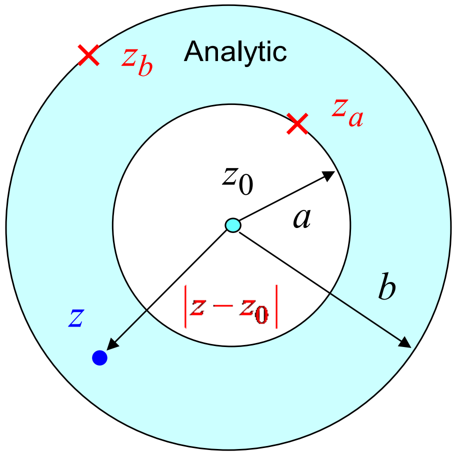


ECE 6382

Fall 2023

David R. Jackson

Notes 7

Power Series Representations

Notes are from D. R. Wilton, Dept. of ECE

Geometric Series

Consider the following sum:

$$S_N = 1 + z + z^2 + \cdots + z^N = \sum_{n=0}^{N} z^n$$

Geometric series (GS):

$$S = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

Note that

$$zS_N = z + z^2 + \cdots + z^N + z^{N+1}$$

Hence, we have

$$S_N - zS_N = (1-z)\,S_N = 1 - z^{N+1}$$

$$\Rightarrow \quad S_N = \frac{1 - z^{N+1}}{1 - z}$$

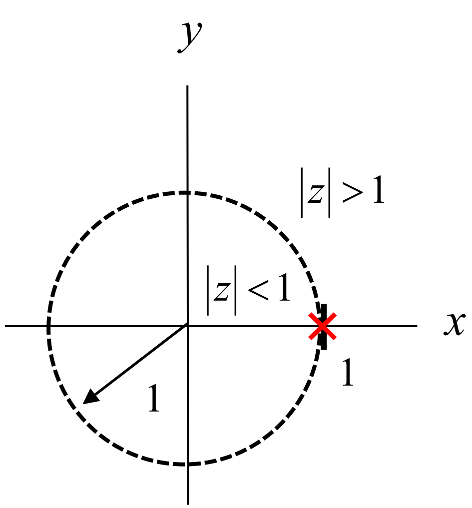
Note: $\left|z^{N+1}\right| = \left|r^{N+1} e^{i\theta(N+1)}\right| = r^{N+1} \xrightarrow{N\to\infty} 0$ **iff** $r = |z| < 1$

Geometric Series (cont.)

Hence, we have:

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots = \frac{1}{1-z} \;,\; |z| < 1$$

(summation of geometric series)



Complex plane for $f(z)$

We can write this as:

$$f(z) \equiv \frac{1}{1-z} = 1 + z + z^2 + \cdots \;,\; |z| < 1$$

(a power series expansion of the function $f(z)$)

The power series for $f(z)$ converges inside the unit circle and diverges outside the unit circle. It oscillates (does not converge) on the unit circle.

Geometric Series (cont.)

For $|z| > 1$ we can use another series representation:

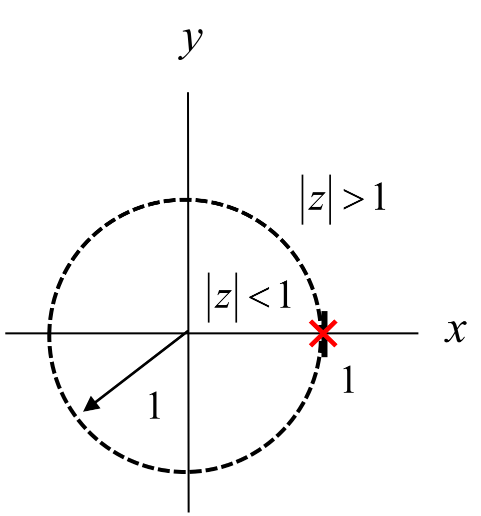
$$\frac{1}{1-z} = \frac{-1}{z\left(1-\frac{1}{z}\right)} \overset{\text{GS}}{=} -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\cdots\right)$$

$$= \left(-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}\cdots\right)$$

This is the geometric series with $z \to 1/z$.

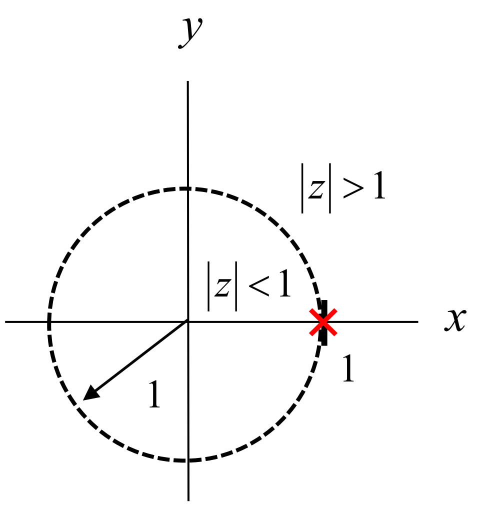
This series converges iff

$$\left|\frac{1}{z}\right| < 1, \quad \textbf{i.e.,} \quad |z| > 1$$



Geometric Series (cont.)

Summary of Geometric Series Results



$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n \,, \quad |z| < 1$$

(Geometric series)

$$\frac{1}{1-z} = -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \cdots\right) = -\frac{1}{z}\sum_{n=0}^{\infty} z^{-n} \,, \quad |z| > 1$$

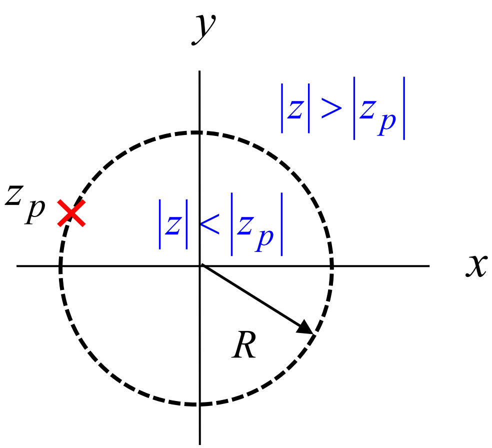
This is an extension (or “continuation”) of the original geometric series.

The geometric series is important by itself, and it is also useful for later derivations.

Geometric Series (cont.)

Generalize: (pole at z_p):

$$\frac{1}{z-z_p}=\frac{-1}{z_p\left(1-\dfrac{z}{z_p}\right)}=-\frac{1}{z_p}\left[1+\left(\frac{z}{z_p}\right)+\left(\frac{z}{z_p}\right)^2+\left(\frac{z}{z_p}\right)^3+\cdots\right]$$ **if** $\left|\dfrac{z}{z_p}\right|<1$ **, i.e.** $|z|<|z_p|$



Taylor series (↑)

Laurent series (↓)

$$\frac{1}{z-z_p}=\frac{1}{z\left(1-\dfrac{z_p}{z}\right)}=\frac{1}{z}\left[1+\left(\frac{z_p}{z}\right)+\left(\frac{z_p}{z}\right)^2+\left(\frac{z_p}{z}\right)^3+\cdots\right]$$ **if** $\left|\dfrac{z_p}{z}\right|<1$ **, i.e.** $|z|>|z_p|$

Geometric Series (cont.)

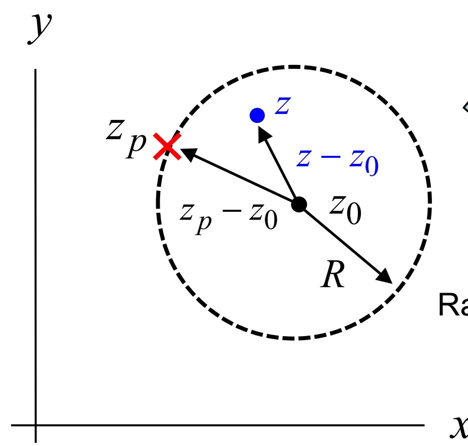
The previous series were expanded about the origin. We can also expand about another point.

$$\frac{1}{z-z_p}=\frac{1}{(z-z_0)-(z_p-z_0)}=\frac{-1}{(z_p-z_0)\left(1-\dfrac{z-z_0}{z_p-z_0}\right)}=-\frac{1}{(z_p-z_0)}\left[1+\left(\frac{z-z_0}{z_p-z_0}\right)+\left(\frac{z-z_0}{z_p-z_0}\right)^2+\left(\frac{z-z_0}{z_p-z_0}\right)^3+\cdots\right]$$

Valid if $\left|\dfrac{z-z_0}{z_p-z_0}\right|<1$ **, i.e.** $|z-z_0|<|z_p-z_0|$

$|z-z_0|<|z_p-z_0|$

This is the case illustrated here.



Radius of convergence :

$R=|z_p-z_0|$

Decide whether $|z_p\text{-}z_0|$ or $|z\text{-}z_0|$ is larger (i.e., if z is inside or outside the circle at right), and then factor out the term with largest magnitude!

Similarly,

$$\frac{1}{z-z_p}=\frac{1}{(z-z_0)-(z_p-z_0)}=\frac{1}{(z-z_0)\left(1-\dfrac{z_p-z_0}{z-z_0}\right)}=\frac{1}{(z-z_0)}\left[1+\left(\frac{z_p-z_0}{z-z_0}\right)+\left(\frac{z_p-z_0}{z-z_0}\right)^2+\left(\frac{z_p-z_0}{z-z_0}\right)^3+\cdots\right]$$

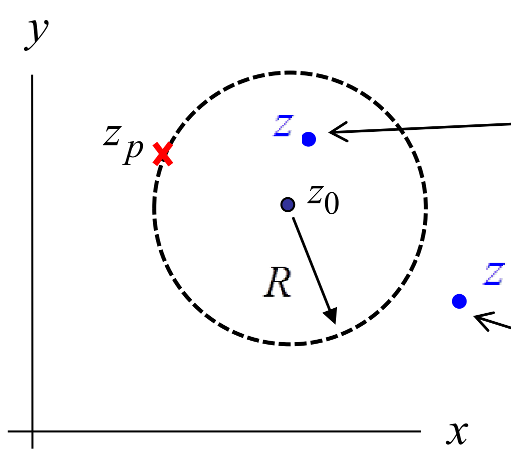
Valid if $\left|\dfrac{z_p-z_0}{z-z_0}\right|<1$ **, i.e.** $|z-z_0|>|z_p-z_0|$

Geometric Series (cont.)

