Name: ____

Student Number: ______________________________

ELEE 6382
Fall 2013
Oct. 21, 2013

MIDTERM EXAM

INSTRUCTIONS:

This exam is open-book (*Arfken, Weber, Harris* or approved substitute) and open-notes. You may also use your class notes. Please show **all steps of your work** and **write neatly and legibly** in order to receive full credit.

Please write all of your work on the attached sheets. If a problem continues onto the workspace pages at the end, please indicate this.

Useful identities and integrals:

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2 \qquad \int x \sin x \, dx = \sin x - x\cos x$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \qquad \int x \cos x \, dx = \cos x + x \sin x$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2 \qquad \int x e^x dx = x e^x - e^x$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\sin iz = i \sinh z, \qquad \cos iz = \cosh z$$

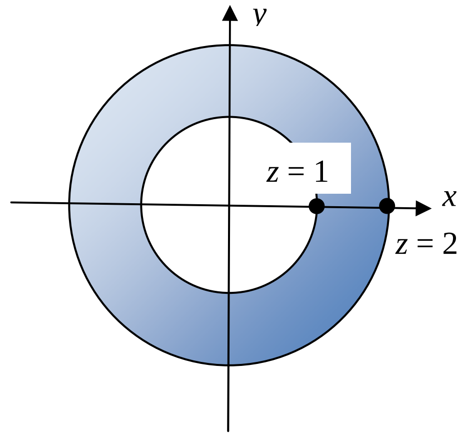
$$\sinh iz = i \sin z, \qquad \cosh iz = \cos z$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$\frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z,$$

1. **(25 pts) (Show all work!)**

Expand $\dfrac{1}{z(z-1)(z-2)}$ in a Laurent series in the region $1<|z|<2$; also, **sketch the region** of convergence of the series.



Since (see fig.) expansion is about $z = 0$, we seek an expansion in z^n:

$$\frac{1}{z(z-1)(z-2)}=\frac{1}{z}\left[\frac{1}{(z-1)(z-2)}\right]=\frac{1}{z}\left[\frac{-1}{(z-1)}+\frac{1}{(z-2)}\right]$$

$$=\frac{1}{z}\left[\frac{-1}{z(1-1/z)}-\frac{1}{2(1-z/2)}\right]$$ (note $1/z$ factor already has the form desired)

$$=\frac{1}{z}\left[\frac{-1}{z}\overbrace{\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)}^{|z|>1}-\frac{1}{2}\overbrace{\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3\right)}^{|z|<2}\right]$$

$$=\frac{-1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)-\frac{1}{2z}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3\right)$$

$$=\cdots-\frac{1}{z^4}-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{2z}-\frac{1}{4}-\frac{z}{8}-\frac{z^2}{16}-\frac{z^3}{32}-\cdots$$

$$=-\sum_{n=-\infty}^{-2}z^n-\sum_{n=-1}^{\infty}\frac{z^n}{2^{n+2}}$$

Alternatively, we have the partial fraction expansion,

$$\frac{1}{z(z-1)(z-2)}=\frac{1/2}{z}-\frac{1}{(z-1)}+\frac{1/2}{(z-2)}$$ (note each numerator is just the residue of a pole!)

$$=\frac{1/2}{z}-\frac{1}{z}\frac{1}{(1-1/z)}-\frac{1}{4}\frac{1}{(1-z/2)}$$ (expand last two terms as geometric series)

$$=\frac{1/2}{z}-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots-\frac{1}{4}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\frac{z^4}{16}+\cdots\right)$$

$$=\cdots-\frac{1}{z^4}-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{2z}-\frac{1}{4}-\frac{z}{8}-\frac{z^2}{16}-\frac{z^3}{32}-\cdots=-\sum_{n=-\infty}^{-2}z^n-\sum_{n=-1}^{\infty}\frac{z^n}{2^{n+2}}$$

2. **(25 pts)** **(Show all work!)**

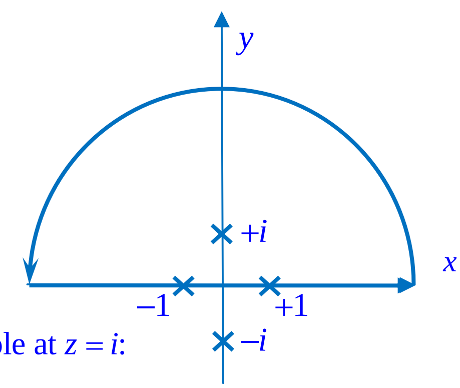
Calculate the value of the following definite integral. **Sketch any contours used**, including their orientation, closures, and singularity locations, if any. Note there are two cases.

