ECE 3317
Applied Electromagnetic Waves

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Notes 2
Complex Vectors

Adapted from notes by Prof. Stuart A. Long
Circuit quantities:

- $v(t)$ is a time-varying function.
- $V$ is a phasor (complex number).

Field quantities:

- $\mathcal{E}(t)$ is a time-varying vector function.
- $\mathcal{E}$ is a phasor vector (complex vector).
- $\mathcal{E}_x(t)$ is a time-varying component of a vector function.
- $E_x$ is a phasor component of a vector function.

Note:

"Handscript SF" font is used for time-domain vector quantities. (This font has been placed on Blackboard for you.)

Appendices A, B, C, and D in the Shen & Kong text book list frequently used symbols, constants, and units. Appendix B in the Hayt & Buck book discusses units in some detail.
Complex Numbers

\[ c = a + j b = |c| e^{j\phi} \]

\[ j = \sqrt{-1} \]

Phase (always in radians)

Real part \quad Imaginary part \quad Magnitude

\[ |c| = \sqrt{a^2 + b^2} \]

Euler’s identity allows us to write the polar form.

Euler’s identity:

\[ e^{j\phi} = \cos \phi + j \sin \phi \]

\[ c = |c| (\cos \phi + j \sin \phi) \]

\[ a = |c| \cos \phi \]

\[ b = |c| \sin \phi \]
Complex conjugate

\[ c_1 = (a_1 + jb_1) = |c_1| e^{j\phi} \]

\[ c_2 = c_1^* = (a_1 + jb_1)^* \equiv a_1 - jb_1 = |c_1| e^{j(-\phi)} \quad (\phi_2 = -\phi_1) \]
Complex Algebra

\[ c_1 = a_1 + jb_1 = |c_1|e^{j\phi_1} \]
\[ c_2 = a_2 + jb_2 = |c_2|e^{j\phi_2} \]

Addition

\[ c_1 + c_2 = (a_1 + a_2) + j(b_1 + b_2) \]

Subtraction

\[ c_1 - c_2 = (a_1 - a_2) + j(b_1 - b_2) \]
\[ c_1 = a_1 + jb_1 = \left| c_1 \right| e^{j\phi_1} \]
\[ c_2 = a_2 + jb_2 = \left| c_2 \right| e^{j\phi_2} \]

**Multiplication**

\[ (c_1)(c_2) = (a_1 + jb_1)(a_2 + jb_2) = (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1) \]
\[ (c_1)(c_2) = \left| c_1 \right| \left| c_2 \right| e^{j(\phi_1 + \phi_2)} \]

**Division**

\[ \frac{c_1}{c_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{a_1 + jb_1}{a_2 + jb_2} \left( \frac{a_2 - jb_2}{a_2 - jb_2} \right) = \frac{(a_1 + jb_1)(a_2 - jb_2)}{a_2^2 + b_2^2} = \frac{(a_1a_2 + jb_1b_2) + j(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2} \]
\[ \frac{c_1}{c_2} = \frac{\left| c_1 \right|}{\left| c_2 \right|} e^{j(\phi_1 - \phi_2)} \]
**Square Root**

Principal square root of a complex number (denoted by radical sign):

$$\sqrt{c} = \left( |c| \ e^{j\phi} \right)^{1/2} \quad -\pi < \phi \leq \pi$$

(\text{the principal branch})

Note:

For a positive real number \( x \), the principal square root is

$$\sqrt{x} > 0, \text{ for } x > 0$$

Example: \( \sqrt{4} = 2 \)
Square Root

$c = |c| e^{j\phi}$

Illustration of principal square root:

$\sqrt{c} = \sqrt{|c|} e^{j(\phi/2)}$

$-\pi < \phi \leq \pi$

Here we approach the point $c = -1$ from above or below the negative real axis (the two red dots).