Summary of Series

$$\frac{1}{z-z_p} = -\frac{1}{(z_p-z_0)}\left[1+\left(\frac{z-z_0}{z_p-z_0}\right)+\left(\frac{z-z_0}{z_p-z_0}\right)^2+\left(\frac{z-z_0}{z_p-z_0}\right)^3+\cdots\right] \quad \textbf{if } |z-z_0|<|z_p-z_0|$$

Taylor series



Converges <u>inside</u> circle

Converges <u>outside</u> circle

Laurent series

$$\frac{1}{z-z_p} = \frac{1}{(z-z_0)}\left[1+\left(\frac{z_p-z_0}{z-z_0}\right)+\left(\frac{z_p-z_0}{z-z_0}\right)^2+\left(\frac{z_p-z_0}{z-z_0}\right)^3+\cdots\right] \quad \textbf{if } |z-z_0|>|z_p-z_0|$$

Uniform Convergence

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots, \quad |z| < 1$$

Let's evaluate the geometric series for $z = 10^{-3} + i0,\ 10^{-2} + i0,\ 10^{-1} + i0$.

$z = 10^{-3} + i0$:

$$\frac{1}{1-10^{-3}} = 1.00 + 0.001 + 0.000001 + 0.000000001 + \cdots$$
$$= 1.001001001001001\cdots$$

Clearly, every additional term adds 3 more significant figures to the final result.

$z = 10^{-2} + i0$:

$$\frac{1}{1-10^{-2}} = 1.00 + 0.01 + 0.0001 + 0.000001 + \cdots$$
$$= 1.0101010101\cdots$$

Here, however, each additional term adds only 2 more significant figures to the result.

$z = 10^{-1} + i0$:

$$\frac{1}{1-10^{-1}} = 1.00 + 0.1 + 0.01 + 0.001 + \cdots$$
$$= 1.11111\cdots$$

And here each additional term adds only 1 more significant figure to the result.

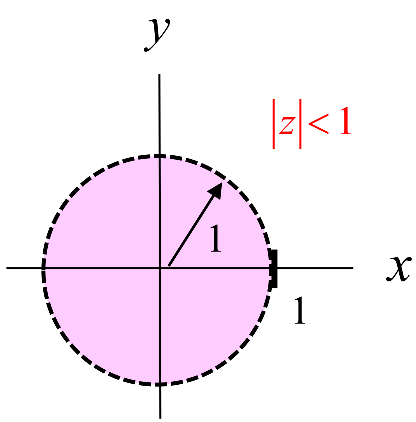
In general, for a given accuracy, the number of terms needed increases as $|z|$ get larger and approaches 1. The series is said to converge non-uniformly in the region $|z| < 1$.

Uniform Convergence (cont.)

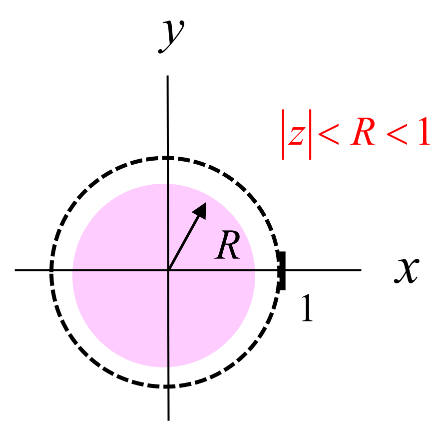
- A series $f(z)=\sum_{n=0}^{\infty} g_n(z)$ is *uniformly convergent* in a region $\mathcal{R}$ if for any $\varepsilon > 0$, there exists a number N_ε, dependent on ε but *independent of* z in the region, such that $N > N_\varepsilon$ implies $\left| f(z) - \sum_{n=0}^{N} g_n(z) \right| < \varepsilon$ for all z in $\mathcal{R}$.

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

The series converges slower and slower as $|z|$ approaches 1.



Non-uniform convergence



Uniform convergence

Key Point:
Term-by-term integration of a series (switching the order of summation and integration) is allowed over any region where it is uniformly convergent. We use this property extensively later!

Uniform Convergence (cont.)

Example

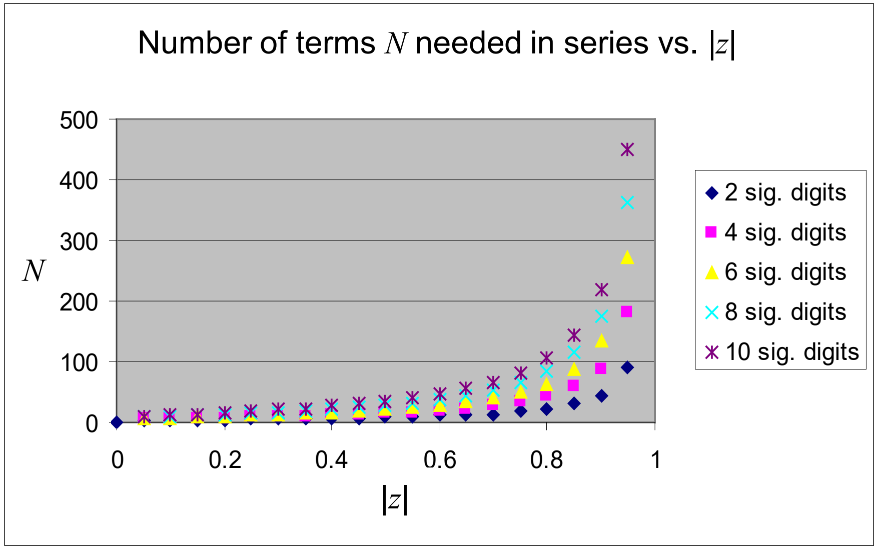
$$S = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \;,\; |z < 1|$$

$$S_N = 1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1-z}$$

The partial sum error is $S - S_N = e_N = \frac{1}{1-z} - \frac{1-z^{N+1}}{1-z} = \left(\frac{1}{1-z}\right) z^{N+1} = S\, z^{N+1}$

The relative error is $\varepsilon_{\text{rel}} = \left|\frac{S - S_N}{S}\right| = |z|^{N+1}$

Note:
A relative error of 10^{-p} means p significant figures of accuracy.

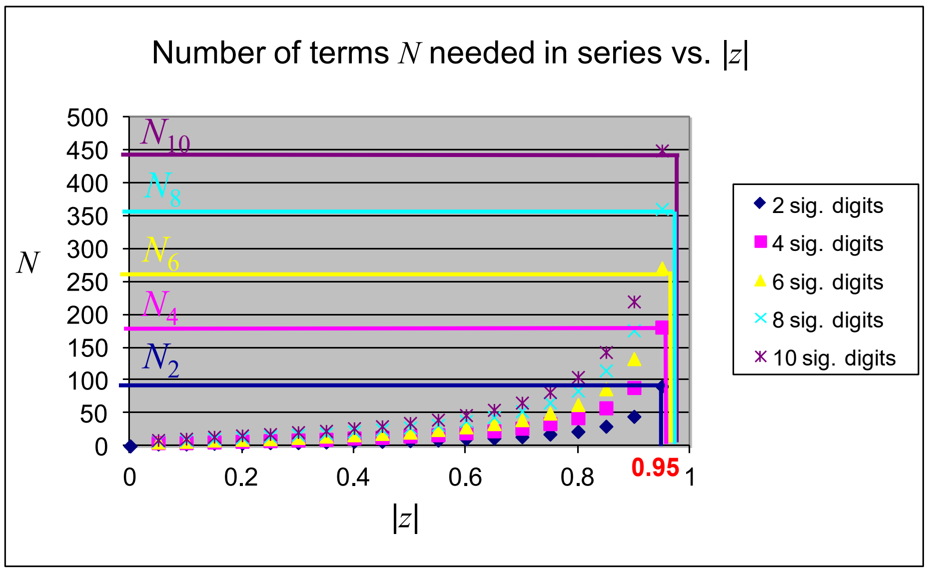


The closer z gets to the boundary of the circle, the more terms we need to get the same level of accuracy (<u>non-uniform</u> convergence).

Uniform Convergence (cont.)

Example (cont.) Assume: $|z| < R < 1$

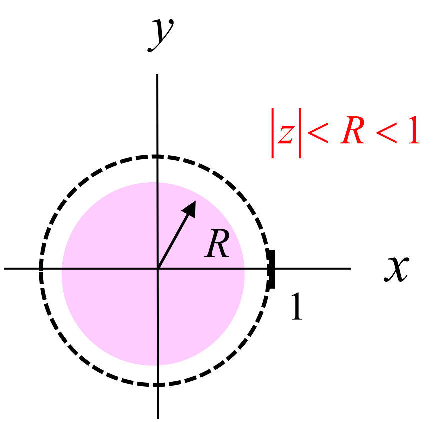
The relative error is $\varepsilon_{\text{rel}} = \left|\dfrac{S - S_N}{S}\right| = |z|^{N+1} < R^{N+1}$



For example:

$R = 0.95$

$\varepsilon_{\text{rel}} < R^{N+1} = (0.95)^{N+1}$



Using $N = 350$ will give 8 significant figures <u>everywhere</u> inside the region.

We now have <u>uniform</u> convergence for $R = 0.95$.

Uniform Convergence (cont.)

Example

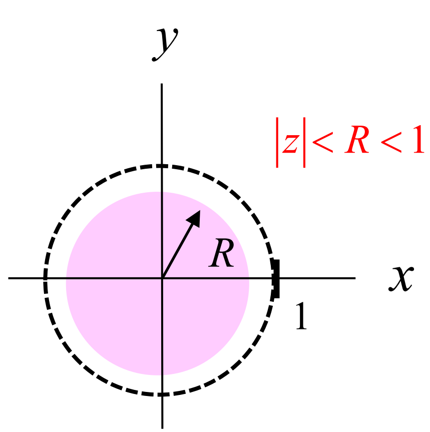
$$\int_0^z \frac{1}{1-z}\,dz, \quad |z| < R < 1$$

$$\int_0^z \frac{1}{1-z}\,dz = \int_0^z \left(\sum_{n=0}^{\infty} z^n\right) dz = \sum_{n=0}^{\infty}\left(\int_0^z z^n\,dz\right)$$

↑ Switch the order of integration and summation (integrate term-by-term)

Hence, we have

$$\int_0^z \frac{1}{1-z}\,dz = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$



Note: This is not valid for $|z| \le 1$.

(We do not have uniform convergence there.)

The Taylor Series Expansion

This expansion assumes we have a function $f(z)$ that is analytic in a disk.

Summary of Taylor Series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

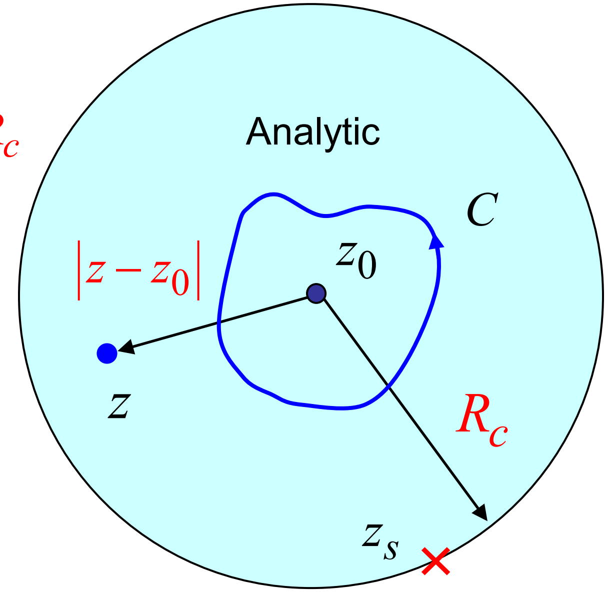
$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

"derivative formula"

Note: Both forms are useful.

