

# ECE 6382

Fall 2023

## Project

Project update date: 10/29/2023

### INSTRUCTIONS

This project is due on Wednesday, Dec. 6, at 7:00 p.m. (this is after the last day of classes). Please submit it by slipping it under the instructor's door by the deadline. Please work individually on the project, and do not discuss it with anyone other than the instructor. Please make sure that your version of the project is the latest one that is on Canvas by checking the "project update date" on the first page. If any corrections need to be made, a revised version of the project will be posted on Canvas and this date will be changed accordingly.

### PROBLEM DESCRIPTION

The purpose of this project is to implement a root-finding algorithm in the complex plane based on the argument principle and the secant method, and to then use this to find the zeros of a function.

### FORMULATION

#### *A. Using the Argument Principle for Root Finding*

The argument principle can be used to find where the zeros of a complex function are located, if we have an analytic function that has zeros (but no poles) in a region. The concept is illustrated below in Fig. 1. We wish to see how many zeros of the analytic function there are in the rectangular region  $S$  that is inside of the rectangular contour  $C_R$ .

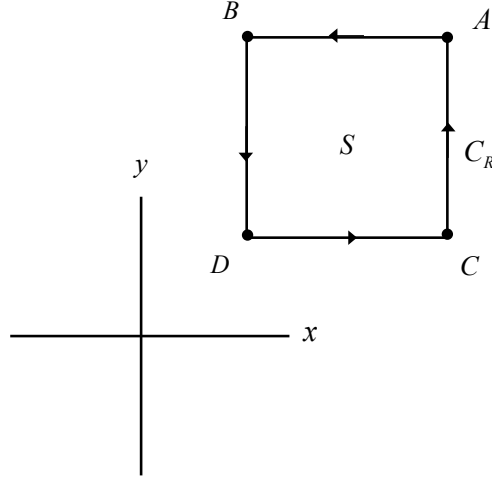


Figure 1. A rectangular contour of integration  $C_R$  on which the argument principle is applied.

According to the argument principle, if there are  $N$  zeros of an analytic function inside  $C_R$ , then we have

$$\frac{1}{2\pi i} \oint_{C_R} \frac{f'(z)}{f(z)} dz = N. \quad (1)$$

In order to find where a zero is located, we can break a region into successive smaller squares as shown below in Fig. 2. The initial square is called  $C_0$ , and has a width  $w$ . If a zero is detected inside the square  $C_0$ , that square is divided into four subsquares. Whichever subsquare has a zero inside of it is then called square  $C_1$ . This square is then divided into four subsquares, and the process continues. Figure 2 shows the process after  $n = 4$  iterations. After  $n$  iterations (i.e.,  $n$  subdividing steps) starting with a square  $C_0$  of side length  $w$ , the squares will have a side length of  $\Delta_n = w / 2^n$ . After  $n$  steps, the estimate of the root position is taken to be the center of square  $C_n$ . After  $n$  steps the location error  $E_n$  in both the  $x$  and  $y$  coordinates of the root in this “iterative square” method will be

$$E_n^{(x,y)} < \frac{w}{2^{n+1}}. \quad (2)$$

The maximum error in the root position will be

$$|\Delta z_n| < \frac{w}{2^{n+1}} \sqrt{2}. \quad (3)$$

Note that a numerical integration around a square is equivalent to four numerical integrations along a straight line in the complex plane (two vertical paths and two horizontal paths).

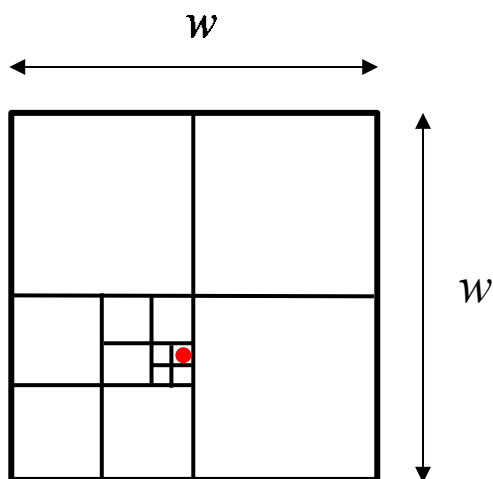


Figure 2. Finding a zero (shown by the red dot) by using a successive nesting of smaller square regions. The starting (largest) square is called  $C_0$ .