Note: Matlab uses the principal square root: $\sqrt{-1} = +j$
Property of principal square root:

\[-\pi < \phi \leq \pi\]

\[-\pi/2 < \phi/2 \leq \pi/2\]

\[\text{Re}\sqrt{c} \geq 0\]

\[c = |c| e^{j\phi}\]

\[\sqrt{c} = \sqrt{|c|} e^{j(\phi/2)}\]

Examples:

\[\sqrt{4} = 2\]

\[\sqrt{j} = \sqrt{1}e^{j\pi/2} = \sqrt{1}e^{j\pi/4} = \frac{1+j}{\sqrt{2}}\]
General square root of a complex number:

\[ c^{1/2} = \left( |c| e^{j\phi} \right)^{1/2} \]

\[ = \left( |c| e^{j(\phi_p + 2\pi n)} \right)^{1/2} \quad (-\pi < \phi_p \leq \pi, \text{ } n \text{ is any integer}) \]

\[ = \sqrt{|c|} \ e^{j(\phi_p/2 + 2\pi n/2)} \]

\[ = (\sqrt{|c|} \ e^{j\phi_p/2}) e^{j\pi n} \]

\[ = \pm \sqrt{|c|} \ e^{j\phi_p/2} \quad (+ \text{ for } n \text{ even}, \text{ - for } n \text{ odd}) \]

\[ = \pm \sqrt{c} \]

The general square root has two possible values.

Examples:

\[ 4^{1/2} = \pm 2 \]

\[ j^{1/2} = \pm \left( \frac{1+j}{\sqrt{2}} \right) \]
Define the phasor:  \( V \equiv A e^{j\phi} \)

We then have

\[ v(t) = \text{Re}\left\{ A e^{j\phi} e^{j\omega t} \right\} \]
\[ \nu(t) = A \cos(\omega t + \phi) \]

\[
\begin{align*}
V & \equiv A e^{j\phi} & \text{going from time domain to phasor domain} \\
\nu(t) &= \text{Re} \left\{ Ve^{j\omega t} \right\} & \text{going from phasor domain to time domain}
\end{align*}
\]

Time-domain $\Leftrightarrow$ Phasor domain

\[ \nu(t) \Leftrightarrow V \]
The complex number $V$ is given by $V = A e^{j\phi}$. This can be expressed as:

$$v(t) = \text{Re}\{Ve^{j\omega t}\} = \text{Re}\{Ae^{j\phi} e^{j\omega t}\}$$
Time-Harmonic Quantities (cont.)

\[ \nu(t) \iff V \]

Note:

\[ u(t) + \nu(t) \iff U + V \quad \text{This assumes that the two sinusoidal signals are at the same frequency.} \]

\[ \frac{\partial}{\partial t} \nu(t) \iff j\omega V \quad \text{There are no time derivatives in the phasor domain!} \]

However:

\[ u(t) \nu(t) \nsim UV \quad \left( \text{i.e., } u(t) \nu(t) \neq \text{Re} \left( UVe^{j\omega t} \right) \right) \]

All phasors are complex numbers, but not all complex numbers are phasors!
Vectors in the Phasor Domain

\[ \mathbf{E}(t) = \hat{x}E_x(t) + \hat{y}E_y(t) + \hat{z}E_z(t) \] (assumed to be sinusoidal)

\[ = \hat{x}A_x \cos(\omega t + \phi_x) + \hat{y}A_y \cos(\omega t + \phi_y) + \hat{z}A_z \cos(\omega t + \phi_z) \]

Convert to phasor domain:

\[ \mathbf{E}(t) = \hat{x}A_x \cos(\omega t + \phi_x) + \hat{y}A_y \cos(\omega t + \phi_y) + \hat{z}A_z \cos(\omega t + \phi_z) \]

\[ = \text{Re} \left\{ \left( \hat{x}A_x e^{j\phi_x} + \hat{y}A_y e^{j\phi_y} + \hat{z}A_z e^{j\phi_z} \right) e^{j\omega t} \right\} \]

\[ = \text{Re} \left\{ \left( \hat{x}E_x + \hat{y}E_y + \hat{z}E_z \right) e^{j\omega t} \right\} \]

\[ = \text{Re} \left\{ \mathbf{E} e^{j\omega t} \right\} \]

where

\[ \mathbf{E} \equiv \hat{x}E_x + \hat{y}E_y + \hat{z}E_z \]

