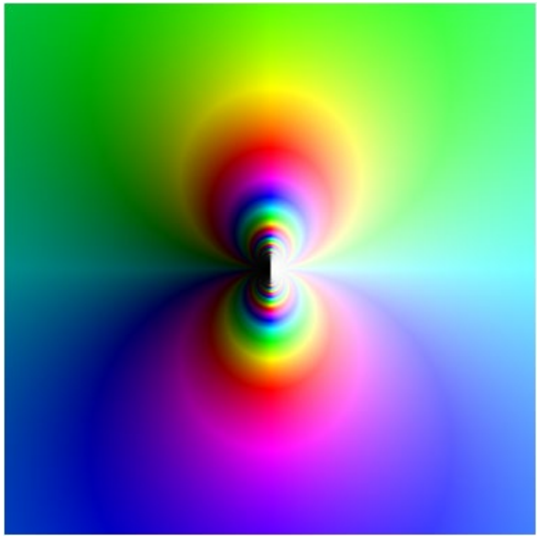


ECE 6382

Fall 2023

David R. Jackson

Notes 9

Singularities

Notes are from D. R. Wilton, Dept. of ECE

Singularity

A point z_s is a singularity of the function $f(z)$ if the function is not analytic at z_s.

(The function does not necessarily have to be infinite there.)

- Recall from Liouville's theorem that the only function that is analytic and bounded everywhere in the complex plane is a constant.
- Hence, all non-constant functions that are analytic everywhere in the complex plane must be unbounded at infinity and hence have a singularity at infinity.

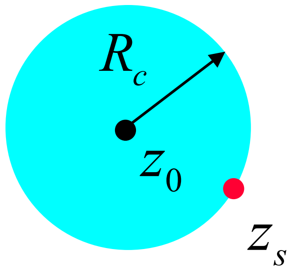
Example: $f(z) = e^z$ $\left(\text{tends to infinity as } z = x \to \infty\right)$

Taylor Series

The radius of convergence R_c is the distance to the closest singularity.

If $f(z)$ is analytic in the region

$$\left|z - z_0\right| < R_c$$



then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for} \quad \left|z - z_0\right| < R_c$$

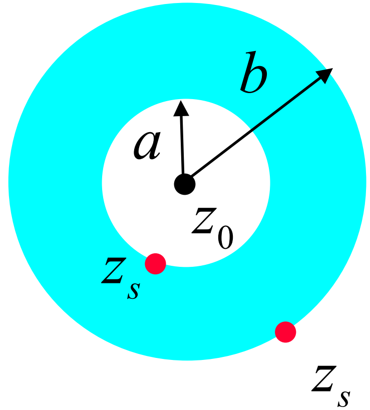
$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

The series converges for $|z-z_0| < R_c$.

The series diverges for $|z-z_0| > R_c$ (proof omitted).

Laurent Series

If $f(z)$ is analytic in the region $a < |z - z_0| < b$



then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad \text{for} \qquad a < |z - z_0| < b$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

The series <u>converges</u> inside the annulus.

The series <u>diverges</u> outside the annulus (proof omitted).

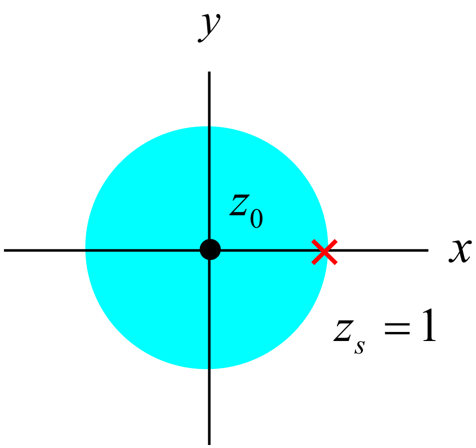
Taylor Series Example

Example:

$$f(z)=\frac{1}{z-1}$$

$$f(z)=\frac{-1}{1-z}=-\left[1+z+z^2+z^3+\ldots\right]$$

$$=-\sum_{n=0}^{\infty} z^n$$



The point $z=1$ is a singularity (a first-order pole).