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{\left(x^2+1\right)\left(x^2-1\right)} dx = \text{Im} \int_{-\infty}^{\infty} \frac{x e^{imx}}{\left(x^2+1\right)\left(x^2-1\right)} dx \ ; \ \textbf{Two cases}: m>0, m<0.$$

a. $\boxed{m>0 \text{ case}}$

Let $f(z) = \dfrac{ze^{imz}}{\left(z^2+1\right)\left(z^2-1\right)} = \dfrac{ze^{imz}}{(z+i)(z-i)(z+1)(z-1)}$,

and $I = \int_{-\infty}^{\infty} f(x)dx$ (**don't forget we want** Im I **later**). Note that $e^{im(x+iy)} = e^{imx}e^{-my}$ decays in the UHP for $m>0$. Hence by Jordan's lemma we have no contribution from the semicircular path in the UHP. We then have **three (equivalent) choices**:



I. Close contour *above* two poles on x-axis, enclosing only simple pole at $z=i$:

$$\oint_{C_I} f(z)dz = I - \pi i \text{Res}\left[f(z),-1\right] - \pi i \text{Res}\left[f(z),+1\right] = 2\pi i \text{Res}\left[f(z),i\right]$$

II. Close contour *below* two poles on x-axis, enclosing simple poles at $z=i,+1,-1$:

$$\oint_{C_{II}} f(z)dz = I + \pi i \text{Res}\left[f(z),-1\right] + \pi i \text{Res}\left[f(z),+1\right] = 2\pi i \text{Res}\left[f(z),i\right] + 2\pi i \text{Res}\left[f(z),-1\right] + 2\pi i \text{Res}\left[f(z),+1\right]$$

III. Close contour *along* the x-axis (c.f. figure), enclosing "half-residues" from x-axis poles:

$$\oint_{C_{III}} f(z)dz = I = 2\pi i \text{Res}\left[f(z),i\right] + \pi i \text{Res}\left[f(z),-1\right] + \pi i \text{Res}\left[f(z),+1\right]$$

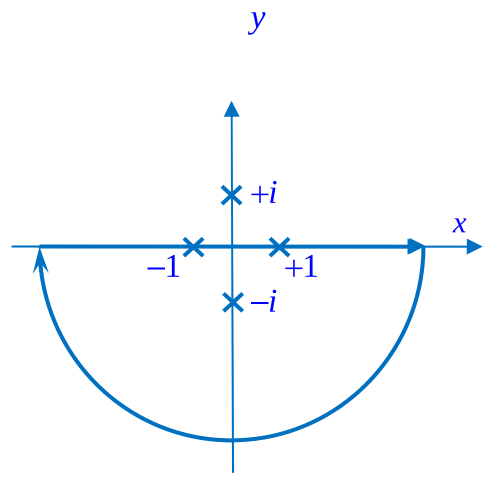
$$\boxed{\begin{aligned}
\text{Res}[f;i] &= \lim_{z\to i} \frac{\cancel{(z-i)}\, ze^{imz}}{(z+i)\,\cancel{(z-i)}\left(z^2-1\right)} = \frac{ie^{i^2 m}}{(2i)(-2)} = -\frac{e^{-m}}{4} \\
\text{Res}[f;-1] &= \lim_{z\to -1} \frac{\cancel{(z+1)}\, ze^{imz}}{\left(z^2+1\right)\cancel{(z+1)}\,(z-1)} = \frac{(-1)e^{-im}}{(2)(-2)} = \frac{e^{-im}}{4} \\
\text{Res}[f;+1] &= \lim_{z\to 1} \frac{\cancel{(z-1)}\, ze^{imz}}{\left(z^2+1\right)(z+1)\,\cancel{(z-1)}} = \frac{(1)e^{im}}{(2)(2)} = \frac{e^{im}}{4}
\end{aligned}}$$

$\Rightarrow$ Using case III above:

$$I = 2\pi i \left[\text{Res}[f;i] + \frac{1}{2}\text{Res}[f;-1] + \frac{1}{2}\text{Res}[f;+1]\right] = 2\pi i \left[-\frac{e^{-m}}{4} + \frac{e^{-im}}{8} + \frac{e^{im}}{8}\right] = \frac{\pi}{2} i \left(-e^{-m} + \cos m\right)$$