$|z - z_0| < R_c$



Here z_s is the closest singularity to z_0.

The path C is any counterclockwise closed path within the disk that encircles the point z_0.

R_c = radius of convergence of the Taylor series (distance to closest singularity)

The Taylor series will converge within the radius of convergence and diverge outside.

The Taylor Series Expansion (cont.)

Taylor's theorem is named after the mathematician Brook Taylor, who stated a version of it in 1712. Yet an explicit expression of the error was not provided until much later on by Joseph-Louis Lagrange. An earlier version of the result was already mentioned in 1671 by James Gregory.

Brook Taylor (1685-1731)

From Wikipedia

The Laurent Series Expansion (cont.)

Review of Cauchy's Integral Formula (from Notes 3):

$$f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

Another way to write it:

$$f(z)=\frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}\,dz'$$

Take derivative n times.

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$

Taylor Series Expansion of an Analytic Function (cont.)

- **Write the Cauchy integral formula in the form**

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)-(z-z_0)}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)\left[1-\dfrac{z-z_0}{z'-z_0}\right]}dz' \quad \left(|z'-z_0|>|z-z_0|\right)$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n dz'$$

uniform convergence

$$= \sum_{n=0}^{\infty}(z-z_0)^n \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

derivative formulas

$$= \sum_{n=0}^{\infty}(z-z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

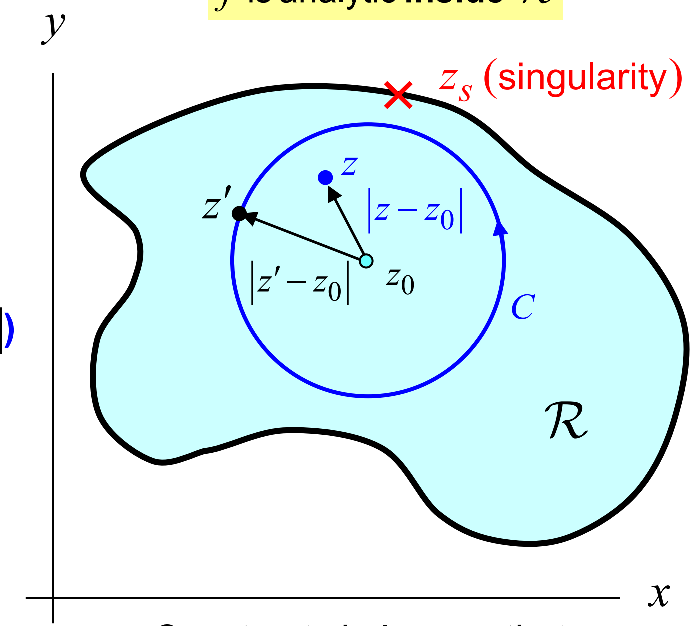
$\left(\textbf{recall } f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)$

$$\Rightarrow \boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n} \Leftrightarrow$$ **Taylor series expansion of** $f(z)$ **about** z_0

where

$$\boxed{a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz' = \frac{f^{(n)}(z_0)}{n!}}$$

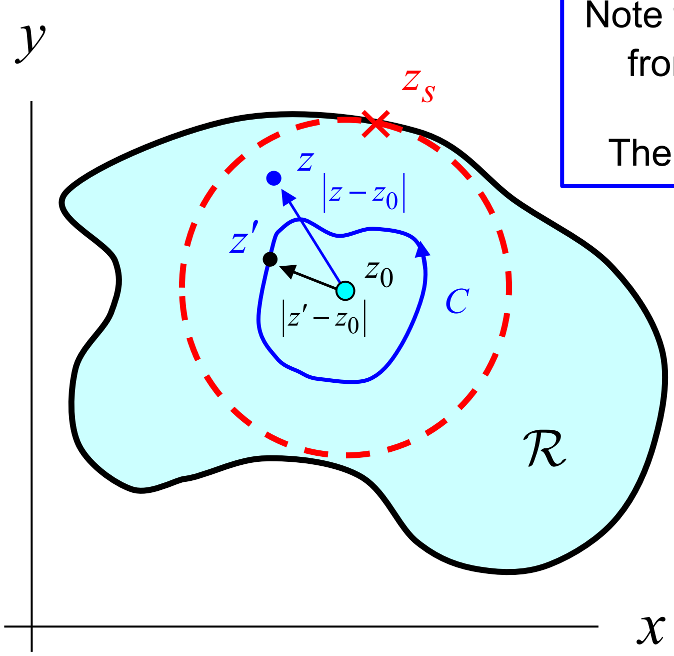
f is analytic **inside** $\mathcal{R}$



Construct circle C so that

$$|z-z_0| < |z'-z_0| < |z_s-z_0|$$

Taylor Series Expansion of an Analytic Function (cont.)



Note that in the result for a_n, the integrand is analytic inside $\mathcal{R}$ away from z_0, and hence (from Cauchy's theorem) the path C is now arbitrary, as long as it encircles z_0 and stays inside $\mathcal{R}$.
The point z can even be outside the path C (see the note below).

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

(Note: This integral does not have z in it.)

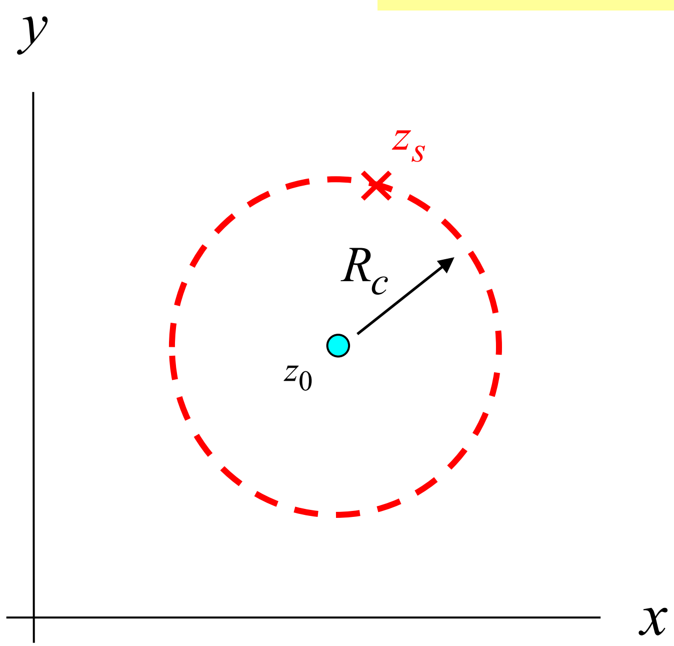
- **Note that the result is valid for any** $|z - z_0| < |z_s - z_0|$ **where** z_s **is the singularity nearest** z_0 **; hence the series will converge if**

$$|z - z_0| < |z_s - z_0|$$

Note: It can also be shown that the series will diverge for $|z - z_0| > |z_s - z_0|$

Taylor Series Expansion of an Analytic Function (cont.)

The radius of convergence of a Taylor series is the distance out to the closest singularity.



Key point:
The point z_0 about which the expansion is made is *arbitrary*, but It determines the region of convergence of the Taylor series.

$$\text{Converges for: } |z - z_0| < R_c \equiv |z_s - z_0|$$

Taylor Series Expansion of an Analytic Function (cont.)

Properties of Taylor Series

R_c = radius of convergence = distance to closest singularity

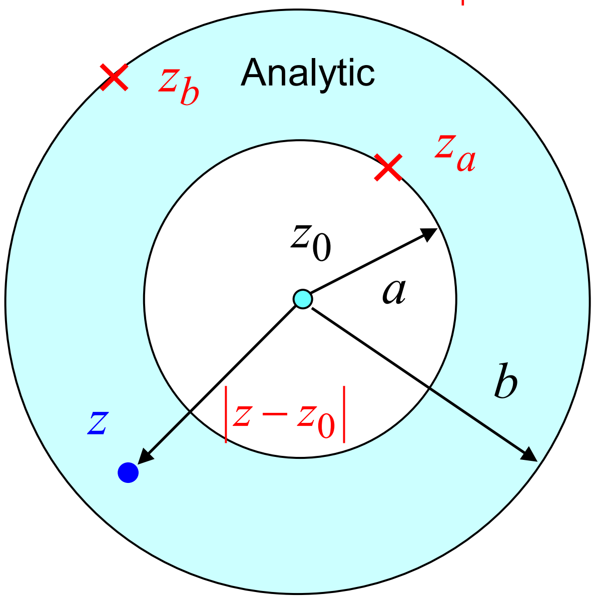
- A Taylor series will converge for $|z-z_0| < R_c$ (i.e., inside the radius of convergence).
- A Taylor series will diverge for $|z-z_0| > R_c$ (i.e., outside the radius of convergence).
- A Taylor series *converges uniformly* for $|z-z_0| \leq R < R_c$.
- A Taylor series may be differentiated or integrated term-by-term within the radius of convergence. This does not change the radius of convergence.
- A Taylor series *converges absolutely* inside the radius of convergence (i.e., the series of absolute values converges).
- Within the common region of convergence, we can add and multiply Taylor series, collecting terms to find the resulting Taylor series.
- When a Taylor series converges, the resulting function is an analytic function.

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 9th Ed., McGraw-Hill, 2013.

The Laurent Series Expansion

This generalizes the concept of a Taylor series to include cases where the function is analytic in an annulus.

$$a < |z - z_0| < b$$



Final Result:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

or

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}$$

where $b_n = a_{-n}$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$ (derived later)

(This is the same formula as for the Taylor series, but with negative n allowed.)

Here z_a and z_b are two singularities.

Note:
The point z_a may be the point z_0.
The point z_b may be at infinity.

The path C is any counterclockwise closed path that stays inside the annulus and encircles the point z_0.

