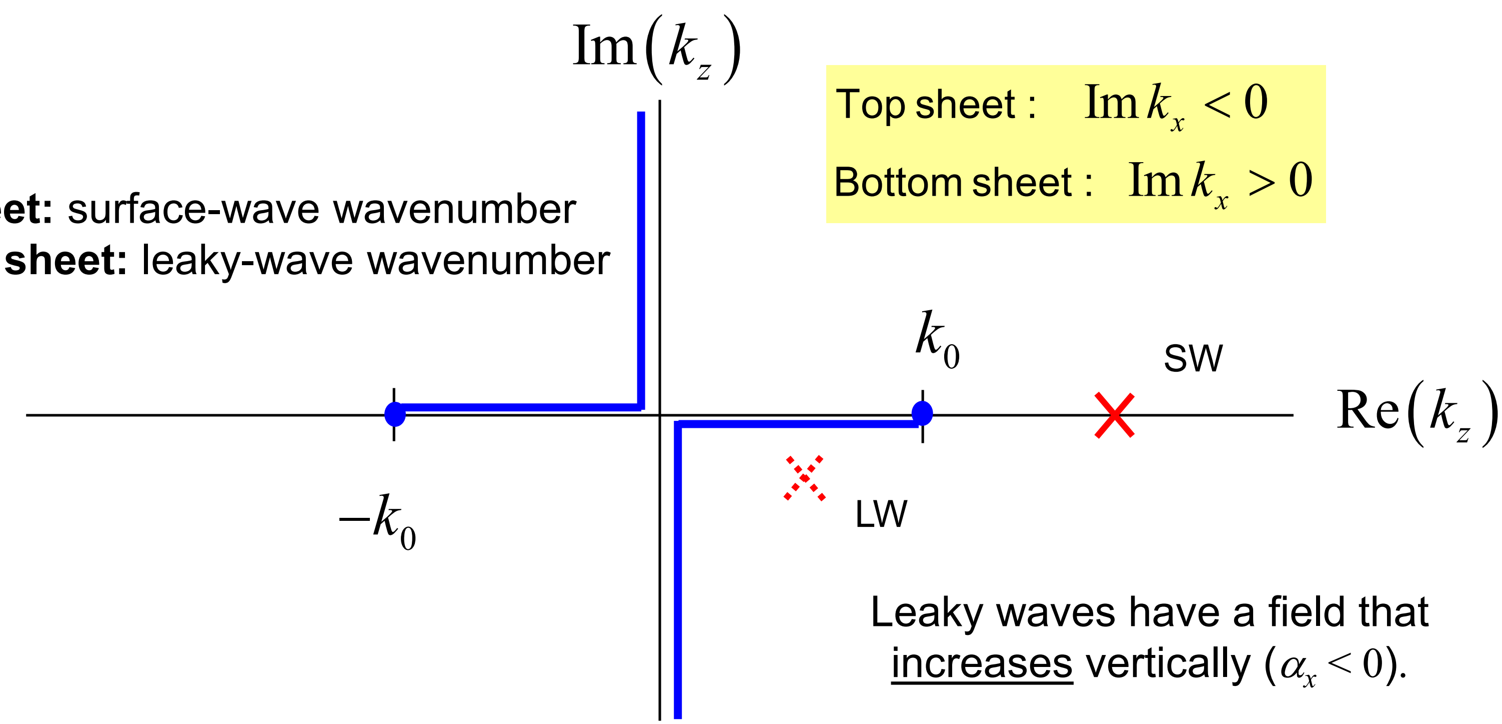
$$k_x = \left(k_0^2 - k_z^2\right)^{1/2} = -j\left(k_z^2 - k_0^2\right)^{1/2}$$

$k_x = \beta_x - j\alpha_x$ = *vertical* propagation wavenumber of mode

Choose Sommerfeld (hyperbolic) shape of escalators.

Top sheet :	$\operatorname{Im} k_x < 0$
Bottom sheet :	$\operatorname{Im} k_x > 0$

Top sheet: surface-wave wavenumber
Bottom sheet: leaky-wave wavenumber



Leaky waves have a field that <u>increases</u> vertically ($\alpha_x < 0$).

Other Multiple-Branch Functions

$$f\left(z\right)=z^{1/3}=r^{1/3}\,e^{i\theta/3}$$ 3 sheets

$$f\left(z\right)=z^{4/5}=r^{4/5}\,e^{i4\theta/5}$$ 5 sheets

$$f\left(z\right)=z^{p/q}=r^{p/q}\,e^{ip\theta/q}$$ q sheets

$$f\left(z\right)=\ln(z)=\ln\left(r\,e^{i\theta}\right)=\ln\left(r\right)+i\theta$$ Infinite number of sheets

$$f\left(z\right)=z^{\pi}=\left(e^{\ln z}\right)^{\pi}=e^{\pi\ln z}=e^{\pi\left(\ln r+i\theta\right)}=e^{\pi\ln r}\,e^{i\pi\theta}$$

Infinite number of sheets

The power π is an irrational number.

Other Multiple-Branch Functions (cont.)

Riemann surface for $\ln(z)$

Note: There are no escalator pairs here: as we keep going in one direction (clockwise or counterclockwise), we never return to the original sheet.