Complex vector!
Complex Vectors (cont.)

\[ \vec{E}(t) = \hat{x}E_x(t) + \hat{y}E_y(t) + \hat{z}E_z(t) \quad \text{(sinusoidal)} \]

We have proven:

\[ \vec{E}(t) = \text{Re}\left\{ E e^{j\omega t} \right\} \]

where

\[ \vec{E} \equiv \hat{x}E_x + \hat{y}E_y + \hat{z}E_z \]

Complex vector!

So we work with phasor vectors the same way as we do with phasor scalars!

Notation:

\[ \vec{E}(t) \iff \vec{E} \]

\[ \vec{E}_x(t) \iff E_x \]

\[ \vec{E}_y(t) \iff E_y \]

\[ \vec{E}_z(t) \iff E_z \]
Example 1.15 (Shen & Kong)

Assume \( A = \hat{x} + j\hat{y} \)

Find the corresponding time-domain vector

\[
\mathcal{A}(t) = \text{Re}\left\{ A e^{j\omega t} \right\} = \text{Re}\left\{ (\hat{x} + j\hat{y}) e^{j\omega t} \right\} = \text{Re}\left\{ (\hat{x} + j\hat{y})(\cos \omega t + j \sin \omega t) \right\} = \hat{x} \cos \omega t - \hat{y} \sin \omega t
\]

Note: \( |\mathcal{A}(t)| = \sqrt{\mathcal{A}_x^2(t) + \mathcal{A}_y^2(t)} = \sqrt{\cos^2 \omega t + \sin^2 \omega t} = 1 \)

The vector rotates (clockwise) with time!
Example 1.15 (cont.)

Practical application:
A circular-polarized plane wave (discussed later)

\[ \mathbf{E}(t, z) = \hat{x} \cos(\omega t - kz) - \hat{y} \sin(\omega t - kz) \]

For a fixed value of position \( z \), the electric field vector rotates clockwise in time.

\[ \mathbf{E}(0, z) = \hat{x} \cos(kz) + \hat{y} \sin(kz) \]

For a fixed value of position \( z \), the electric field vector rotates clockwise in time.

\[ \mathbf{E}(t, 0) = \hat{x} \cos(\omega t) - \hat{y} \sin(\omega t) \]

Note
For a fixed value of time, the field rotates counterclockwise in space.
Time Average of Time-Harmonic Quantities

\[ v(t) = A \cos(\omega t + \phi) \]

\[
\langle v(t) \rangle \equiv \frac{1}{T} \int_{0}^{T} A \cos(\omega t + \phi) \, dt = 0
\]

\[
T = \frac{1}{f} = \frac{2\pi}{\omega}
\]

Note: \( \cos^2 x = \left[1 + \cos(2x)\right]/2 \)

\[
\langle v^2(t) \rangle = \frac{1}{T} \int_{0}^{T} A^2 \cos^2(\omega t + \phi) \, dt
\]

\[
\langle v^2(t) \rangle = \frac{A^2}{T} \int_{0}^{T} \left\{ \frac{1 + \cos[2(\omega t + \phi)]}{2} \right\} \, dt
\]

\[
\langle v^2(t) \rangle = \frac{A^2}{T} \left( \frac{T}{2} \right) = \frac{A^2}{2}
\]

Sinusoidal (time ave = 0)

Hence

\[
\langle v^2(t) \rangle = \frac{A^2}{2}
\]

(The average value of \( \cos^2 \) is 1/2.)
Next, consider the time average of a product of sinusoids:

\[
\langle v(t)i(t) \rangle = \frac{1}{T} \int_0^T [A \cos(\omega t + \alpha)] [B \cos(\omega t + \beta)] dt
\]

