Blast From the Past!

Exam 1
Fall 2020

Problem 1 (30 pts)

Fields exists inside of a coaxial type of region that is described in cylindrical coordinates. The fields exists in the coaxial region $a < \rho < b$, $0 < \phi < 2\pi$, $0 < z < L$. This region is nonmagnetic ($\mu = \mu_0$) having a relative permittivity ε_r. The magnetic field in this region is given by

$$\underline{\mathscr{H}}(\rho, \phi, z, t) = \underline{\hat{\phi}}\left(\frac{1}{\rho}\right)\cos(kz)\cos\omega t \quad [\text{A/m}],$$

where $k = k_0\sqrt{\varepsilon_r}$ and $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$.

a) Find the magnetic field vector in the phasor domain.

b) Find the electric field vector in the phasor domain.

c) Find the complex power going through the surface S in the downward sense. The surface S is an annular region that lies in the $z = L$ plane and is defined by $a < \rho < b$ and $0 < \phi < 2\pi$. Also find the time-average power in watts and the reactive power in vars going downward though the surface S.

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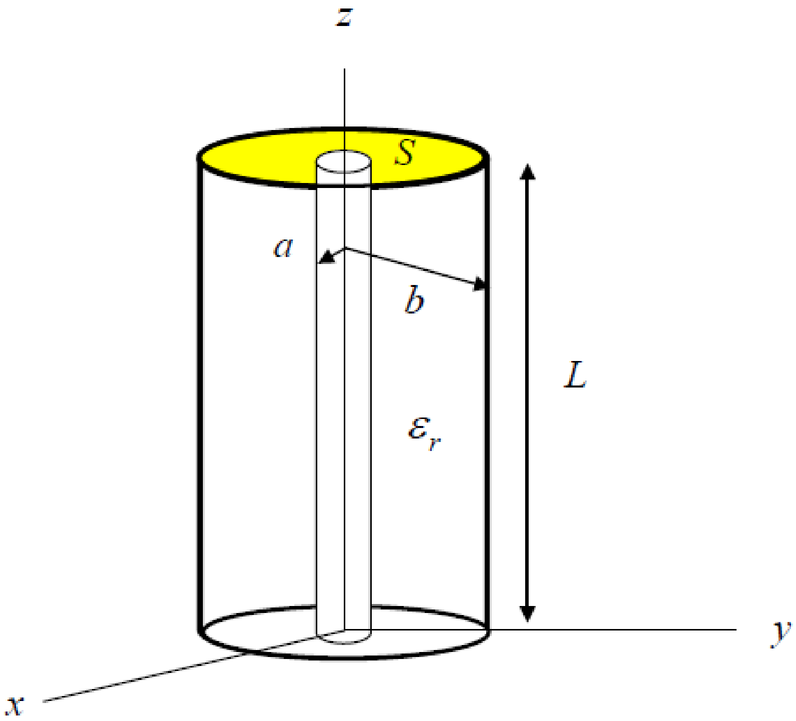

Note:
This problem corresponds to a coaxial cable transmission line that is shorted circuited at $z = 0$.

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In the phasor domain we have

$$\underline{H}(\rho,\phi,z) = \underline{\hat{\phi}}\left(\frac{1}{\rho}\right)\cos(kz) \quad [\text{A/m}]$$

To find the electric field we use Amperes' law:

$$\nabla \times \underline{H}(\rho,\phi,z) = j\omega\varepsilon \underline{E}$$

so

$$\underline{E} = \frac{1}{j\omega\varepsilon}\nabla \times \underline{H}(\rho,\phi,z) = \frac{1}{j\omega\varepsilon_0\varepsilon_r}\nabla \times \underline{H}(\rho,\phi,z)$$

We have

$$\nabla \times \underline{H} = \underline{\hat{\rho}}\left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right) + \underline{\hat{\phi}}\left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right) + \underline{\hat{z}}\frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right)$$

Keeping only the terms that survive, we have

$$\nabla \times \underline{H} = \underline{\hat{\rho}}\left(-\frac{\partial H_\phi}{\partial z}\right) + \underline{\hat{z}}\frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial \rho}\right)$$

so

$$\nabla \times \underline{H} = \underline{\hat{\rho}}\left(-\frac{\partial}{\partial z}\left(\left(\frac{1}{\rho}\right)\cos(kz)\right)\right) + \underline{\hat{z}}\frac{1}{\rho}\left(\frac{\partial\left(\rho\left(\frac{1}{\rho}\right)\cos(kz)\right)}{\partial \rho}\right)$$

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$$\underline{E} = -j\underline{\hat{\rho}}\left(\frac{1}{\rho}\right)\frac{k}{\omega\varepsilon_0\varepsilon_r}\sin(kz) \quad [\mathrm{V/m}]$$

The Poynting vector is:

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \frac{1}{2}\left(\underline{\hat{\rho}}E_\rho\right)\times\left(\underline{\hat{\phi}}H_\phi\right)^* = \frac{1}{2}\underline{\hat{z}}\left(E_\rho H_\phi^*\right)$$

Hence, we have

$$\underline{S} = \frac{1}{2}\underline{\hat{z}}\left(-j\left(\frac{1}{\rho}\right)\frac{k}{\omega\varepsilon_0\varepsilon_r}\sin(kz)\right)\left(\left(\frac{1}{\rho}\right)\cos(kz)\right)^*$$

or

$$\underline{S} = -j\underline{\hat{z}}\frac{1}{2}\left(\frac{1}{\rho^2}\right)\left(\frac{k}{\omega\varepsilon_0\varepsilon_r}\right)\left(\sin(kz)\cos(kz)\right) \quad \left[\mathrm{VA/m^2}\right]$$

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$$\underline{S} = -j\underline{\hat{z}}\frac{1}{2}\left(\frac{1}{\rho^2}\right)\left(\frac{k}{\omega\varepsilon_0\varepsilon_r}\right)\left(\sin(kz)\cos(kz)\right) \quad \left[\text{VA/m}^2\right]$$

We then have

$$P = \int_S \underline{S}\cdot(-\underline{\hat{z}})\,dS = \int_0^{2\pi}\int_a^b -S_z\,\rho\,d\rho\,d\phi = j\frac{1}{2}\left(\frac{k}{\omega\varepsilon_0\varepsilon_r}\right)\left(\sin(kL)\cos(kL)\right)\int_0^{2\pi} d\phi \int_a^b \frac{1}{\rho}\,d\rho$$

This gives us the complex power flowing down through the surface as

$$P = j\frac{1}{2}\left(\frac{k}{\omega\varepsilon_0\varepsilon_r}\right)\left(\sin(kL)\cos(kL)\right)\left[(2\pi)\ln\left(\frac{b}{a}\right)\right]$$

Hence,

$$\text{Time-average power} = 0 \quad [\text{W}]$$

$$\text{Reactive power} = \frac{1}{2}\left(\frac{k}{\omega\varepsilon_0\varepsilon_r}\right)\left(\sin(kL)\cos(kL)\right)(2\pi)\ln\left(\frac{b}{a}\right) \quad [\text{Vars}]$$