Inductors and Capacitors

We introduce here the two basic circuit elements we have not considered so far: the inductor and the capacitor.

Inductors and capacitors are energy storage devices, which means energy can be stored in them. But they cannot generate energy, so these are passive devices. The inductor stores energy in its magnetic field; the capacitor stores energy in its electric field.

A Bit of Physics

The ***behavior of the inductor*** is based on the properties of the magnetic field generated in a coil of wire. In fact, the inductor is basically a coil of wire.

Ampere's Law: current in a coil → magnetic field

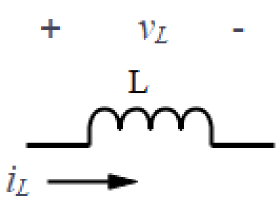
Faraday's Law: Time-varying magnetic field → induced voltage (emf)

In circuits that we will study, the time-varying magnetic field is produced by a changing current.

The ***behavior of the capacitor*** is based on the properties of the electric field created in a dielectric (non-conductor) placed between two conductors. The capacitor is basically a non-conductor sandwiched between two conductors.

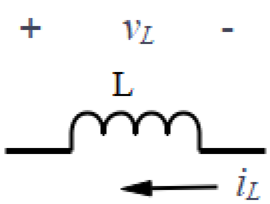
The Inductor

Circuit symbol



There is a relationship between current and voltage for an inductor, just as there is for a resistor. However, for the inductor, the voltage is related to the ***change*** in the current:

$$v_L = L\frac{di_L}{dt}.$$



This relationship holds when the voltage and current are drawn in the passive sign relationship. When they are in the active sign relationship, we need a '-' sign:

$$v_L = -L\frac{di_L}{dt}.$$

The relationship between current and voltage involves the time derivative of the current. This is because a changing current produces a changing magnetic field, which induces a voltage.

Units

$$[L] = \frac{voltage}{current/time} = \frac{volt - \sec}{Amp} = Henry[H]$$

Construction: We can make an inductor by wrapping a coil of wire around a core of magnetic material.

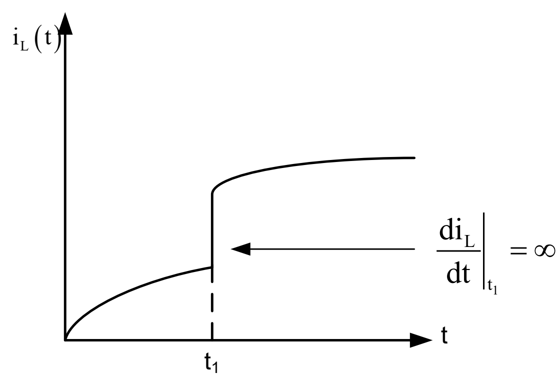
Modeling: Any physical device that involves a coil of wire can be modeled using inductance. An obvious example is a motor, whose windings have an inductance. More generally, a device with a current-induced magnetic field that interacts like an inductor will have inductance.

Important Points

1. $v_L = L\frac{di_L}{dt} \Rightarrow$ if current i_L is **constant** (NOT necessarily 0), there is no voltage. In other words, **under constant current conditions, the inductor is a short**.

2. An instantaneous change in current would generate an infinite voltage! Therefore, we assume (and in reality this is always the case) that **in an inductor, there cannot be an instantaneous change in current.**

Here is what we mean by "instantaneous change in current":



Here, the current changes from one value to another over a time span of 0 [s] at t_1, i.e., instantaneously. This produces a derivate, and hence a voltage, that is infinity large. We can't have this!!

Current in terms of voltage

The current-voltage relationship we discussed above tells us the inductor voltage if we know the inductor current. But sometimes we have the inductor voltage and need to find the current, so we need to integrate…

$$v_L(t) = L\frac{di_L}{dt}$$

Integrate both sides:

$$\int_{t_0}^{t} v_L(t)dt = \int_{i_L(t_0)}^{i_L(t)} L di_L$$

$$\therefore i_L(t) - i_L(t_0) = \frac{1}{L}\int_{t_0}^{t} v_L(t)dt$$

Do a little algebra, and voila!

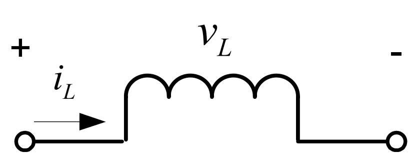
$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L(t)dt + i_L(t_0)$$

We have integrated the voltage from an "initial" time t_0 to the "final" time t (which is arbitrary). If we know the value of the current at the initial time t_0, we can find the current as a function of time.

The current-voltage relationship is a ***first-order differential equation*** for the current $i_L(t)$. To solve it (meaning that we find a numerical expression for the current as a function of time) we need to know the initial condition $i_L(t_0)$. This will be given, or there will be a way to find it.