Remembering to take the imaginary part! $\Rightarrow \boxed{\text{Im}\, I = \frac{\pi}{2}\left(\cos m - e^{-m}\right)}, \quad m>0$

b. $\boxed{m<0 \text{ case}}$



For $m<0$, $e^{im(x+iy)} = e^{imx}e^{-my}$ decays in the LHP,
and hence by Jordan's lemma, Case III in part a) becomes

$$\oint_C f(z)dz = I$$

$$= -2\pi i \text{Res}[f(z), i] - \pi i \text{Res}[f(z), -1] - \pi i \text{Res}[f(z), +1]$$

where the new residue is

$$\text{Res}[f(z); -i] = \lim_{z\to -i} \frac{\cancel{(z+i)}\, z e^{imz}}{\cancel{(z+i)}(z-i)\left(z^2-1\right)} = \frac{(-i)e^{-i^2 m}}{(-2i)(-2)} = \frac{e^{+m}}{-4}$$

and the other two residues just produce the negative of their contributions in part a):

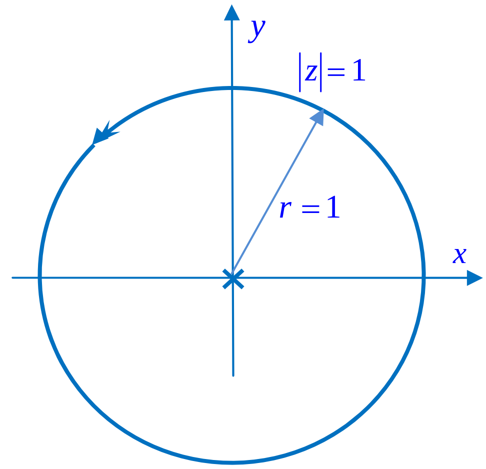
$$I = -2\pi i\left(\frac{e^{+m}}{-4}\right) - \frac{i\pi}{2}\cos m \text{ (where the last term is from part a))}$$

$$= \frac{i\pi}{2}\left(e^{+m} - \cos m\right)$$

$$\Rightarrow \boxed{\text{Im}\, I = \frac{\pi}{2}\left(e^{+m} - \cos m\right)}, \quad m<0$$

3. **(25 pts)** **(Show all work!)**

Calculate the value of the following **two** definite integrals ((a) and (b)). Sketch any contours used, including their orientation, closures, and singularity locations, if any, for each case.



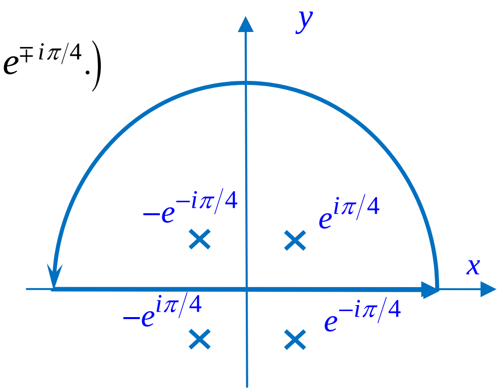
a) $\int_0^{2\pi} e^{e^{i\theta}} d\theta$

Let $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta = izd\theta$:

$$\int_0^{2\pi} e^{e^{i\theta}} d\theta = \int_{|z|=1} \frac{e^z}{iz} dz = \frac{2\pi i}{i} \text{Res}\left[\frac{e^z}{z};0\right] = 2\pi$$

since $\text{Res}\left[\frac{e^z}{z};0\right] = \lim_{z\to 0} \frac{ze^z}{z} = e^0 = 1$. Note since $|z| = |e^{i\theta}| = 1$, the integration path is the unit circle in the z-plane enclosing the pole at $z = 0$.

b) $\int_{-\infty}^{\infty} \frac{dx}{(x^4+1)}$ (It may be helpful to note that $e^{\pm i3\pi/4} = -e^{\mp i\pi/4}$.)



$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^4+1)} = \oint_C \frac{dz}{(z^4+1)} = \oint_C f(z)dz$$

where $f(z) = \frac{1}{(z^4+1)} = \frac{1}{(z^2+i)(z^2-i)} = \frac{1}{(z+e^{-i\pi/4})(z-e^{-i\pi/4})(z+e^{i\pi/4})(z-e^{i\pi/4})} \sim O(z^{-4})$

has simple poles at $z = \pm e^{-i\pi/4}, \pm e^{i\pi/4}$ (note $e^{\pm i3\pi/4} = -e^{\mp i\pi/4}$). Closing the path in the UHP,

$$I = \oint_C \frac{dz}{(z^4+1)} = 2\pi i \text{Res}\left[(z^4+1)^{-1};\, e^{i\frac{\pi}{4}}\right] + 2\pi i \text{Res}\left[(z^4+1)^{-1};\, -e^{-i\frac{\pi}{4}}\right]$$

where

$$\text{Res}\left[(z^4+1)^{-1};\, e^{i\frac{\pi}{4}}\right] = \lim_{z\to e^{i\pi/4}} \frac{(z-e^{i\pi/4})}{(z^2+i)(z+e^{i\pi/4})(z-e^{i\pi/4})} = \frac{1}{(2i)(2e^{i\pi/4})} = \frac{e^{-i\pi/4}}{4i}$$

