A point $z_s$ is a singularity of the function $f(z)$ if the function is not analytic at $z_s$.

(The function does not necessarily have to be infinite there.)

- Recall from Liouville’s theorem that the only function that is analytic and bounded in the entire complex plane is a constant.

- Hence, all non-constant analytic functions have singularities somewhere (possibly at infinity).
Taylor Series

The radius of convergence $R_c$ is the distance to the closest singularity.

Since $f(z)$ is analytic in the region

$$|z - z_0| < R_c$$

then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $|z - z_0| < R_c$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{C} \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

The series converges for $|z-z_0| < R_c$.

The series diverges for $|z-z_0| > R_c$ (proof omitted).
If $f(z)$ is analytic in the region $a < |z - z_0| < b$ then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $a < |z - z_0| < b$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \quad n = 0, \pm 1, \pm 2 \ldots$$

The series **converges** inside the annulus.

The series **diverges** outside the annulus (proof omitted).
Taylor Series Example

Example: \( f(z) = \frac{1}{z-1} \)

\[
f(z) = \frac{-1}{1-z} = -\left[1 + z + z^2 + z^3 + \ldots\right]
\]

From the property of Taylor series we have:

\[
\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1
\]

\[
\sum_{n=0}^{\infty} z^n \text{ diverges}, \quad |z| > 1
\]

The point \( z = 1 \) is a singularity (a first-order pole).
Taylor Series Example

Example: \( f(z) = \sqrt{z} \)

Expand about \( z_0 = 1 \): \( f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n \)

\[
R_c = 1
\]

\begin{align*}
a_0 &= 1 \\
a_1 &= \left. \frac{1}{1!} \left( \frac{1}{2} z^{-3/2} \right) \right|_{z=1} = \frac{1}{2} \\
a_2 &= \left. \frac{1}{2!} \left( \frac{1}{2} \left( -\frac{3}{2} \right) z^{-5/2} \right) \right|_{z=1} = -\frac{3}{8} \\
\end{align*}

etc.

\( \sqrt{z} = 1 + \frac{1}{2} (z-1) - \frac{3}{8} (z-1)^2 + \ldots \)

The series converges for \(|z-1| < 1\)

The series diverges for \(|z-1| > 1\)
Example: \( f(z) = \frac{\sqrt{z}}{z-1} \quad -\pi < \theta < \pi \)

Expand about \( z_0 = 1 \): \( f(z) = \sum_{n=-\infty}^{\infty} a_n (z-1)^n \)

Using the previous example, we have:

\[
\begin{align*}
  a_{-1} &= 1 \\
  a_0 &= \frac{1}{2} \\
  a_1 &= -\frac{3}{8} \\
  \text{etc.}
\end{align*}
\]

(The coefficients are shifted by 1 from the previous example.)

\[
\frac{\sqrt{z}}{z-1} = \frac{1}{z-1} + \frac{1}{2} - \frac{3}{8}(z-1) + \ldots
\]

The series converges for \( 0 < |z-1| < 1 \)

The series diverges for \( |z-1| > 1 \)
Isolated singularity:

The function is singular at $z_s$ but is analytic for $0 < |z - z_s| < \delta$

(for some $\delta$)

Examples: $\frac{\sin z}{z}, \frac{1}{z}, e^{1/z}, \frac{1}{\sin z}$ at $z = 0$

A Laurent series expansion about $z_s$ is always possible!

This is a special case of a Laurent series with $a > 0, a < b < \delta$. 
Non-Isolated Singularity:

By definition, this is a singularity that is not isolated.

Example:

\[ f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)} \]

Simple poles at:

\[ z = \frac{1}{m\pi} \]

(Distance between successive poles decreases with \( m \) !)

Note: The function is not analytic in any region \( 0 < |z| < \delta \).

Note: A Laurent series expansion in a neighborhood of \( z_s = 0 \) is not possible!
Branch Point:

This is a type of non-isolated singularity.

Example:

\[ f(z) = z^{1/2} \]

Not analytic at the branch point.

**Note:** The function is not analytic in any region \( 0 < |z| < \delta \).

**Note:** A Laurent series expansion in a neighborhood of \( z_s = 0 \) is not possible!
Examples of Singularities

Examples:
(These will be discussed in more detail later.)

- \( \frac{\sin(z)}{z} \) removable singularity at \( z = 0 \)
- \( \frac{1}{(z - z_s)^p} \) pole of order \( p \) at \( z = z_s \) (if \( p = 1 \), pole is a simple pole)
- \( e^{1/z} \) essential singularity at \( z = z_0 \) (pole of infinite order)
- \( \frac{1}{\sin\left(\frac{1}{z}\right)} \) non-isolated singularity \( z = 0 \)
- \( z^{1/2} \) branch point (not an isolated singularity)
Classification of Isolated Singularities

Isolated singularities

Removable singularities

\[
\frac{\sin(z)}{z}, \quad \frac{1 - \cos(z)}{z^2}
\]

Poles of finite order

\[
\frac{1}{z}, \quad \frac{1}{z^2}, \quad \frac{1}{(z - 1)^m}
\]

\[
\frac{2z + 3}{(z - 1)^2 (z + 2)}
\]

Essential singularities (poles of infinite order)

\[
\sin\left(\frac{1}{z}\right), \quad e^{1/z}
\]

These are each discussed in more detail next.
Removable singularity:

The limit $z \to z_0$ exists and $f(z)$ is made analytic by defining

$$f(z_0) \equiv \lim_{z \to z_0} f(z)$$

Example: $\frac{\sin(z)}{z}$

$$\lim_{z \to 0} \frac{\sin(z)}{z} = \lim_{z \to 0} \frac{\cos(z)}{1} = 1$$

Laurent series $\to$ Taylor series
Pole of finite order (order $P$):

$$f(z) = \sum_{n=-P}^{\infty} a_n (z - z_s)^n$$

The Laurent series expanded about the singularity terminates with a finite number of negative exponent terms.