Power and Energy

Not surprisingly, we will sometimes want to know about energy stored in the inductor and about power delivered to/from it. In what follows we assume passive sign convention. Here's the analysis.



$$p_{abs} = v_L \cdot i_L$$
$$= i_L \cdot L\frac{di_L}{dt}$$
$$= Li_L\frac{di_L}{dt} = \frac{dw}{dt}$$

Now we can integrate…

$$\int_0^w dw = \int_0^{i_L} Li_L \, di_L$$

So the energy stored in an inductor that is carrying a current i_L is…

$$W = \frac{1}{2}Li_L^2$$
$$units\ of\ w = [L]\cdot[i_L]^2$$
$$= H \cdot Amp^2 = Volt.\sec.Amp = Volt \cdot Coul$$

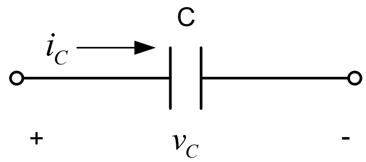
But we know that

$$1\,Volt = 1\,{}^{Joul}\!/_{Coul}$$
$$units\,of\;w = [Joul]$$

So w is an energy, as expected.

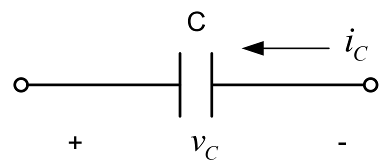
The Capacitor

Circuit symbol



There is a relationship between current and voltage for a capacitor, just as there is for a resistor. However, for the capacitor, the current is related to the change in the voltage, as follows.

$$i_C = C\frac{dv_C}{dt}$$



This relationship holds when the voltage and current are drawn in the passive sign convention. When they are in the active sign convention, we need a '-' sign:

$$i_C = -C\frac{dv_C}{dt}$$

The relationship between current and voltage involves the time derivative of the voltage. This is because a changing voltage produces a changing electric field, which induces a current.

Construction: We can make a capacitor by sandwiching an insulator between two conductors. The conductors can be metal or metallic foil, as is often used in construction of capacitors used for discrete circuit elements in the lab. It can also be an insulating material between two semiconductors, or between a metal and a semiconductor, as it is in integrated circuits.

Modeling: Any physical device that involves conducting plates or wires with insulation between them (note that air is an insulator) can be modeled using capacitance. Two wires stranded together that connect two devices will have capacitance. Semiconductor devices made from some combination of metal and semiconductor layers have capacitance.

Important Points

1. $i_C = C\frac{dv_C}{dt} \Rightarrow$ if voltage v_C is constant (NOT necessarily 0), there is no current. In other words, **under constant voltage conditions, the capacitor is an open circuit**.

2. An instantaneous change in voltage would generate an infinite current! Therefore, we assume (and in reality this is always the case) that **in a capacitor, there cannot be an instantaneous**

change in voltage. What is meant by “instantaneous change in voltage” can be seen by looking at the graph above for instantaneous change in inductor current – just substitute v_C for i_L.

Units

$$\text{units of C} = \frac{\text{Amp}\cdot\text{sec}}{\text{Volt}} = \frac{\text{Coul}}{\text{Volt}} = \text{Farad}\,[F]$$

Voltage in terms of Current

The current-voltage relationship we discussed above gives the capacitor current if we know the capacitor voltage. But sometimes we have the capacitor current and need to find the voltage. So we need to integrate…

$$i_C = C\frac{dv_C}{dt}$$

$$\Rightarrow \int_{t_0}^{t} i_C dt = \int_{v_C(t_0)}^{v_C(t)} C dv_C$$

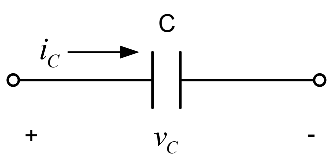
$$\therefore v_C(t) = v_C(t_0) = \frac{1}{C}\int_{t_0}^{t} i_C dt$$

$$v_C(t) = \frac{1}{C}\int_{0}^{t} i_C dt + v_C(0)$$

Power and Energy

We sometimes want to know about energy stored in the capacitor and about power delivered to/from the capacitor. In what follows we assume passive sign convention.

Here is the analysis.