$$\text{Res}\left[(z^4+1)^{-1};\, -e^{-i\frac{\pi}{4}}\right] = \lim_{z\to -e^{-i\pi/4}} \frac{(z+e^{-i\pi/4})}{(z+e^{-i\pi/4})(z-e^{-i\pi/4})(z^2-i)} = \frac{1}{(-2e^{-i\pi/4})(-2i)} = \frac{e^{i\pi/4}}{4i}$$

$$\Rightarrow \boxed{I = 2\pi i\left[\frac{e^{-i\pi/4}}{4i} + \frac{e^{i\pi/4}}{4i}\right] = \pi\left[\frac{e^{-i\pi/4}}{2} + \frac{e^{i\pi/4}}{2}\right] = \pi\cos(\pi/4) = \frac{\pi\sqrt{2}}{2}}$$

A simpler method to compute residues of the simple poles is to note $f(z)$ has the form $h(z)/g(z)$ where $h(z)$ $(=1)$ is analytic and $g(z)$ $(= z^4+1)$ has a simple zero at $z = z_0 (= \pm e^{\pm i\frac{\pi}{4}})$. Hence the residue, by the special formula for simple poles, is $h(z_0)/g'(z_0)$ where the prime denotes differentiation.

$$\Rightarrow \text{Res}\left[(z^4+1)^{-1};\, e^{i\frac{\pi}{4}}\right] = \left.\frac{1}{4z^3}\right|_{z=e^{i\frac{\pi}{4}}} = \frac{e^{-i\frac{3\pi}{4}}}{4} = \frac{-i\,e^{-i\frac{\pi}{4}}}{4} \quad \text{(Same as above!)}$$

$$\text{Res}\left[(z^4+1)^{-1};\, -e^{-i\frac{\pi}{4}}\right] = \left.\frac{1}{4z^3}\right|_{z=-e^{-i\frac{\pi}{4}}} = \frac{-e^{i\frac{3\pi}{4}}}{4} = \frac{-ie^{i\frac{\pi}{4}}}{4} \quad \text{(Same as above!)}$$

4. (25 pts) (Show all work!)

For each of the following functions (in (a), (b), and (c)) , give the following information for *each* singularity in the *finite* complex plane:

i. Location of singularity
ii. Singularity type: pole (*specify order and residue*), removable singularity, branch point (*specify order*), essential singularity (*specify whether isolated or non-isolated*),
iii. Residue for any poles or (isolated) essential singularities

a) $z^2 \sin\left(\frac{\pi}{z}\right)^3$

$$z^2 \sin\left(\frac{\pi}{z}\right)^3 = z^2\left[\left(\frac{\pi}{z}\right)^3 - \frac{1}{3!}\left(\frac{\pi}{z}\right)^9 + \frac{1}{5!}\left(\frac{\pi}{z}\right)^{15} - \cdots\right] = \frac{\pi^3}{z} - \frac{1}{3!}\left(\frac{\pi^9}{z^7}\right) + \frac{1}{5!}\left(\frac{\pi^{15}}{z^{13}}\right) - \cdots$$

$\Rightarrow$ i) at $z=0$ we have an ii) isolated essential singularity with iii) residue $=\pi^3$

b) $\frac{z}{\sin z}$

$\frac{z}{\sin z}$ has singularities at $z = n\pi$;

The singularities are simple poles at $z = n\pi$, $n = \pm 1, \pm 2, \cdots$,

but the singularity at $z = 0$ is removable since

$$\lim_{z\to 0}\frac{z}{\sin z} = \lim_{z\to 0}\frac{z}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)} = \lim_{z\to 0}\frac{1}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)} = 1$$

The residues at $z = n\pi$, $n = \pm 1, \pm 2, \cdots$ are

$$\lim_{z\to n\pi}\frac{z}{\frac{d}{dz}(\sin z)} = \frac{n\pi}{\cos n\pi} = (-1)^n n\pi$$

c) $\frac{z^{\frac{1}{3}}}{(z+27)}$

$z^{\frac{1}{3}}/(z+27)$ has a branch point (third order) at $z = 0$, and a simple pole at $z = -27$.

The residue at the pole is

$$\text{Res}\left[\frac{z^{\frac{1}{3}}}{(z+27)}; -27\right] = \lim_{z\to -27}\frac{(z+27)z^{\frac{1}{3}}}{(z+27)} = (-27)^{\frac{1}{3}} = \left(27e^{i\pi}\right)^{\frac{1}{3}} = 3e^{i\pi/3} \text{ (principal branch);}$$

other residues (on other branches) are -3 and $3e^{-i\pi/3}$.

ROOM FOR EXTRA WORK

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