### *B. Using the Secant Method for the Final Iteration*

The square division process can continue until the desired accuracy is reached. However, once we are very close to the zero, it is more efficient to use an iterative method such as the secant method. In the secant method, an estimate of the zero location  $z_n$  of a function  $f(z)$  is found from two previous estimates  $z_{n-1}$  and  $z_{n-2}$  by using

$$z_n = z_{n-1} - (z_{n-1} - z_{n-2}) \left( \frac{f(z_{n-1})}{f(z_{n-1}) - f(z_{n-2})} \right). \quad (4)$$

The process can start by taking  $n = 2$  in the formula and letting  $z_0$  and  $z_1$  to be the centers of two successive subsquares. The above formula is then used to generate  $z_2$ . We then let  $n = 3$  in

the formula and use  $z_1$  and  $z_2$  to generate  $z_3$ . By continuing this process, the secant method can then be used to successively generate increasingly accurate estimates for the zero of the function. The advantage of the secant method is that it usually converges very fast (much faster than the method of subdividing squares). However, it is not guaranteed to converge. Convergence will usually occur if the initial guesses  $z_0$  and  $z_1$  are close enough to the correct zero location. That is why it is good to let the iterative square method proceed until the error is fairly small before applying the secant method.

## APPLICATION

Assume that we wish to find the zeros of the following function:

$$f(z) \equiv (\sin(\cos(z))e^z + A_0 \sinh(z^2)), \quad (5)$$

where  $A_0 = 1.7241209734 + i(2.8187614099)$ .

We wish to find all zeros (if there is more than one) of the function  $f(z)$  that lie within a square region of the complex plane defined by  $0 < x < 1$ ,  $0 < y < 1$ . (The function is analytic within this region.) Do this by numerically integrating the function  $f'(z)/f(z)$  around the boundary of successive squares as shown in Fig. 2. The starting square  $C_0$  is the square defined by the region  $0 < x < 1$ ,  $0 < y < 1$ .

### Task 1

Make a table showing the coordinates of the centers of squares  $C_n$  for  $n = 0, 1, 2 \dots 5$ , for those squares that contain a zero. Your table should also indicate the numerical value of the integration result for each  $n$  in the table ( $n = 0, 1, 2 \dots 5$ ). That is, your table should also show the values for

$$I_n = \frac{1}{2\pi i} \oint_{C_n} \frac{f'(z)}{f(z)} dz. \quad (6)$$

Also, for each  $n$  ( $n = 0, 1, 2 \dots 5$ ), give in your table the values of  $(I_n^r, I_n^t, I_n^l, I_n^b)$ . There are the integrals along the (right, top, left, bottom) sides of square  $C_n$ . (The sum of these four values is the value of  $I_n$ .)

Please keep at least 6 significant figures in your results.

## Task 2

Start with the centers of your last two squares,  $C_4$  and  $C_5$ , as your initial guesses for  $z_0$  and  $z_1$  in the secant method. Then make a table showing  $z_0, z_1 \dots z_{10}$ . The final value  $z_{10}$  in your table will be your final estimate of the location of the zero of  $f(z)$  that it is inside the square  $C_0$ . Please keep at least 8 significant figures in your results.

If you find that there is more than one zero inside the starting square  $C_0$ , then please repeat Tasks 1 and 2 for each of the zeros.

## NUMERICAL INTEGRATION

You may use whatever numerical integration scheme that you prefer. For example, you can parametrize the integrals in terms of  $t$  and integrate in the real variable  $t$  from 0 to 1 (as discussed in class), using MATLAB or any method/tool you wish for performing the numerical integration in  $t$ . If you program the numerical integration yourself (e.g., using 6-point Gaussian quadrature), please make sure that you have enough sample points along each side of the squares to ensure that the numerical integration along each side of the squares is accurate in Task 1. Please make sure that the numerical integrations are accurate to at least 6 significant figures.

## FORMAT GUIDELINES

Your project should consist of a write-up that contains an Abstract, an Introduction section, a Formulation section, a Results section, and a Conclusions Section. You may also include a References section and an Appendix, if appropriate. The project should be done on a word

processor, with the equations done in the word processor. You may use any word processor that you wish (e.g., Microsoft Word or Latex). Please make your equations look professional. If you are using Microsoft Word, you may wish to consider using MathType to do your equations: <https://www.wiris.com/en/mathtype/>.

The Results section should provide the results that are required, and also provide a thorough discussion of the results. All tables should look professional.

A very significant part of your grade will depend on the accuracy of your results, so you are encouraged to do as much numerical checking as possible to have confidence in your results.

You will also be graded on the neatness and quality of your write-up (including grammar, format, and appearance).