From the property of Taylor series we have:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z|<1$$

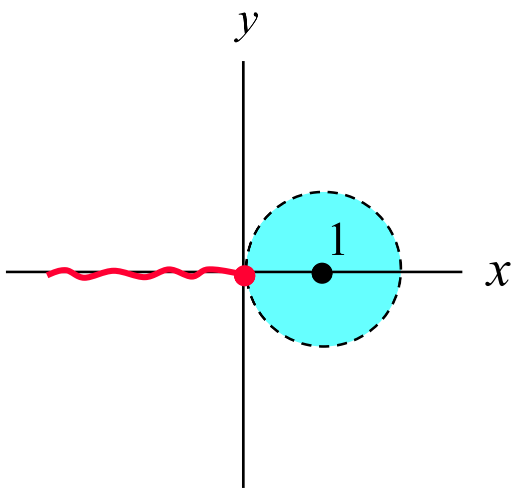
$$\sum_{n=0}^{\infty} z^n \quad \text{diverges}, \quad |z|>1$$

Taylor Series Example

Example:

$$f(z) = \sqrt{z}$$

Expand about $z_0 = 1$: $\quad f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n$



$$R_c = 1$$

$$-\pi < \theta < \pi$$

$$a_0 = z^{1/2}\Big|_{z=1} = 1$$

$$a_1 = \frac{1}{1!}\left(\frac{1}{2} z^{-1/2}\right)\Big|_{z=1} = \frac{1}{2}$$

$$a_2 = \frac{1}{2!}\left(\frac{1}{2}\left(-\frac{1}{2}\right) z^{-3/2}\right)\Big|_{z=1} = -\frac{1}{8}$$

etc.

$$\sqrt{z} = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \ldots$$

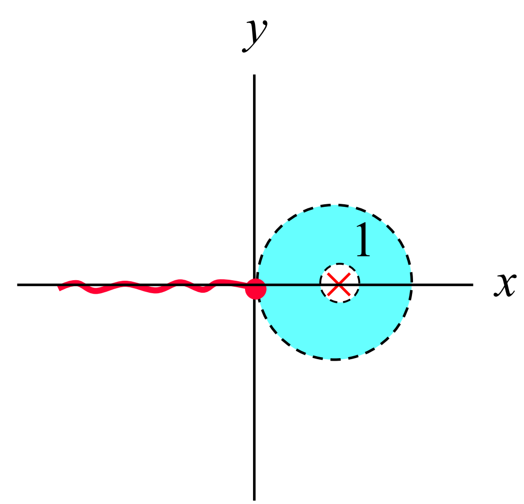
The series converges for $|z-1| < 1$

The series diverges for $|z-1| > 1$

Laurent Series Example

Example:

$$f(z) = \frac{\sqrt{z}}{z-1} \qquad -\pi < \theta < \pi$$



Expand about $z_0 = 1$: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-1)^n$

Using the previous example, we have:

$a_{-1} = 1$

$a_0 = \frac{1}{2}$

$a_1 = -\frac{1}{8}$

etc.

(The coefficients are shifted by 1 from the previous example.)

$$\frac{\sqrt{z}}{z-1} = \frac{1}{z-1} + \frac{1}{2} - \frac{1}{8}(z-1) + \ldots$$

The series converges for $0 < |z-1| < 1$

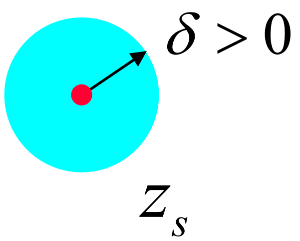
The series diverges for $|z-1| > 1$

Isolated Singularity

Isolated singularity:

The function is singular at z_s but is analytic for $0 < |z - z_s| < \delta$

(for some δ)



Examples: $\dfrac{\sin z}{z}, \; \dfrac{1}{z}, \; e^{1/z}, \; \dfrac{1}{\sin z}$ **at** $z = 0$

A Laurent series expansion about z_s is always possible!