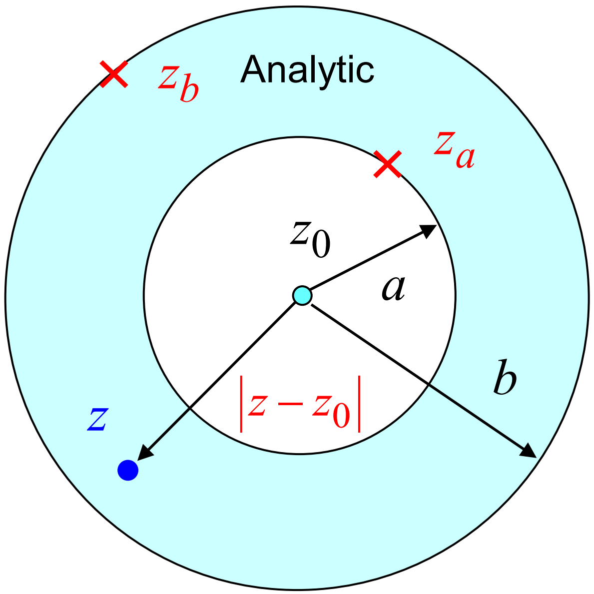
Note: We no longer have the "derivative formula" as we do for a Taylor series.

The Laurent Series Expansion (cont.)

Summary of Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$



The Laurent series converges <u>inside</u> the region

$$a < |z - z_0| < b$$

The Laurent series diverges <u>outside</u> this region if there are singularities at $|z - z_0| = a$ and b.

The Laurent Series Expansion (cont.)

The Laurent series was named after and first published by Pierre Alphonse Laurent in 1843. Karl Weierstrass may have discovered it first in a paper written in 1841, but it was not published until after his death.

Pierre Alphonse Laurent (1813 -1854)

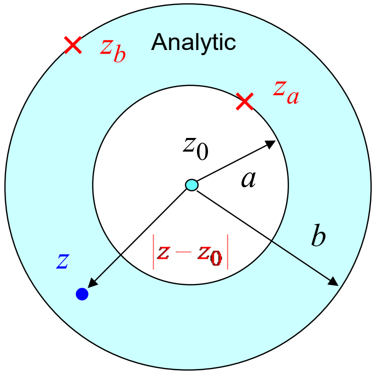
From Wikipedia

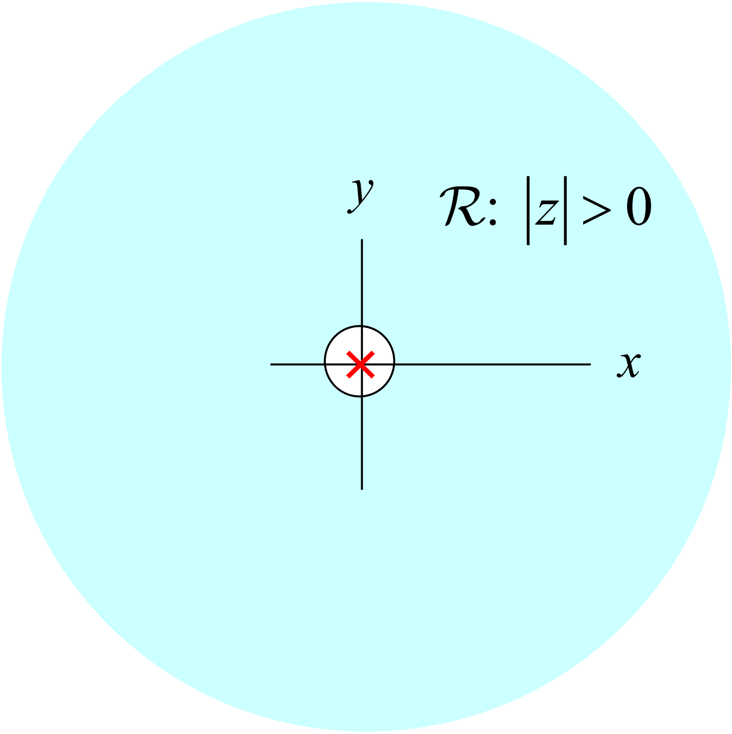
The Laurent Series Expansion (cont.)

The Laurent series is particularly useful for functions that have poles.

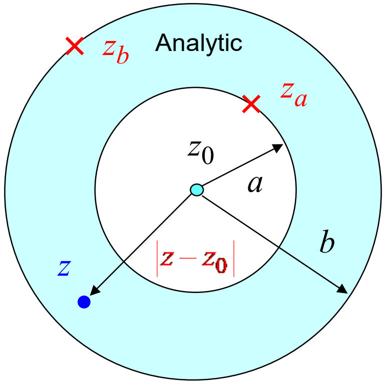
Examples of functions with poles, and how we can choose a Laurent series:

$$f(z) = \frac{1}{z} \qquad \left(\text{Choose } z_0 = 0\text{:}\;\; a = 0,\; b < \infty\right)$$



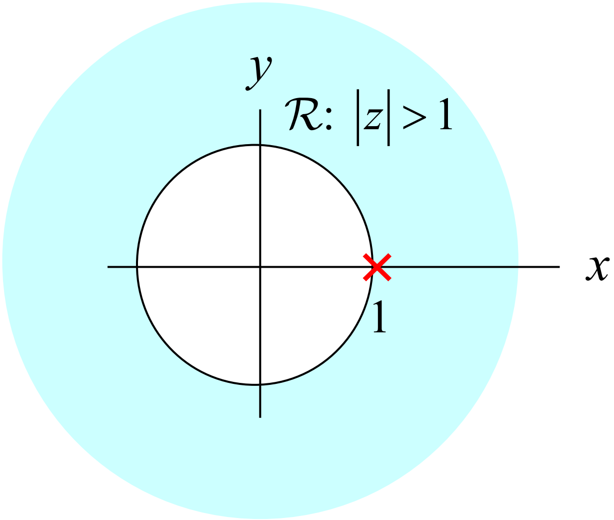


The Laurent Series Expansion (cont.)

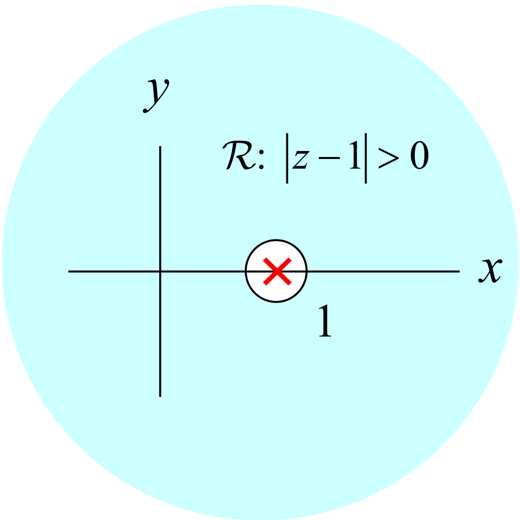


Examples of functions with poles, and how we can choose a Laurent series:

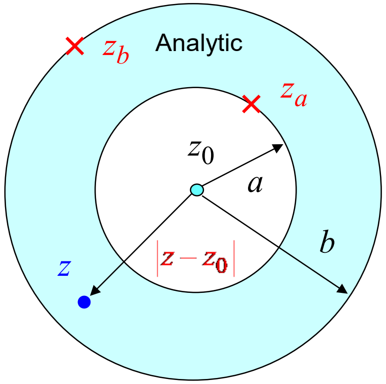
$$f(z)=\frac{z}{z-1} \quad (\text{Choose } z_0=0:\ a=1,\ b<\infty)$$



$$f(z)=\frac{z}{z-1} \quad (\text{Choose } z_0=1:\ a=0,\ b<\infty)$$



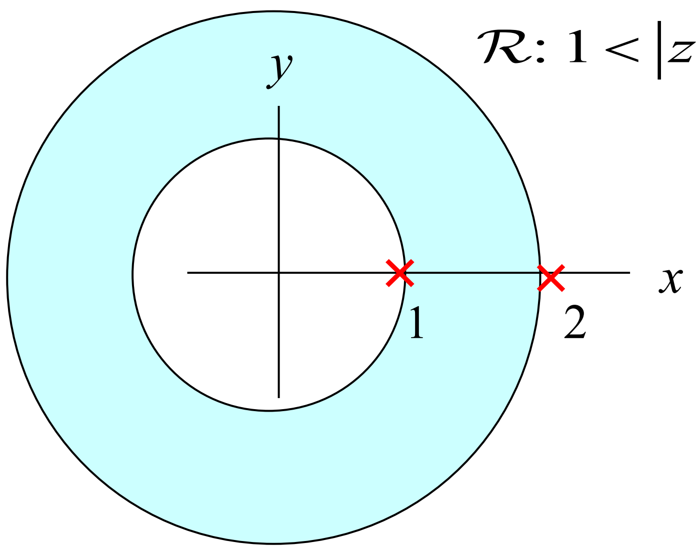
The Laurent Series Expansion (cont.)



Examples of functions with poles, and how we can choose a Laurent series:

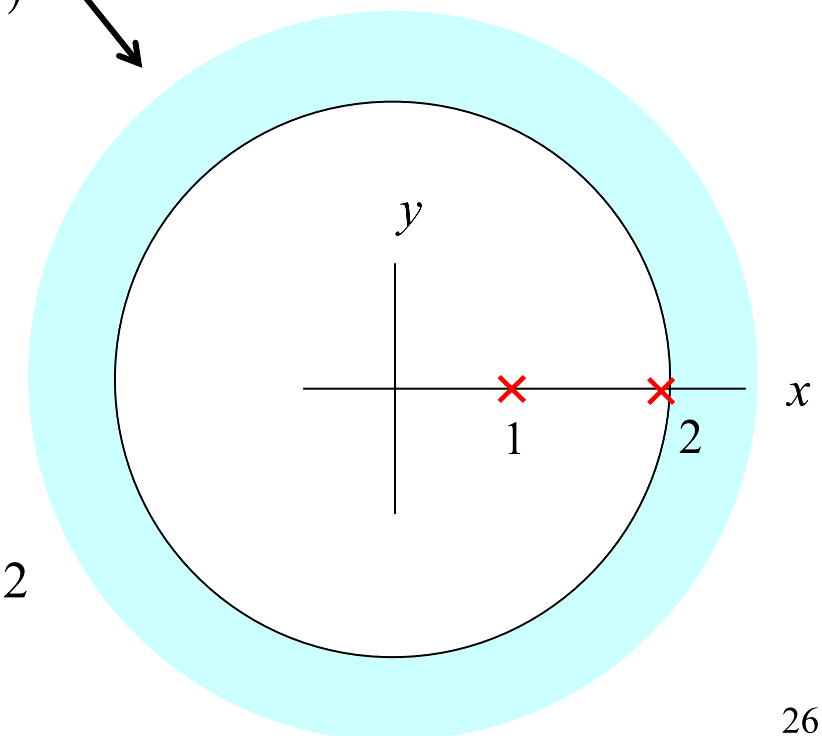
$$f(z) = \frac{z}{(z-1)(z-2)} \qquad \left(\text{Choose } z_0 = 0\text{: } a = 1,\ b = 2\right)$$

$\mathcal{R}: 1 < |z| < 2$



$$f(z) = \frac{z}{(z-1)(z-2)} \qquad \left(\text{Choose } z_0 = 0\text{: } a = 2,\ b < \infty\right)$$

$\mathcal{R}: |z| > 2$



The Laurent Series Expansion (cont.)

