NAME: ____________________KEY__________________________

ELEE 6382
Fall 2011
Oct. 27, 2011

MIDTERM EXAM

INSTRUCTIONS:

This exam is open-book (*Arfken and Weber* or approved substitute) and open-notes. You may also use your class notes, and a calculator. Please show *all steps of your work* and *write neatly and legibly* in order to receive full credit.

Please write all of your work on the attached sheets. If a problem is continued onto the workspace pages at the end, please indicate this.

Useful identities and integrals:

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2 \qquad \int x \sin x \, dx = \sin x - x\cos x$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \qquad \int x \cos x \, dx = \cos x + x\sin x$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2 \qquad \int xe^x dx = xe^x - e^x$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\sin iz = i \sinh z, \qquad \cos iz = \cosh z$$

$$\sinh iz = i \sin z, \qquad \cosh iz = \cos z$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$\frac{d}{dz}\sinh z = \cosh z, \;\; \frac{d}{dz}\cosh z = \sinh z,$$

Problem 1 (25 pts)

The vectors $\mathbf{a} = (\alpha_1, \alpha_2, \alpha_3)$, $\mathbf{b} = (\beta_1, \beta_2, \beta_2)$ are in C^3 with inner product defined as

$$< \mathbf{a}, \mathbf{b} > \ = (\alpha_1 \beta_1^* + \alpha_2 \beta_2^* + \alpha_3 \beta_3^*)$$

where the asterisk ($*$) denotes complex conjugate. Construct a vector $\mathbf{b}'$ that is 1) a linear combination of $\mathbf{b}$ and $\mathbf{a}$ and 2) is orthogonal to $\mathbf{a}$. That is, let $\mathbf{b}' = \mathbf{b} - c\mathbf{a}$ and find a constant c such that $< \mathbf{b}', \mathbf{a} > = 0$. (You may want to refer to the Gram-Schmidt process.)

a. Using inner product notation, derive an expression for c.

$\mathbf{b}' = \mathbf{b} - c\mathbf{a}$

Taking the inner product with **a** on both sides of the equation,

$< \mathbf{b}', \mathbf{a} > \equiv \ 0 = < \mathbf{b}, \mathbf{a} > - c < \mathbf{a}, \mathbf{a} >$

$$\Rightarrow c = \frac{< \mathbf{b}, \mathbf{a} >}{< \mathbf{a}, \mathbf{a} >} = \frac{\beta_1 \alpha_1{}^* + \beta_2 \alpha_2{}^* + \beta_3 \alpha_3{}^*}{\alpha_1 \alpha_1{}^* + \alpha_2 \alpha_2{}^* + \alpha_3 \alpha_3{}^*}$$

b. If $\mathbf{a} = (1, i, 2)$, $\mathbf{b} = (1, i, 0)$, determine $\mathbf{b}'$ and check that $< \mathbf{b}', \mathbf{a} > = 0$

$\mathbf{a} = (1, i, 2)$, $\mathbf{b} = (1, i, 0)$,

$$c = \frac{< \mathbf{b}, \mathbf{a} >}{< \mathbf{a}, \mathbf{a} >} = \frac{\beta_1 \alpha_1{}^* + \beta_2 \alpha_2{}^* + \beta_3 \alpha_3{}^*}{\alpha_1 \alpha_1{}^* + \alpha_2 \alpha_2{}^* + \alpha_3 \alpha_3{}^*} = \frac{1 \cdot 1 + i \cdot (-i) + 0 \cdot 2}{1 \cdot 1 + i \cdot (-i) + 2 \cdot 2} = \frac{2}{6} = \frac{1}{3}$$

$$\mathbf{b}' = \mathbf{b} - c\mathbf{a} = (1, i, 0) - \frac{1}{3}(1, i, 2) = (\frac{2}{3}, \frac{2}{3} i, -\frac{2}{3}) = \frac{2}{3}(1, i, -1)$$

Check: $$(\mathbf{a}, \mathbf{b}') = (1, i, 2) \cdot (\frac{2}{3}, -\frac{2}{3} i, -\frac{2}{3}) = \frac{2}{3}(1, i, 2) \cdot (1, -i, -1) = \frac{2}{3} + \frac{2}{3} - \frac{4}{3} = 0$$

Problem 2 (25 pts)

A function $w(z) = u(x, y) + iv(x, y)$ of the complex variable $z = x + iy$ has imaginary part $v(x, y) = \left(y\cos y + x\sin y\right)e^{x}$. Find $u(x, y)$ if $w(0 + i0) = 0 + i0$.