This is a special case of a Laurent series with $a > 0, \; a < b < \delta$.

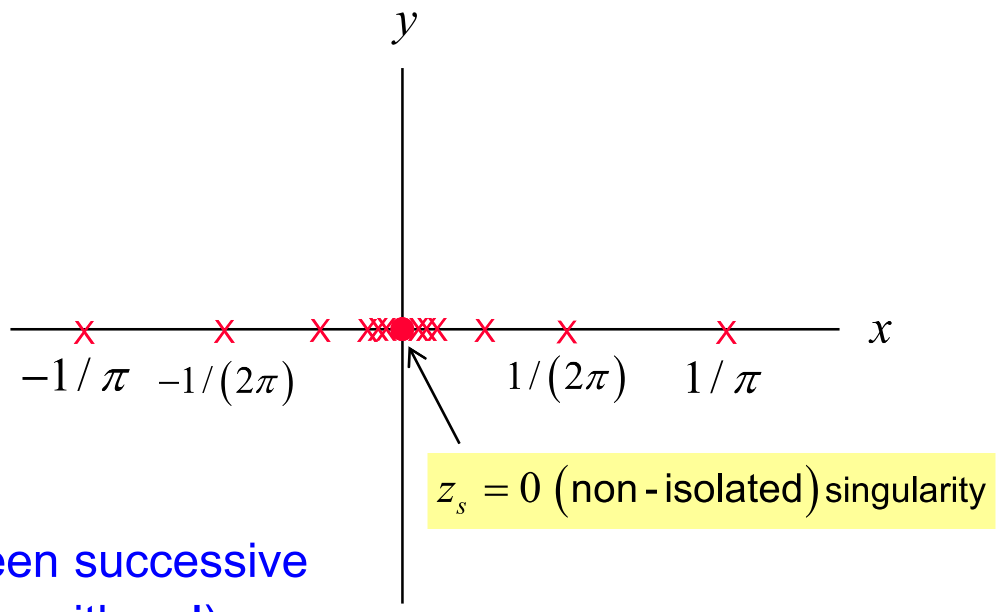
Non-Isolated Singularity

Non-Isolated Singularity:

By definition, this is a singularity that is not isolated.

Example:

$$f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$$



Simple poles at:

$$z = \frac{1}{m\pi}$$

(Distance between successive poles *decreases* with m !)

Note: The function is not analytic in any region $0 < |z| < \delta$.

Note: A Laurent series expansion about $z = 0$ with $a = 0$ is not possible!

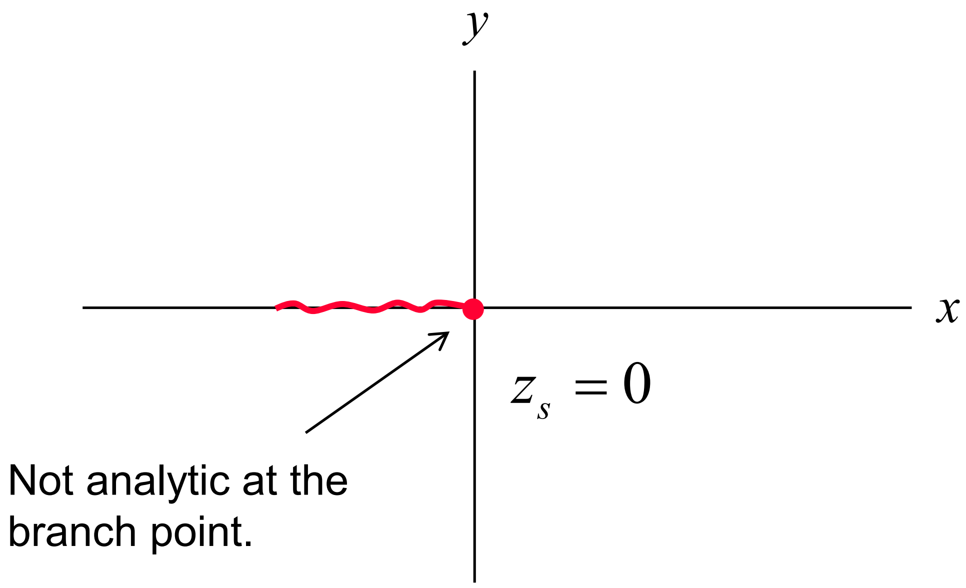
Non-Isolated Singularity (cont.)