$$p_{abs} = v_C \cdot i_C$$

$$p_{abs} = v_C \cdot C\frac{dv_C}{dt}$$

$$p_{abs} = Cv_C\frac{dv_C}{dt} = \frac{dw}{dt}$$

Now we integrate…

$$\int_0^w dw = \int_0^t Cv_C dv_C$$

So the energy stored in a capacitor that has a voltage v_C across it is

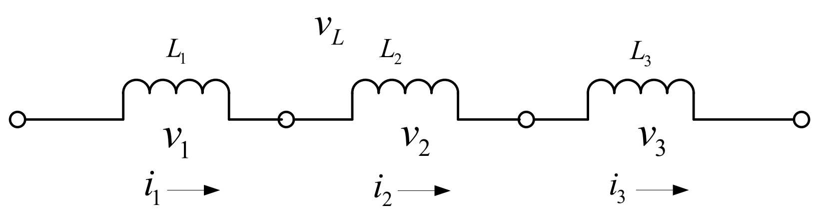
$$w = \frac{1}{2}Cv_C^2$$

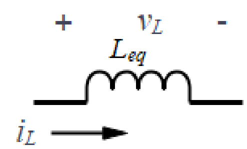
$$units\ of\ w = [C]\cdot[v_C]^2$$

$$= F \cdot Volt^2 = Coul \cdot Volt$$
$$= [Joul]$$

Series – Parallel Combinations of Inductance and Capacitance

Inductors in Series





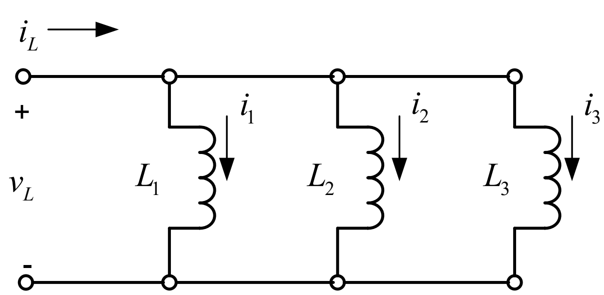
Since $i_1 = i_2 = i_3 = i_L$, we have

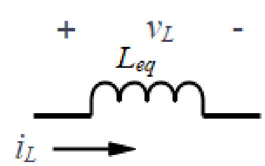
$$v_L = v_1 + v_2 + v_3 = (L_1 + L_2 + L_3)\frac{di_L}{dt}$$
$$v_L = L_{eq}\frac{di_L}{dt}$$
$$\Rightarrow L_{eq} = L_1 + L_2 + L_3$$

In general

$$L_{eq} = \sum_n L_n$$

Inductors in Parallel





We have

$$i_L = i_1 + i_2 + i_3 = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int_{t_0}^{t} v_L(t)\,dt + i_1(t_0) + i_2(t_0) + i_3(t_0).$$

Also,

$$i_L = \frac{1}{L_{eq}}\int_{t_0}^{t} v_L(t)\,dt + i_L(t_0)$$

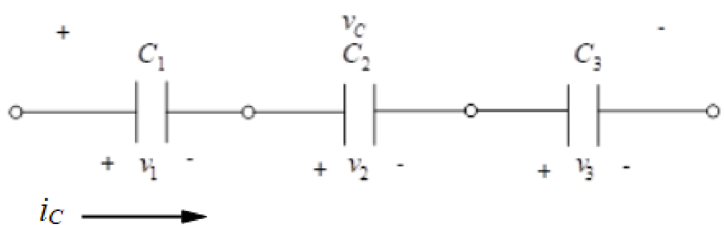
$$\Rightarrow L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \right)$$

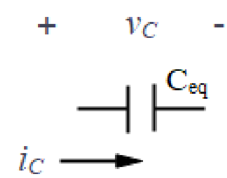
$$i_L(t_0) = i_1(t_0) + i_2(t_0) + i_3(t_0)$$

In general

$$\frac{1}{L_{eq}} = \sum_n \frac{1}{L_n}$$

Capacitors in Series





$$v_C = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right) \int_{t_0}^{t} i_C dt + v_1(t_0) + v_2(t_0) + v_3(t_0)$$

$$v_C = \frac{1}{C_{eq}} \int_{t_0}^{t} i_C dt + v_C(t_0)$$

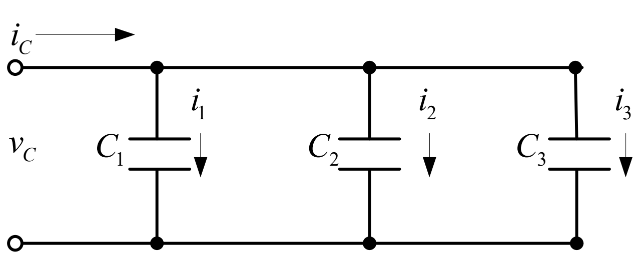
$$\Rightarrow \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

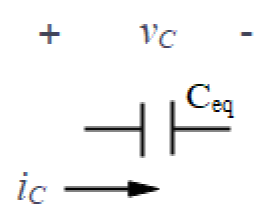
$$v_C(t_0) = v_1(t_0) + v_2(t_0) + v_3(t_0)$$

In general

$$\frac{1}{C_{eq}} = \sum_n \frac{1}{C_n}$$

Capacitors in Parallel





$$i_C = i_1 + i_2 + i_3 = \left(C_1 + C_2 + C_3\right)\frac{dv_C}{dt}$$

$$i_C = C_{eq}\frac{dv_C}{dt}$$

In general

$$\Rightarrow C_{eq} = \sum_n C_n$$