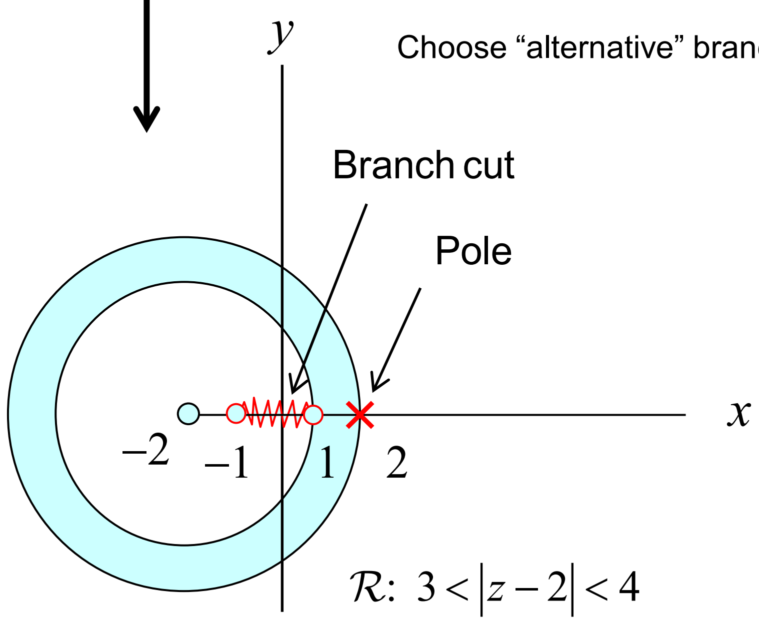
Example: The singularity does not have to be a pole.

$$f(z)=\frac{z}{(z-2)\left(z^2-1\right)^{1/2}} \qquad \left(\text{Choose } z_0=-2:\ a=3,\ b=4\right)$$

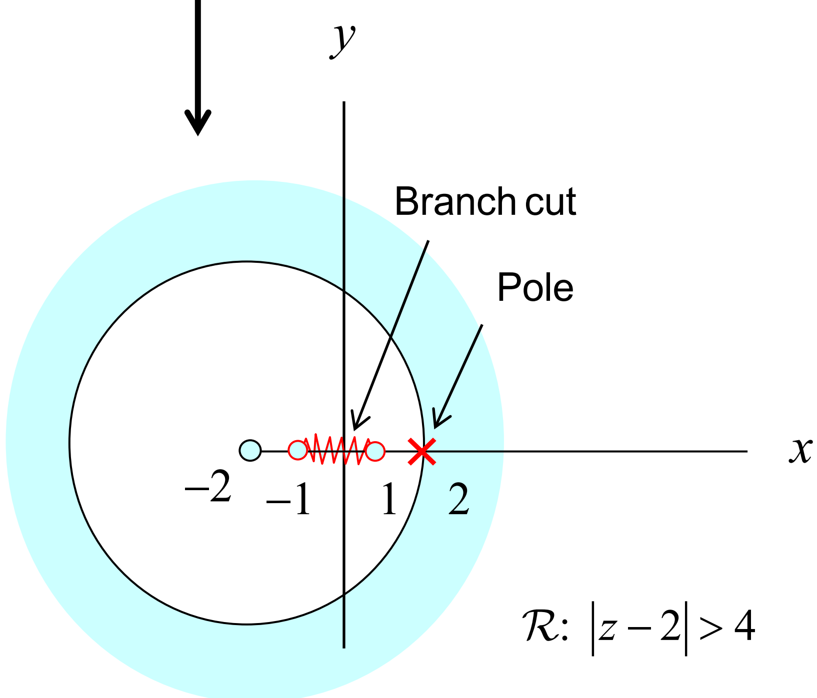
$z_0=-2$

$\left(\text{or choose } z_0=-2:\ a=4,\ b<\infty\right)$

Choose “alternative” branch cut.



$\mathcal{R}:\ 3<|z-2|<4$



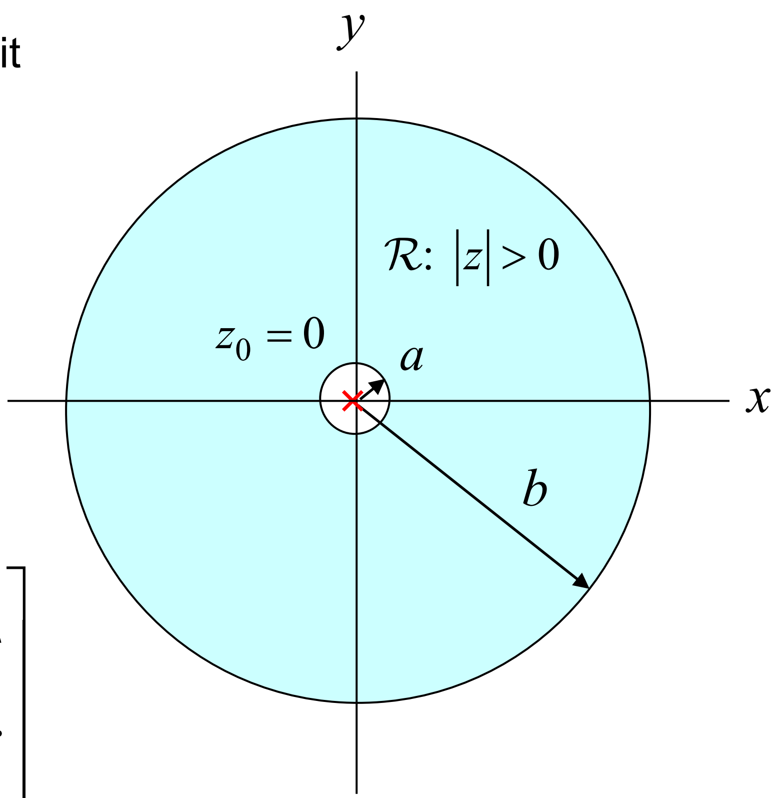
$\mathcal{R}:\ |z-2|>4$

The Laurent Series Expansion (cont.)

Theorem:
The Laurent series expansion in the annulus region is unique.

(So it doesn't matter how we get it; once we obtain it by any series of valid steps, it is correct!)

This is justified by our Laurent series expansion formula, derived later.



Example:

$$f(z) = \frac{\cos(z)}{z} \qquad (z_0 = 0,\ a = 0,\ b < \infty)$$

$$\Rightarrow f(z) = \underbrace{\frac{1}{z}}_{\text{analytic for } |z|>0} \left[\underbrace{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots}_{\text{valid for } |z|<\infty} \right]$$

Hence $f(z) = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} \ldots, \quad 0 < |z| < \infty$

The Laurent Series Expansion (cont.)

A Taylor series is a special case of a Laurent series.

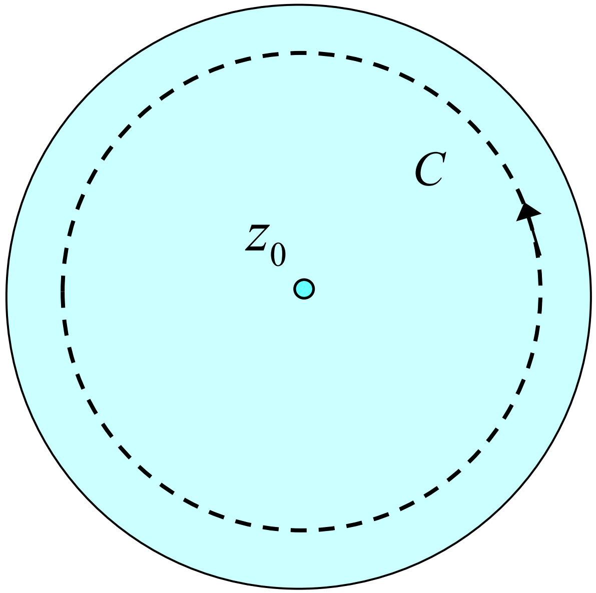
Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

If $f(z)$ is analytic:

⇨ $a_n = 0, \ n = -1, -2, -3 \ldots$



Here f is assumed to be analytic within C.

If $f(z)$ is analytic within C, the integrand is analytic for negative values of n.
Hence, all coefficients a_n for negative n become zero (by Cauchy's theorem).

The Laurent Series Expansion (cont.)

Derivation of Laurent Series

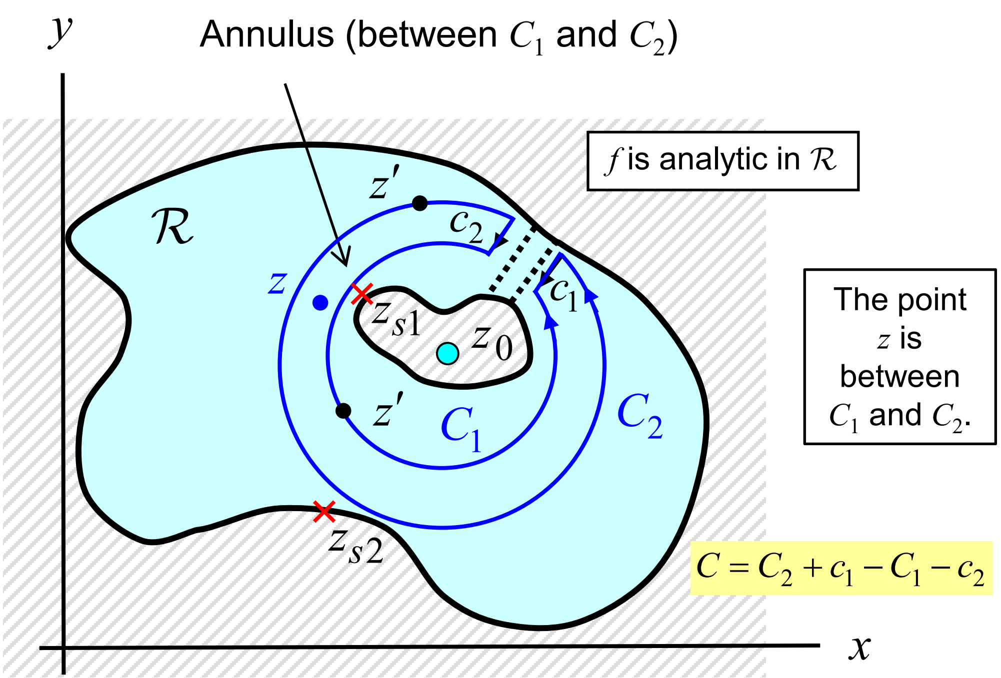
We use the "bridge" principle again

Pond, island, & bridge

Pond: Domain of analyticity
Island: Region containing singularities
Bridge: Region connecting island and boundary of pond

The Laurent Series Expansion (cont.)