$$v(x, y) = \left(y\cos y + x\sin y\right)e^{x}$$

From the C.R. conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \left(\cos y - y\sin y + x\cos y\right)e^{x} \quad \overset{\substack{\text{integration}\\ \text{w.r.t. } x}}{\Rightarrow} \quad u(x, y) = \left[\cancel{\cos y} - y\sin y + (x - \cancel{1})\cos y\right]e^{x} + f(y)$$

$$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \left(-y\cos y - (x+1)\sin y\right)e^{x} \quad \overset{\substack{\text{integration}\\ \text{w.r.t. } y}}{\Rightarrow} \quad u(x, y) = \left[-\cancel{\cos y} - y\sin y + (x + \cancel{1})\cos y\right]e^{x} + g(x)$$

Since neither expression contains functions of x or y alone

$$\Rightarrow g(x) = f(y) = C, \quad u(x, y) = \left[-y\sin y + x\cos y\right]e^{x} + C$$

$$w(0 + i0) = 0 + i0 = u(0, 0) + iv(0, 0) = C = 0 \quad \Rightarrow \boxed{u(x, y) = \left[-y\sin y + x\cos y\right]e^{x}}$$

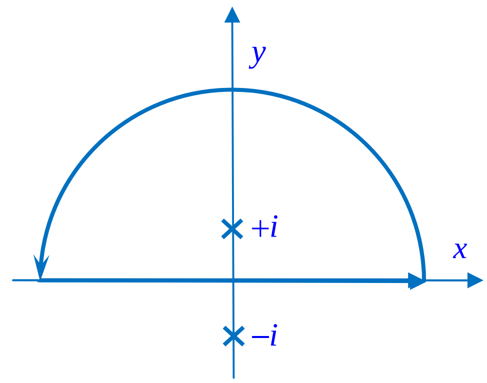
$$\Rightarrow w(z) = u(x, y) + iv(x, y) = \left[-y\sin y + x\cos y\right]e^{x} + i\left(y\cos y + x\sin y\right)e^{x}$$

$$= z\,e^{z}$$

Problem 3 (25 pts)

Calculate the value of the following **three** definite integrals. Sketch any contours used, including their orientation, closures, and singularity locations, if any, for each case.

a) $\int_{-\infty}^{\infty} \frac{x \sin mx}{(x^2+1)} dx = \text{Im} \int_{-\infty}^{\infty} \frac{x e^{imx}}{(x^2+1)} dx$; two cases: $m > 0, m < 0$



$\boxed{m > 0}$

Consider

$$\oint_C \frac{z e^{imz}}{(z^2+1)} dz = 2\pi i \text{Res}\left[\frac{z e^{imz}}{(z^2+1)}, i\right]$$

since $e^{im(x+iy)} = e^{imx} e^{-my}$ decays in the UHP for $m > 0$, since by Jordan's lemma the contribution from the semicircular path vanishes,

$$I = \int_{-\infty}^{\infty} \frac{x e^{imx}}{(x^2+1)} dx = 2\pi i \lim_{z \to i} \frac{(z-i) z e^{imz}}{(z+i)(z-i)} = 2\pi i \frac{i e^{i^2 m}}{(2i)} = \pi i e^{-m}$$

$$\Rightarrow \text{Im} \int_{-\infty}^{\infty} \frac{x e^{imx}}{(x^2+1)} dx = \boxed{\int_{-\infty}^{\infty} \frac{x \sin mx}{(x^2+1)} dx = \pi e^{-m}}$$

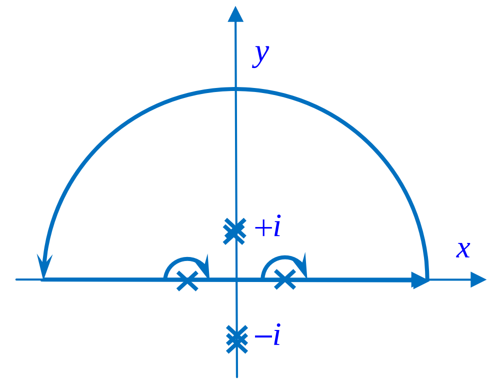
$\boxed{m < 0}$

For $m < 0$, $e^{im(x+iy)} = e^{imx} e^{-my}$ decays in the LHP, and hence by Jordan's lemma, we have

$$\oint_C \frac{z e^{imz}}{(z^2+1)} dz = -2\pi i \text{Res}\left[\frac{z e^{imz}}{(z^2+1)}, -i\right] = -2\pi i \lim_{z \to -i} \frac{(z+i) z e^{imz}}{(z+i)(z-i)} = -2\pi i \frac{(-i) e^{+m}}{-2i} = -i\pi e^{+m}$$

$$\Rightarrow \text{Im} \int_{-\infty}^{\infty} \frac{x e^{imx}}{(x^2+1)} dx = \boxed{\int_{-\infty}^{\infty} \frac{x \sin mx}{(x^2+1)} dx = -\pi e^{m}}; \text{ Two cases} \Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{x \sin mx}{(x^2+1)} dx = \text{sgn}(m) \pi e^{-|m|}}$$

b) $\displaystyle\int_0^\infty \frac{dx}{\left(x^2+1\right)^2\left(x^2-1\right)}$



$$I=\int_0^\infty \frac{dx}{\left(x^2+1\right)^2\left(x^2-1\right)}=\frac{1}{2}\int_{-\infty}^\infty \frac{dx}{\left(x^2+1\right)^2\left(x^2-1\right)}=\frac{1}{2}\int_{-\infty}^\infty f(x)dx$$

where $f(z)$ has double poles at $z=\pm i$ and simple, x-axis poles at $z=\pm 1$. Also, $f(z)\sim\dfrac{1}{z^6}$.