Branch Point:

This is a type of non-isolated singularity.

Example:

$$f(z) = z^{1/2}$$



Note: The function is not analytic in any region $0 < |z| < \delta$.

Note: A Laurent series expansion in any neighborhood of $z = 0$ is not possible!

Examples of Singularities

Examples:

(These will be discussed in more detail later.)

If expanded about the singularity, we can have:

T = Taylor
L = Laurent
N = Neither

T $\quad \dfrac{\sin(z)}{z}$ removable singularity at $z = 0$

(isolated singularity)

L $\quad \dfrac{1}{(z - z_s)^p}$ pole of order p at $z = z_s$ (if $p = 1$, pole is a *simple pole*)

(isolated singularity)

L $\quad e^{1/z}$ essential singularity at $z = 0$ (pole of infinite order)

(isolated singularity)

N $\quad \dfrac{1}{\sin\left(\dfrac{1}{z}\right)}$ <u>non-isolated</u> singularity $z = 0$ (for $a = 0$)

N $\quad z^{1/2}$ <u>non-isolated</u> singularity $z = 0$ (branch point)

Classification of Isolated Singularities

Isolated singularities

→ Removable singularities

→ Poles of finite order

→ Essential singularities (poles of infinite order)

Removable singularities

$$\frac{\sin(z)}{z}, \quad \frac{1-\cos(z)}{z^2}$$

Poles of finite order

$$\frac{1}{z}, \quad \frac{1}{z^2}, \quad \frac{1}{(z-1)^m},$$

$$\frac{2z+3}{(z-1)^2(z+2)}$$

Essential singularities
(poles of infinite order)

$$\sin\left(\frac{1}{z}\right), \quad e^{1/z}$$

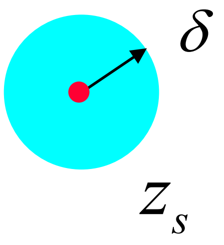
These are each discussed in more detail next.

Isolated Singularity: Removable Singularity

Removable singularity:

The limit $z \to z_0$ exists and $f(z)$ is made analytic by defining

$$f(z_0) \equiv \lim_{z \to z_0} f(z)$$



Example:

$$\frac{\sin(z)}{z}$$

$$\lim_{z\to 0}\frac{\sin(z)}{z} \overset{\text{L'Hospital's Rule}}{=} \lim_{z\to 0}\frac{\cos(z)}{1} = 1$$

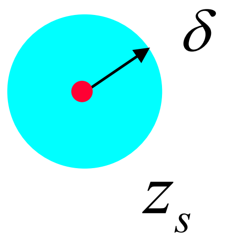
$$\frac{\sin(z)}{z} = \frac{z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \ldots}{z} = 1 - \frac{1}{3}z^2 + \frac{1}{5}z^4 - \ldots$$

Laurent series → Taylor series

Isolated Singularity: Pole of Finite Order

Pole of finite order (order P):

$$f(z)=\sum_{n=-P}^{\infty} a_n (z-z_s)^n$$



The Laurent series expanded about the singularity terminates with a finite number of negative exponent terms.