Contributions from the paths c_1 and c_2 cancel!



The point z is between C_1 and C_2.

- **By Cauchy's Integral Formula,**

$$f(z) = \frac{1}{2\pi i} \oint_{C_2 + c_1 - c_2 - C_1} \frac{f(z')}{z' - z} dz'$$

$$= \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z' - z} dz'$$

On C_2, $|z' - z_0| > |z - z_0|$,

$$\Rightarrow \frac{1}{z' - z} = \frac{1}{(z' - z_0) - (z - z_0)} = \frac{1}{(z' - z_0)\left(1 - \frac{z - z_0}{z' - z_0}\right)} = \frac{1}{z' - z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z' - z_0)^n} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z' - z_0)^{n+1}}$$

On C_1, $|z' - z_0| < |z - z_0|$ (note the convergence regions for C_1, C_2 overlap!)

$$\Rightarrow \frac{1}{z' - z} = \frac{1}{(z' - z_0) - (z - z_0)} = \frac{-1}{(z - z_0)\left(1 - \frac{z' - z_0}{z - z_0}\right)} = -\sum_{n=0}^{\infty} \frac{(z' - z_0)^n}{(z - z_0)^{n+1}} \overset{\substack{n \to -n'-1, \\ n' \to n}}{=} -\sum_{n=-1}^{-\infty} \frac{(z - z_0)^n}{(z' - z_0)^{n+1}}$$

The Laurent Series Expansion (cont.)

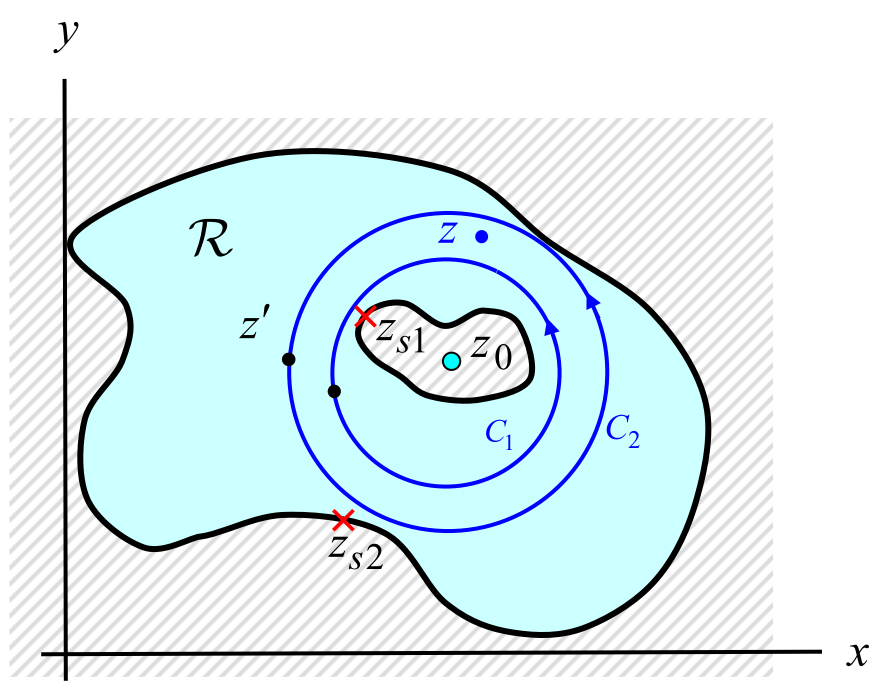
Hence,

$$f(z)=\frac{1}{2\pi i}\oint_{C_2+\cancel{C_1}-\cancel{C_2}-C_1}\frac{f(z')}{z'-z}dz'$$

uniform convergence

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_2}\frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

$$-\frac{1}{2\pi i}\left(-\sum_{n=-1}^{-\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)$$

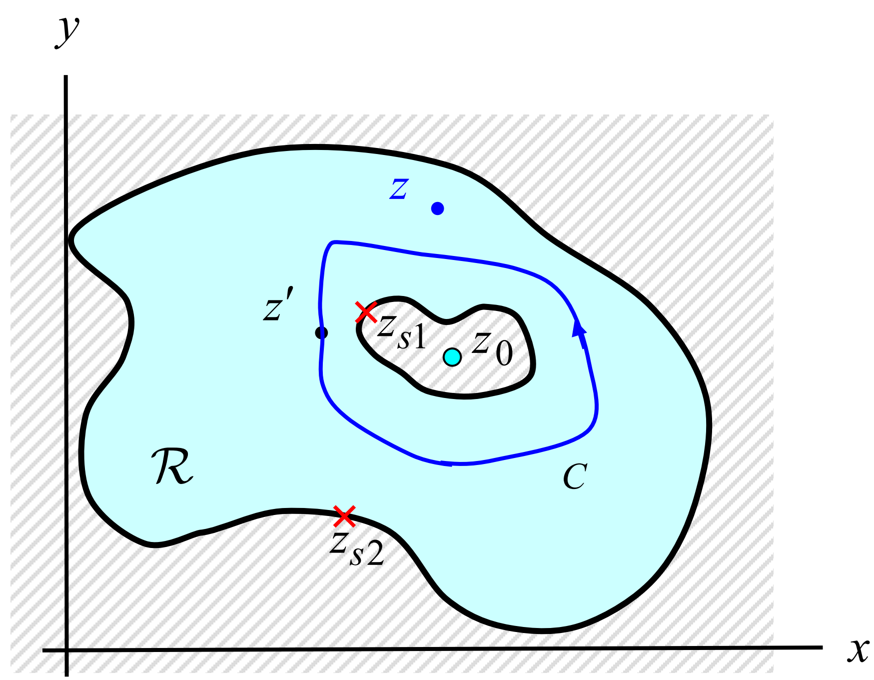


Note: The integrands are analytic in the blue region.

Because the integrands for the coefficient are analytic with $\mathcal{R}$ (the blue region), the paths are arbitrary, as long as they stay within $\mathcal{R}$. We can use the same path C, which is arbitrary as long as it stays within $\mathcal{R}$.

The Laurent Series Expansion (cont.)

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' + \frac{1}{2\pi i} \sum_{n=-1}^{-\infty} (z - z_0)^n \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$



We thus have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

The path C is now arbitrary, as long as it stays in the analytic (blue) region $\mathcal{R}$.

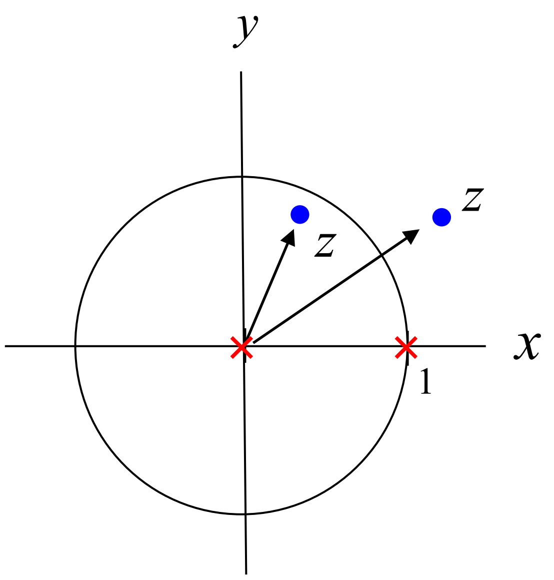
Examples of Taylor and Laurent Series Expansions

Example 1: Obtain **all** expansions of $f(z) = \dfrac{1}{z(z-1)}$ about the origin.

Use the integral formula for the a_n coefficients.

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

$$z_0 = 0$$



The path C can be inside the circle or outside of it (parts (a) and (b)).

Examples of Taylor and Laurent Series Expansions

a) Laurent series with $a = 0,\ b = 1$

From geometric series

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'^{\,n+1}}dz' = \frac{1}{2\pi i}\oint_C \frac{1}{(z'-1)z'^{\,n+2}}dz' = \frac{1}{2\pi i}\oint_C \frac{-1}{z'^{\,n+2}}\sum_{m=0}^{\infty} z'^m dz', \quad (|z'|<1)$$

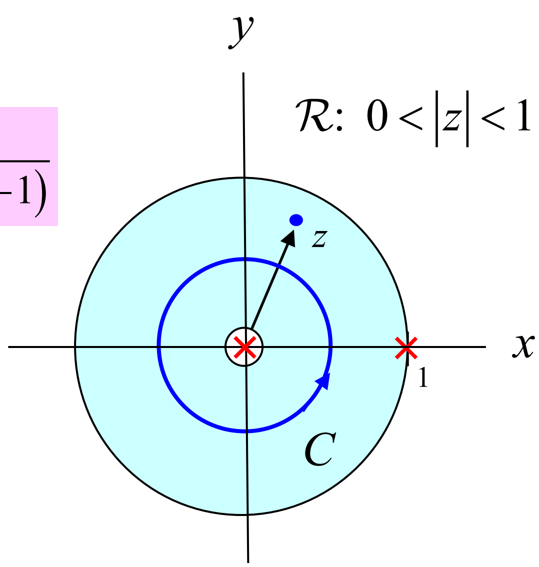
$$= \frac{1}{2\pi i}\oint_C \sum_{m=0}^{\infty} \frac{-1}{z'^{\,n-m+2}}\, dz' = \frac{-1}{2\pi i}\sum_{m=0}^{\infty}\oint_C \frac{1}{z'^{\,n-m+2}}\, dz' = \frac{-1}{2\pi i}\cdot\begin{cases} 2\pi i, & m = n+1 \quad (n-m+2=1) \\ 0, & m \neq n+1 \quad (n-m+2 \neq 1)\end{cases}$$

From uniform convergence   From previous example in Notes 3

$= -1$ (for $n+1 = m \geq 0$)

$\Rightarrow$ $a_n = -1, \quad n \geq -1$

$$f(z) = \frac{1}{z(z-1)}$$



Hence

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots, \quad 0<|z|<1$$

The path C is inside the blue region.