Examples:

$$f(z) = \frac{1}{z}, \quad (P = 1) \quad \text{simple pole at } z = 0$$

$$f(z) = \frac{3}{(z - 3)^3} + \frac{2}{(z - 3)^2} + \frac{1}{(z - 3)} + 1 + (z - 3) + \cdots, \quad (P = 3)$$

pole of order 3 at $z = 3$
Isolated Singularity: Essential Singularity

Essential Singularity
(pole of infinite order):

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_s)^n \]

The Laurent series expanded about the singularity has an infinite number of negative exponent terms.

Examples:

\[ f(z) = \sin\left(\frac{1}{z}\right) = \sum_{n=1 \text{ odd}}^{\infty} \frac{1}{n!} \left(-1\right)^{n+1} \left(\frac{1}{z}\right)^n = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} + \ldots \]

\[ f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \ldots \]
Graphical Classification of an Isolated Singularity at \( z_s \)

Laurent series:

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_s)^n = \cdots + a_{-p} (z-z_s)^{-p} + \cdots + a_{-1} (z-z_s)^{-1} + a_0 + a_1 (z-z_s) + a_2 (z-z_s)^2 \cdots
\]

Isolated singularities

\[
\begin{cases}
\text{Analytic or removable} \\
\text{Simple pole} \\
Pole of order \( p \) \\
\text{Essential singularity}
\end{cases}
\]
Picard’s Theorem

The behavior near an essential singularity is pretty wild.

Picard’s theorem:

In any neighborhood of an essential singularity, the function will assume every complex number (with possibly a single exception) an infinite number of times.

For example:

$$f(z) = e^{1/z}$$

No matter how small $\delta$ is, this function will assume all possible complex values (except possibly one).

(Please see the next slide.)
Picard’s Theorem (cont.)

Example: \( f(z) = e^{1/z} \)

Set

\[
e^{1/z} = w_0 = R_0 e^{i\Theta_0} \quad \text{(a given arbitrary complex number)}
\]

\[
z = r e^{i\theta} \implies e^{1/z} = e^{\frac{e^{-i\theta}}{r}} = e^{\frac{\cos \theta}{r}} e^{-\frac{i\sin \theta}{r}} = w_0 = R_0 e^{i(\Theta_0 + 2n\pi)}
\]

Take the ln of both sides.

\[
\implies \cos \theta = r \ln R_0, \quad \sin \theta = -r(\Theta_0 + 2n\pi)
\]

\[
\implies \cos^2 \theta + \sin^2 \theta = 1 = r^2 \left[ \ln^2 R_0 + (\Theta_0 + 2n\pi)^2 \right]
\]

Hence

\[
r = \frac{1}{\sqrt{\ln^2 R_0 + (\Theta_0 + 2n\pi)^2}} \quad \overset{n \to \infty}{\to} \quad 0, \quad \theta = \tan^{-1} \frac{-(\Theta_0 + 2n\pi)}{\ln R_0} \quad \overset{n \to \infty}{\to} \quad -\pi / 2
\]

The “exception” here is \( w_0 = 0 \ (R_0 = 0) \).
Picard’s Theorem (cont.)

Example (cont.)

This sketch shows that as \( n \) increases, the points where the function \( \exp(1/z) \) equals the given value \( w_0 \) “spirals in” to the (essential) singularity.

You can always find a solution for \( z \) now matter how small \( \delta \) (the “neighborhood”) is!
Example (cont.)

\[ f(z) = e^{1/z} \]

\[ z = re^{i\theta} \]

\[ e^{1/z} = e^{\frac{\cos\theta}{r}} \cdot e^{-i\frac{\sin\theta}{r}} \]

Plot of the function \(\exp(1/z)\), centered on the essential singularity at \(z = 0\). The color represents the phase, the brightness represents the magnitude. This plot shows how approaching the essential singularity from different directions yields different behaviors (as opposed to a pole, which, approached from any direction, would be uniformly white).

https://en.wikipedia.org/wiki/Essential_singularity
We classify the types of singularities at infinity by letting $w = 1/z$ and analyzing the resulting function at $w = 0$.

**Example:**

$$f(z) = z^3$$

$$f(z) = g(w) = \frac{1}{w^3} \quad \text{pole of order 3 at } w = 0$$

The function $f(z)$ has a pole of order 3 at infinity.

**Note:**

When we say “finite plane” we mean everywhere except at infinity. The function $f(z)$ in the example above is analytic in the finite plane.
Other Definitions

Entire: The function is analytic everywhere in the finite plane.

Examples:

\[ f(z) = e^z, \sin z, \quad 2z^2 + 3z + 1 \]

Meromorphic: The function is analytic everywhere in the finite plane except for isolated poles of finite order.

Examples:

\[ f(z) = \frac{\sin z}{(z-1)(z+1)^3}, \quad g(z) = \frac{1}{\sin z} \]

Meromorphic functions can always be expressed as the ratio of two entire functions, with the zeros of the denominator function as the poles (proof omitted).