Examples:

$$f(z)=\frac{1}{z},\quad (P=1)$$ simple pole at $z=0$

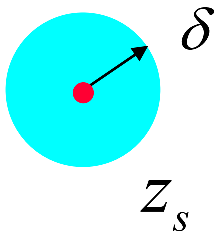
$$f(z)=\frac{3}{(z-3)^3}+\frac{2}{(z-3)^2}+\frac{1}{(z-3)}+1+(z-3)+\cdots,\quad (P=3)$$

pole of order 3 at $z=3$

Isolated Singularity: Essential Singularity

Essential Singularity
(pole of infinite order):

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_s)^n$$



The Laurent series expanded about the singularity has an infinite number of negative exponent terms.

Examples:

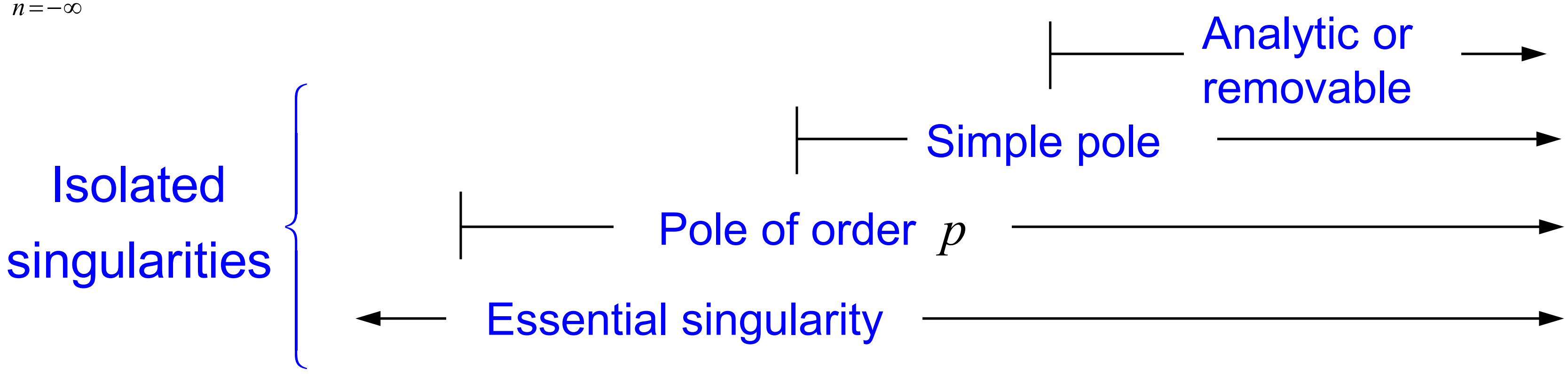
$$f(z) = \sin\left(\frac{1}{z}\right) = \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{1}{n!}(-1)^{n+1}\left(\frac{1}{z}\right)^n = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} + \ldots$$

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \ldots$$

Graphical Classification of an Isolated Singularity at z_s

Laurent series:

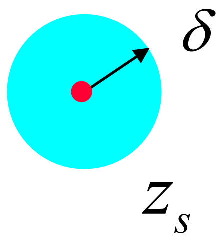
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_s)^n = \cdots + a_{-p}(z - z_s)^{-p} + \cdots + a_{-1}(z - z_s)^{-1} + a_0 + a_1(z - z_s) + a_2(z - z_s)^2 \cdots$$



Picard's Theorem

The behavior near an essential singularity is pretty wild !

Picard's theorem:



> In any neighborhood of an essential singularity, the function will assume every complex number (with possibly a single exception) an infinite number of times.

For example:

$$f(z) = e^{1/z}$$

No matter how small δ is, this function will assume all possible complex values (except possibly one).

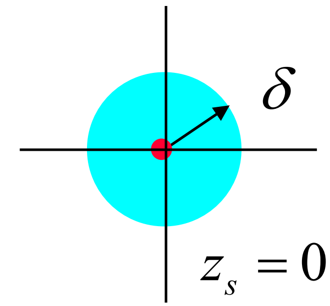
(Please see the next slide.)

Charles Ėmile Picard

Picard's Theorem (cont.)