Examples of Taylor and Laurent Series Expansions (cont.)

b) Laurent series with $a=1,\ b=\infty$

From geometric series

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'^{\,n+1}}dz' = \frac{1}{2\pi i}\oint_C \frac{1}{(z'-1)z'^{\,n+2}}dz' = \frac{1}{2\pi i}\oint_C \frac{1}{\left(1-\frac{1}{z'}\right)z'^{\,n+3}}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{1}{z'^{\,n+3}}\sum_{m=0}^{\infty}\frac{1}{z'^{\,m}}dz', \quad (|z'|>1)$$

From previous example in Notes 3

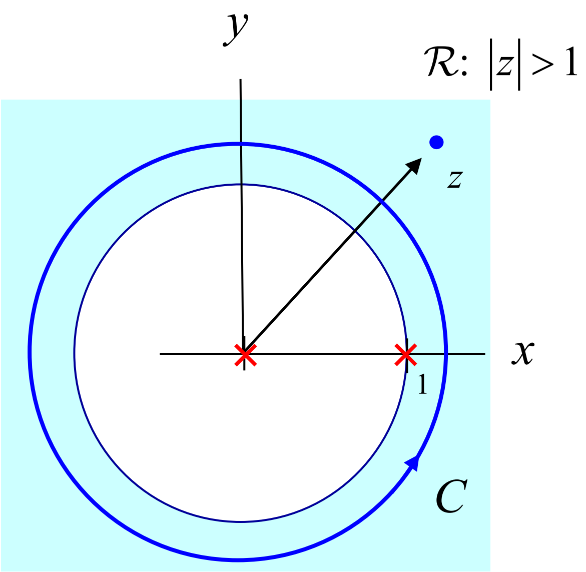
$$= \frac{1}{2\pi i}\oint_C \sum_{m=0}^{\infty}\frac{1}{z'^{\,n+m+3}}dz' = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint_C \frac{1}{z'^{\,n+m+3}}dz' = \frac{1}{2\pi i}\cdot\begin{cases} 2\pi i, & m=-n-2 \quad (n+m+3=1) \\ 0, & m\neq -n-2 \quad (n+m+3\neq 1)\end{cases}$$

From uniform convergence

$$=1 \quad (\text{for } -n-2=m\geq 0)$$

$$\Rightarrow \quad a_n = 1, \quad n\leq -2$$

$$f(z)=\frac{1}{z(z-1)}$$



Hence

$$f(z)=\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots, \quad |z|>1$$

The path C is <u>outside</u> the blue region.

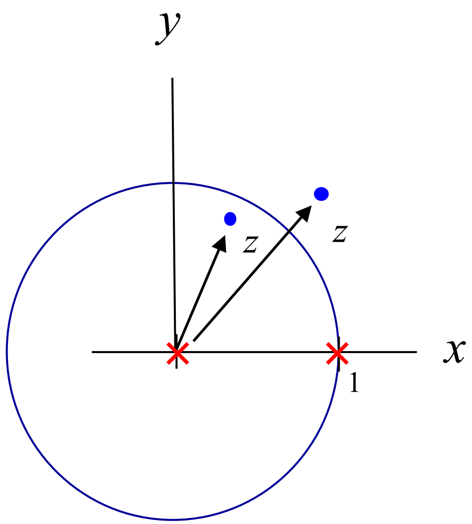
Examples of Taylor and Laurent Series Expansions (cont.)

Summary of results for the example:

$$f(z)=\frac{1}{z(z-1)}$$

$$f(z)=-\frac{1}{z}-1-z-z^{2}-z^{3}-\cdots,\ \ 0<|z|<1$$

$$f(z)=\frac{1}{z^{2}}+\frac{1}{z^{3}}+\frac{1}{z^{4}}+\cdots,\ \ |z|>1$$



Examples of Taylor and Laurent Series Expansions (cont.)

Note:
Often it is easier to directly use the geometric series (GS) formula together with partial fraction expansions and some algebra, instead of the contour integral approach, to determine the coefficients of the Laurent expansion.

This is illustrated next (using the same example as in Example 1).

Examples of Taylor and Laurent Series Expansions (cont.)

◊ **Example 1**

Expand $f(z)=\dfrac{1}{z(z-1)}$ about the origin (we use partial fractions and GS):

$$f(z)=\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

$$0<|z|<1$$

$$A=\lim_{z\to 0} z\,f(z)=\lim_{z\to 0}\frac{\cancel{z}}{\cancel{z}(z-1)}=-1$$

$$B=\lim_{z\to 1}(z-1)\,f(z)=\lim_{z\to 1}\frac{\cancel{z-1}}{z\cancel{(z-1)}}=1$$

$$\Rightarrow f(z)=\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1} = \frac{-1}{z} - \frac{1}{1-z}$$

$$= \frac{-1}{z} - \left(1+z+z^2+\dots\right)$$

Hence

$$f(z) = -\frac{1}{z}-1-z-z^2-z^3-\cdots,\ 0<|z|<1$$

Examples of Taylor and Laurent Series Expansions (cont.)

$|z| > 1$

Alternative expansion ($|z| > 1$):

$$f(z) = \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1} = \frac{-1}{z} + \frac{1}{z}\left(\frac{1}{1-1/z}\right)$$

$$\Rightarrow f(z) = \frac{-1}{z} + \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

Hence

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots, \quad |z| > 1$$

These are the same results that we got in the previous example by using the integral formula for a_n in the Laurent series recipe.

Examples of Taylor and Laurent Series Expansions (cont.)

◊ **Example 2**

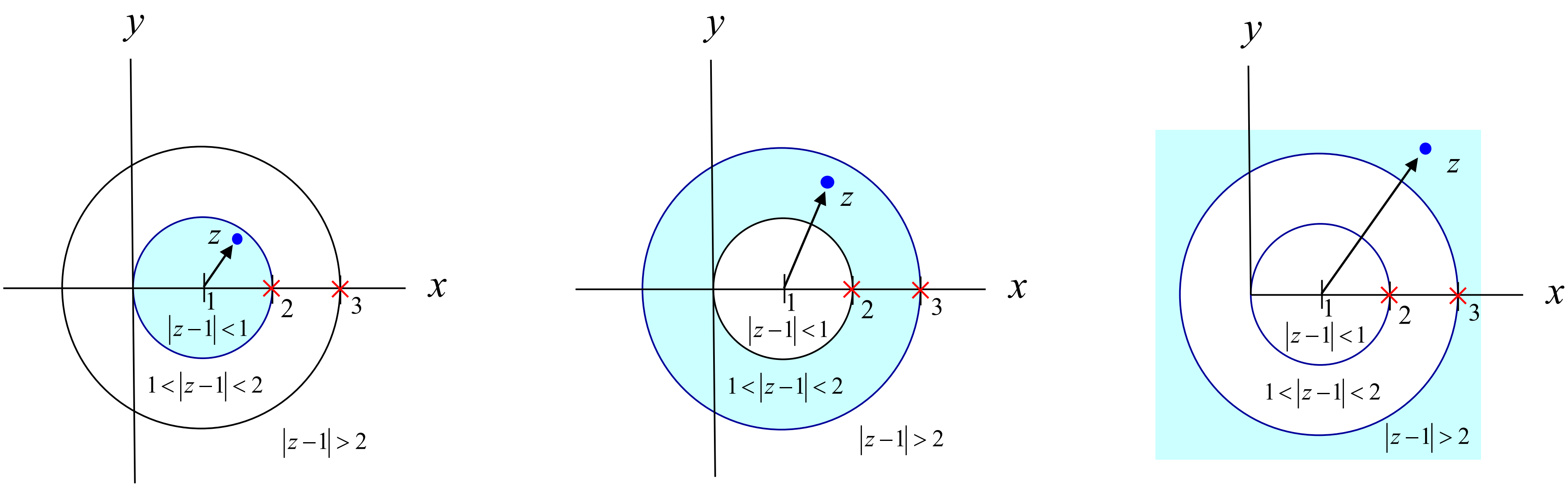
Expand $f(z) = \dfrac{1}{(z-2)(z-3)}$ in a Taylor / Laurent series.

Expand about $z_0 = 1$, valid following in the annular regions:

(a) $0 \le |z-1| < 1$,

(b) $1 < |z-1| < 2$,

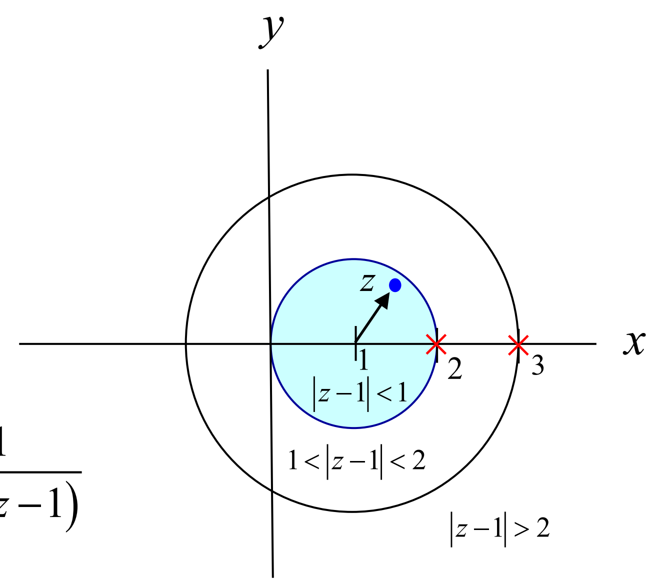
(c) $|z-1| > 2$.



Examples of Taylor and Laurent Series Expansions (cont.)

$$f(z)=\frac{1}{(z-2)(z-3)}$$

Part (a)



a) For $0 \le |z-1| < 1$:

Using partial fraction expansion and GS,

$$f(z)=\frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2}$$

$$= \frac{1}{(z-1)-2} - \frac{1}{(z-1)-1} = \frac{-1}{2\left[1-(z-1)/2\right]} + \frac{1}{1-(z-1)}$$

$$= -\frac{1}{2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]+\left[1+(z-1)+(z-1)^2+\cdots\right]$$

Hence

$$f(z) = \frac{1}{2} + \frac{3}{4}(z-1) + \frac{7}{8}(z-1)^2 + \frac{15}{16}(z-1)^3 + \cdots, \quad 0 \le |z-1| < 1$$

(Taylor series)

Examples of Taylor and Laurent Series Expansions (cont.)

Part (b)

$$f(z)=\frac{1}{(z-2)(z-3)}$$

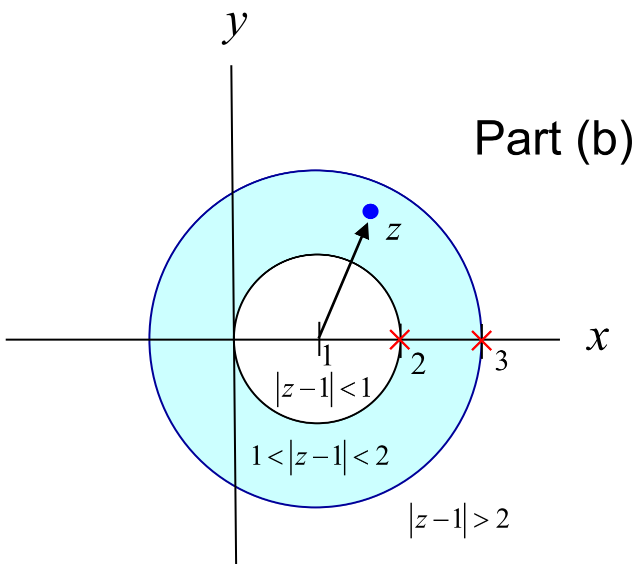
b) For $1<|z-1|<2$:

$$f(z)=\frac{1}{(z-1)-2}-\frac{1}{(z-1)-1}=\frac{-1}{2\left[1-(z-1)/2\right]}-\frac{1}{(z-1)\left[1-1/(z-1)\right]}$$

so

$$f(z)=-\frac{1}{2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]-\frac{1}{(z-1)}\left[1+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\cdots\right]$$

(Laurent series)



Examples of Taylor and Laurent Series Expansions (cont.)