$$\frac{1}{2}\oint_C f(z)dz=I-\frac{\pi}{2}i\text{Res}[\,f;+1]-\frac{\pi}{2}i\text{Res}[\,f;-1]=\pi i\text{Res}[\,f;+i\,]$$

$$\Rightarrow I=\frac{\pi}{2}i\text{Res}[\,f;+1]+\frac{\pi}{2}i\text{Res}[\,f;-1]+\pi i\text{Res}[\,f;+i\,]$$

$$\text{Res}[\,f;+1]=\lim_{z\to 1}\frac{\cancel{(z-1)}}{\left(z^2+1\right)^2\cancel{(z-1)}(z+1)}=\lim_{z\to 1}\frac{1}{\left(z^2+1\right)^2(z+1)}=\frac{1}{8}$$

$$\text{Res}[\,f;-1]=\lim_{z\to -1}\frac{\cancel{(z+1)}}{\left(z^2+1\right)^2\cancel{(z+1)}(z-1)}=\lim_{z\to -1}\frac{1}{\left(z^2+1\right)^2(z-1)}=-\frac{1}{8}$$

$$\text{Res}[\,f;+i\,]=\lim_{z\to +i}\frac{d}{dz}\frac{\cancel{(z-i)^2}}{\cancel{(z-i)^2}(z+i)^2\left(z^2-1\right)}=\lim_{z\to i}\frac{-\left[2(z+i)\left(z^2-1\right)+2z(z+i)^2\right]}{(z+i)^4\left(z^2-1\right)^2}$$

$$=\lim_{z\to i}\frac{-\left[2(2i)(-2)+(2i)(2i)^2\right]}{(2i)^4(-2)^2}=-(2i)\frac{[-4-4]}{(2i)^4(-2)^2}=-i(2)\frac{-8}{16\cdot 4}=+i\frac{1}{4}$$

$$\Rightarrow I=\frac{\pi}{2}i\text{Res}[\,f;+1]+\frac{\pi}{2}i\text{Res}[\,f;-1]+\pi i\text{Res}[\,f;+i\,]=\boxed{\int_0^\infty\frac{dx}{\left(x^2+1\right)^2\left(x^2-1\right)}=-\frac{\pi}{4}}$$

Problem 4 (25 pts)

For each of the following functions, give the following information for each singularity in the *finite* complex plane:

i. Location of singularity
ii. Singularity type: pole (give order and residue), removable singularity, branch point, essential singularity
iii. Residue for any poles or essential singularities

a) $(z-1)^2 e^{\frac{1}{z-1}}$

$$(z-1)^2 e^{\frac{1}{z-1}} = (z-1)^2\left[1+\frac{1}{z-1}+\frac{1}{2!(z-1)^2}+\frac{1}{3!(z-1)^3}+\cdots\right]$$

essential singularity at $z=1$: Residue = $\frac{1}{3!}=\frac{1}{6}$

b) $\dfrac{\sin^2(z-1)}{(z-1)^2}$

removable singularity at $z=1$ since

$$\lim_{z\to 1}\frac{\sin^2(z-1)}{(z-1)^2}=\lim_{z\to 1}\frac{\left[(z-1)-\frac{(z-1)^3}{3!}+\frac{(z-1)^5}{5!}+\cdots\right]^2}{(z-1)^2}=\lim_{z\to 1}\frac{(z-1)^2+\cdots}{(z-1)^2}=1$$

c) $\dfrac{z^{\frac{2}{3}}}{(z+1)}$

branch point at $z=0$; simple pole at $z=-1$

$$\frac{z^{\frac{2}{3}}}{(z+1)}=\lim_{z\to -1}\frac{\cancel{(z+1)}\,z^{\frac{2}{3}}}{\cancel{(z+1)}}=(-1)^{\frac{2}{3}}=\left(e^{i\pi}\right)^{\frac{2}{3}}=e^{i2\pi/3}\ \text{(principal branch)}$$

d) $\cot \pi z$

simple poles at $z=n,\ n=0,\pm1,\pm2,\cdots$

$$\text{Residues of } \cot\pi z=\left.\frac{\cos\pi z}{d(\sin\pi z)/dz}\right|_{z=n}=\left.\frac{\cos\pi z}{\pi\cos\pi z}\right|_{z=n}=\frac{1}{\pi}$$

ROOM FOR EXTRA WORK

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