Example: $f(z) = e^{1/z}$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^{n}$$



Set

$$e^{1/z} = w_0 = R_0 e^{i\Theta_0} \quad \text{(a given arbitrary complex number)}$$

$$z = re^{i\theta} \Rightarrow e^{1/z} = e^{\frac{e^{-i\theta}}{r}} = e^{\frac{1}{r}(\cos\theta - i\sin\theta)} = w_0 = R_0 e^{i(\Theta_0 + 2\pi n)}$$

Take the $\ln$ of both sides, equate real and imaginary parts.

$$\Rightarrow \cos\theta = r\ln R_0, \qquad \sin\theta = -r\left(\Theta_0 + 2n\pi\right)$$

$$\Rightarrow \cos^2\theta + \sin^2\theta = 1 = r^2\left[\ln^2 R_0 + \left(\Theta_0 + 2n\pi\right)^2\right]$$

Any value of n gives a valid solution.

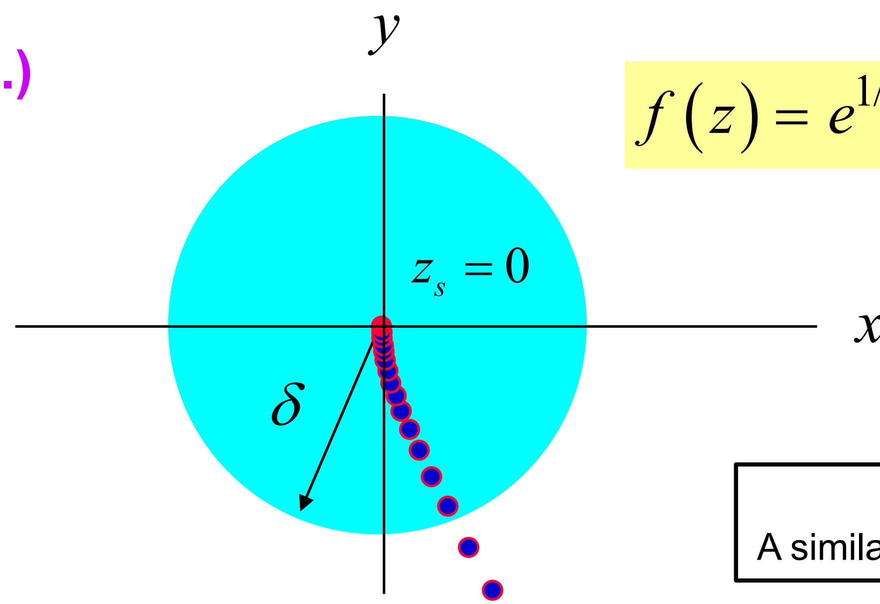
Hence

$$r = \frac{1}{\sqrt{\ln^2 R_0 + \left(\Theta_0 + 2n\pi\right)^2}} \overset{n\to\infty}{\to} 0, \qquad \theta = \tan^{-1}\left(\frac{-\left(\Theta_0 + 2n\pi\right)}{\ln R_0}\right) \overset{n\to\infty}{\to} -\pi/2$$

The "exception" here is $w_0 = 0 \;\; (R_0 = 0)$.

Picard's Theorem (cont.)

Example (cont.)



$$f(z) = e^{1/z}$$

Note:
A similar plot exists for negative n.

$$r = \overset{n\to\infty}{\to} 0, \qquad \theta = \overset{n\to\infty}{\to} -\pi/2$$

This sketch shows that as n increases, the points where the function $\exp(1/z)$ equals the given value w_0 converge to the (essential) singularity at the origin.

You can always find a solution for z now matter how small δ (the "neighborhood") is!