Part (c)

$$f(z)=\frac{1}{(z-2)(z-3)}$$

c) For $|z-1|>2$:

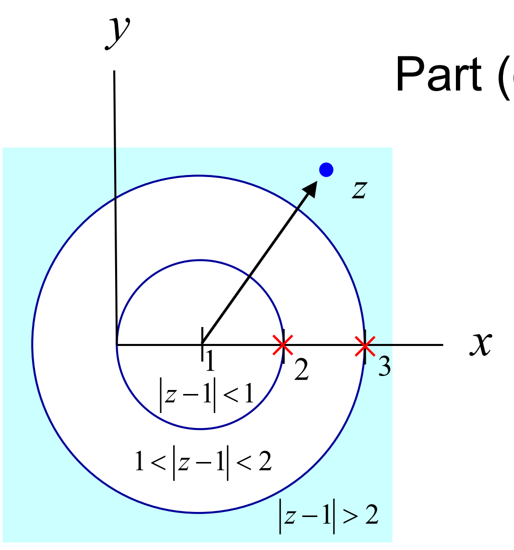
$$f(z)=\frac{1}{(z-1)-2}-\frac{1}{(z-1)-1}=\frac{1}{(z-1)\left[1-2/(z-1)\right]}-\frac{1}{(z-1)\left[1-1/(z-1)\right]}$$

$$=\frac{1}{(z-1)}\left[\cancel{1}+\frac{2}{(z-1)}+\frac{2^2}{(z-1)^2}+\cdots\right]-\frac{1}{(z-1)}\left[\cancel{1}+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\cdots\right]$$

so

$$f(z)=\frac{1}{(z-1)^2}+\frac{3}{(z-1)^3}+\frac{7}{(z-1)^4}+\cdots$$

(Laurent series)



Examples of Taylor and Laurent Series Expansions (cont.)

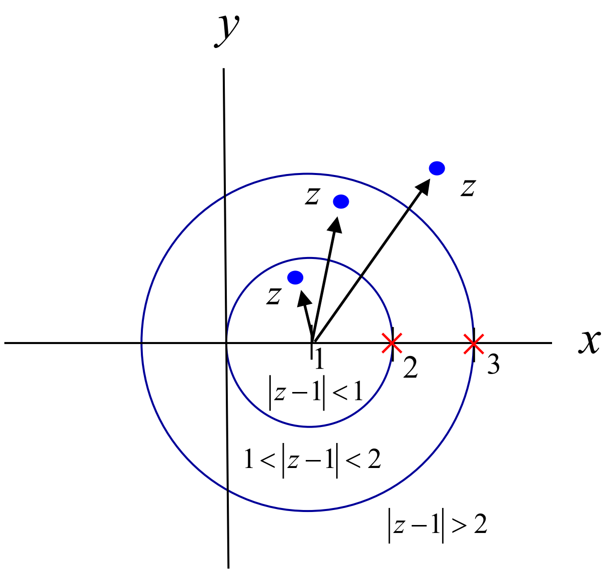
Summary of results for example:

$$f(z)=\frac{1}{(z-2)(z-3)}$$

$$f(z)=\frac{1}{2}+\frac{3}{4}(z-1)+\frac{7}{8}(z-1)^2+\frac{15}{16}(z-1)^3+\cdots,\quad 0\le|z-1|<1$$

$$f(z)=-\frac{1}{2}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]-\frac{1}{(z-1)}\left[1+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\cdots\right],\quad 1<|z-1|<2$$

$$f(z)=\frac{1}{(z-1)^2}+\frac{3}{(z-1)^3}+\frac{7}{(z-1)^4}+\cdots,\quad |z-1|>2$$



Examples of Taylor and Laurent Series Expansions (cont.)

◊ **Example 3**

Find the series expansion about $z=0$:

$$f(z)=\begin{cases}\dfrac{1-\cos z}{z^2}, & z\neq 0\\ 1/2, & z=0\end{cases}$$

($z=0$ is a "removable" singularity)

$$1-\cos z = \not{1}-\left(\not{1}-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)=\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\cdots$$

Hence

$$f(z)=\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\cdots,\quad |z|<\infty$$

Analytic everywhere!

Similarly, we have $f(z)=\dfrac{\sin z}{z}=1-\dfrac{z^2}{3!}+\dfrac{z^4}{5!}-\cdots \quad |z|<\infty$

Note : $\sin z = z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\cdots \quad |z|<\infty$

Examples of Taylor and Laurent Series Expansions (cont.)

◊ **Example 4**

Note: $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots \quad |z| < \infty$

Find the series for $\sin z$ about $z = \pi$.

$$f(z) = \sin z = \sin\left[(z-\pi)+\pi\right]$$

$$= \sin(z-\pi)\cos\pi + \cos(z-\pi)\sin\pi = -\sin(z-\pi)$$

$$f(z) = -(z-\pi) + \frac{(z-\pi)^3}{3!} - \frac{(z-\pi)^5}{5!} + \cdots, \quad |z-\pi| < \infty$$

Alternatively, directly use the derivative formula for Taylor series:

$$f(\pi) = \sin\pi = 0$$

$$f'(\pi) = \cos\pi = -1$$

$$f''(\pi) = -\sin\pi = 0$$

$$f'''(\pi) = -\cos\pi = +1$$

$$f^{(iv)}(\pi) = \sin\pi = 0$$

$$f^{(v)}(\pi) = \cos\pi = -1$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' = \frac{f^{(n)}(z_0)}{n!}$$

$$f(z) = -(z-\pi) + \frac{(z-\pi)^3}{3!} - \frac{(z-\pi)^5}{5!} + \cdots, \quad |z-\pi| < \infty$$

Examples of Taylor and Laurent Series Expansions (cont.)

◊ **Example 5**

Find the first three terms of the series for $\sin^2 z \,\ln(1-z)$ about $z=0$.

Since $\dfrac{1}{1-z} = 1+z+z^2+\cdots,\ |z|<1$ then

$$\int_0^z \frac{1}{1-z}\,dz = -\ln(1-z)\Big|_0^z = \int_0^z \left(1+z+z^2+\cdots\right)dz = z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\cdots,\ |z|<1$$

$$\Rightarrow \ln(1-z) = -z-\frac{z^2}{2}-\frac{z^3}{3}-\frac{z^4}{4}-\cdots,\ |z|<1$$

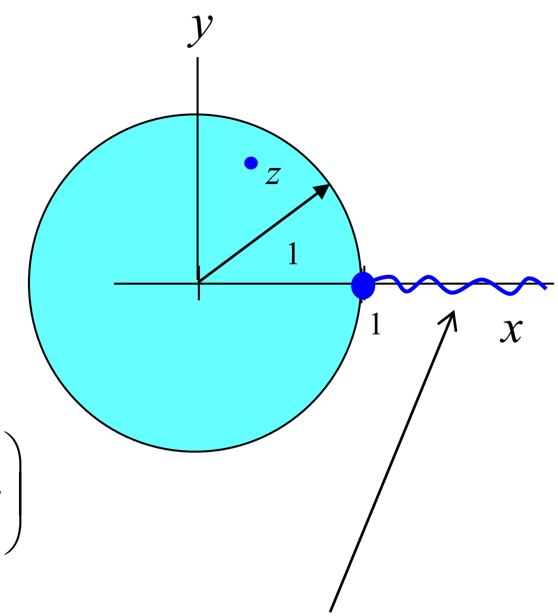
Also $\sin^2 z = \left[z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\cdots\right]\left[z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\cdots\right]$

$$= z^2-\frac{z^4}{3}+\frac{2}{45}z^6+\cdots$$

Hence

$$\sin^2 z\,\ln(1-z) = -\left(z^2-\frac{z^4}{3}+\frac{2}{45}z^6+\cdots\right)\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\cdots\right)$$

$$= -z^3-\frac{z^4}{2}+\left(-\frac{1}{3}+\frac{1}{3}\right)z^5+\left(-\frac{1}{4}+\frac{1}{6}\right)z^6+\cdots,\quad |z|<1$$



The branch cut is chosen away from the blue region.

Summary of Methods for Generating Taylor and Laurent Series Expansions

Summary of Methods

- Taylor (*not* Laurent) series, $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, can be generated using $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

- Taylor *and* Laurent series, $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, can be generated using $a_n = \dfrac{1}{2\pi i} \oint_C \dfrac{f(z')}{(z' - z_0)^{n+1}} dz'$

- To expand about $z = z_0$, first write $f(z)$ in the form $f\left[(z - z_0) + z_0\right]$, rearrange and expand using geometric series or other methods.

- Use partial fraction expansion and geometric series to generate series for rational functions (ratios of polynomials, degree of numerator less than degree of denominator).

Summary of Methods for Generating Taylor and Laurent Series Expansions (cont.)

Summary of Methods (cont.)

- Taylor / Laurent series can be integrated or differentiated term-by-term within their region of convergence.

- Two Taylor series can be multiplied term-by-term *within their common region of convergence*:

$$f(z)=\sum_{n=0}^{\infty}a_n\left(z-z_0\right)^n,\qquad g(z)=\sum_{m=0}^{\infty}b_m\left(z-z_0\right)^m$$

$$\Rightarrow f(z)g(z)=\left(\sum_{n=0}^{\infty}a_n\left(z-z_0\right)^n\right)\left(\sum_{m=0}^{\infty}b_m\left(z-z_0\right)^m\right)=\sum_{p=0}^{\infty}c_p\left(z-z_0\right)^p \text{ where } c_p=\sum_{n=0}^{p}a_n b_{p-n}$$

- Two Laurent series can be multiplied term-by-term *within their common annulus of convergence (if there is one)*:

$$f(z)=\sum_{n=-\infty}^{\infty}a_n\left(z-z_0\right)^n,\qquad g(z)=\sum_{m=\infty}^{\infty}b_m\left(z-z_0\right)^m$$

$$\Rightarrow f(z)g(z)=\left(\sum_{n=-\infty}^{\infty}a_n\left(z-z_0\right)^n\right)\left(\sum_{m=-\infty}^{\infty}b_m\left(z-z_0\right)^m\right)=\sum_{p=-\infty}^{\infty}c_p\left(z-z_0\right)^p \text{ where } c_p=\sum_{n=-\infty}^{\infty}a_n b_{p-n}$$