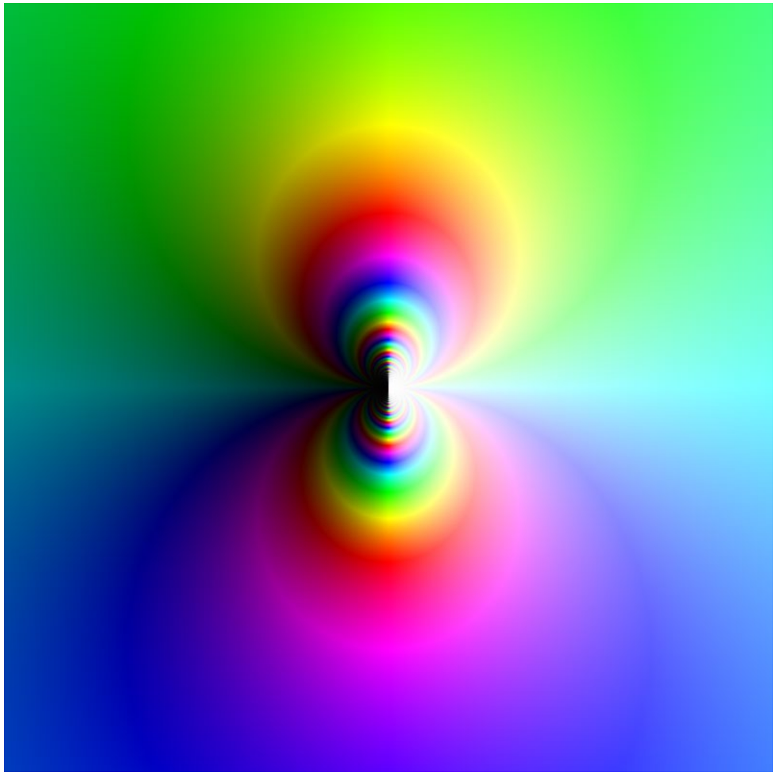
Picard's Theorem (cont.)

Example (cont.)

$$f(z) = e^{1/z}$$

$$z = re^{i\theta}$$

$$e^{1/z} = e^{\frac{\cos\theta}{r}} e^{-i\frac{\sin\theta}{r}}$$



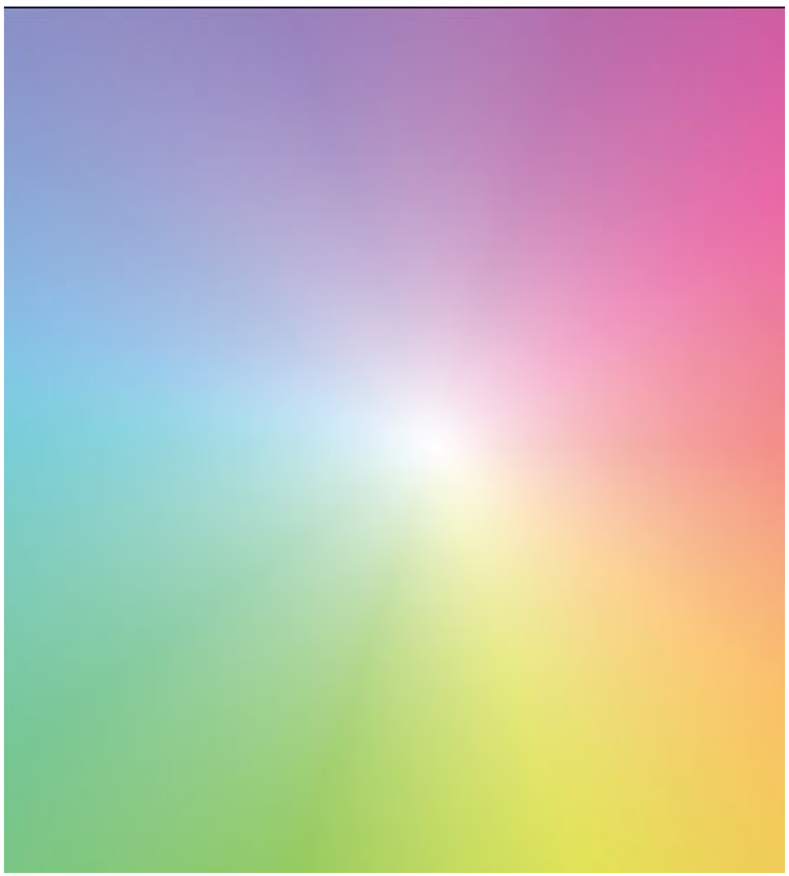
Plot of the function $\exp(1/z)$, centered on the essential singularity at $z = 0$. The color represents the phase, the brightness represents the magnitude. This plot shows how approaching the essential singularity from different directions yields different behaviors (as opposed to a pole, which, approached from any direction, would be uniformly white).

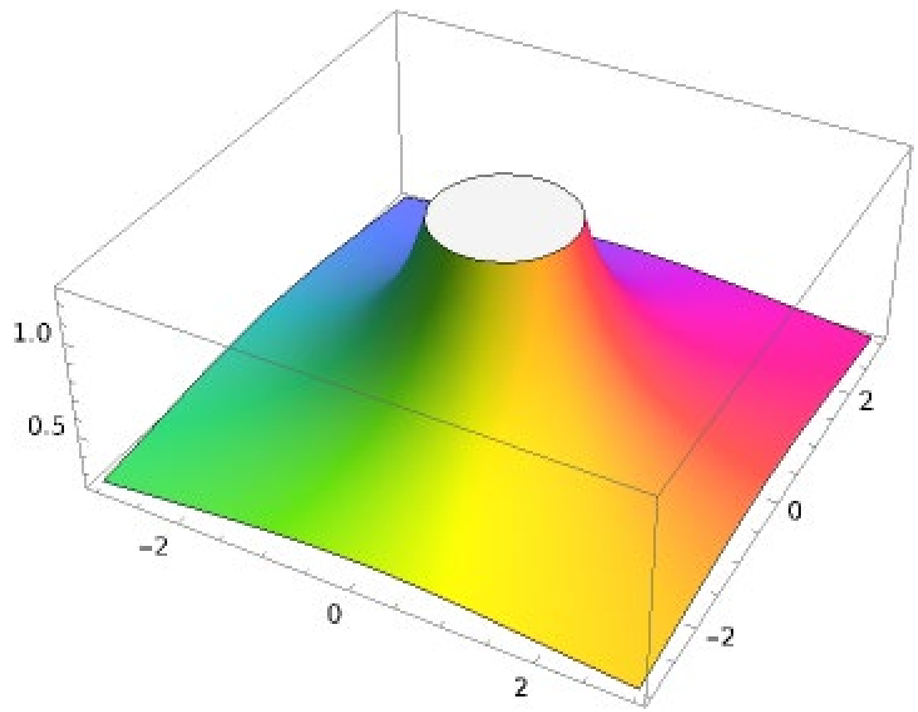
https://en.wikipedia.org/wiki/Essential_singularity

Picard's Theorem (cont.)

Compare with the behavior near a simple pole:

$$f(z) = \frac{1}{z} \qquad \frac{1}{z} = \frac{1}{r} e^{-i\theta}$$





Singularity at Infinity

We classify the types of singularities at infinity by letting $w = 1/z$ and analyzing the resulting function at $w = 0$.

Example:

$$f(z) = z^3$$

$$f(z) = g(w) = \frac{1}{w^3}$$ pole of order 3 at $w = 0$

⇨ The function $f(z)$ has a pole of order 3 at infinity.

Note:
When we say "finite plane" we mean everywhere except at infinity.
The function $f(z)$ in the example above is analytic in the finite plane.

Other Definitions

Entire: The function is analytic everywhere in the finite plane.

Examples:

$$f(z) = e^z, \;\; \sin z, \;\; 2z^2 + 3z + 1$$

Meromorphic: The function is analytic everywhere in the finite plane except for isolated poles of finite order.

Examples: $$f(z) = \frac{\sin z}{(z-1)(z+1)^3}, \quad g(z) = \frac{1}{\sin z}$$

Meromorphic functions can always be expressed as the ratio of two entire functions, with the zeros of the denominator function as the poles (proof omitted).