**Note:** \( \cos x \cos y = \left[ \cos(x - y) + \cos(x + y) \right] / 2 \)

\[
\langle v(t)i(t) \rangle = \frac{1}{T} AB \int_0^T \cos(\omega t + \alpha) \cos(\omega t + \beta) dt
\]

\[
= \frac{AB}{T} \int_0^T \cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta) \] \left[ \frac{2}{2} \right] dt
\]

\[
= \frac{AB}{T} \left[ \int_0^T \cos(\alpha - \beta) \right] \left[ \frac{2}{2} \right]
\]

\[
= AB \left[ \frac{\cos(\alpha - \beta)}{2} \right]
\]

*Sinusoidal (time ave = 0)*
Next, consider

\[ VI^* = (A e^{j\alpha}) (B e^{j\beta})^* \]

\[ = (A e^{j\alpha}) (B e^{-j\beta}) \]

\[ = AB e^{j(\alpha - \beta)} \]

Hence,

\[ \text{Re}(VI^*) = AB \cos(\alpha - \beta) \]

Recall that

\[ \langle v(t)i(t) \rangle = AB \left[ \frac{\cos(\alpha - \beta)}{2} \right] \] (from previous slide)

Hence

\[ \langle v(t)i(t) \rangle = \frac{1}{2} \text{Re}(VI^*) \]

**Question:** Can we put the conjugate on the \( V \) instead of the \( I \)?
The results directly extend to vectors that vary sinusoidally in time.

Consider: \[ \mathcal{D}(t) \cdot \mathcal{E}(t) = \left( \mathcal{D}_x E_x + \mathcal{D}_y E_y + \mathcal{D}_z E_z \right) \]

\[ \mathcal{D}_{x,y,z}(t) = \text{Re}\left[ D_{x,y,z} e^{j\omega t} \right] \text{ etc.} \]

\[ \langle \mathcal{D} \cdot \mathcal{E} \rangle = \langle \mathcal{D}_x E_x \rangle + \langle \mathcal{D}_y E_y \rangle + \langle \mathcal{D}_z E_z \rangle \]

\[ = \frac{1}{2} \text{Re}\left( D_x E_x^* \right) + \frac{1}{2} \text{Re}\left( D_y E_y^* \right) + \frac{1}{2} \text{Re}\left( D_z E_z^* \right) \]

\[ = \frac{1}{2} \text{Re}\left( D_x E_x^* + D_y E_y^* + D_z E_z^* \right) \]

Hence \[ \langle \mathcal{D}(t) \cdot \mathcal{E}(t) \rangle = \frac{1}{2} \text{Re}\left( \mathcal{D} \cdot \mathcal{E}^* \right) \]
The result holds for both dot product and cross products.

\[ \langle D(t) \cdot E(t) \rangle = \frac{1}{2} \text{Re} \left( D \cdot E^* \right) \]

\[ \langle E(t) \times H(t) \rangle = \frac{1}{2} \text{Re} \left( E \times H^* \right) \]

where

\[ D(x, y, z, t) = \text{Re} \left[ D(x, y, z) e^{j\omega t} \right] \] etc.
To illustrate, consider the time-average stored electric energy density \([\text{J/m}^3]\) for a sinusoidal electric field.

\[
U_E(t) = \frac{1}{2} \mathcal{D}(t) \cdot \mathcal{E}(t)
\]

(from ECE 3318)

\[
\langle U_E(t) \rangle = \frac{1}{2} \langle \mathcal{D}(t) \cdot \mathcal{E}(t) \rangle
\]

\[
= \frac{1}{2} \left[ \frac{1}{2} \text{Re}(\mathbf{D} \cdot \mathbf{E}^*) \right]
\]

\[
\langle U_E(t) \rangle = \frac{1}{4} \text{Re}(\mathbf{D} \cdot \mathbf{E}^*)
\]

“Stored electric energy density”
Similarly,

\[ U_H(t) = \frac{1}{2} \mathbf{B}(t) \cdot \mathbf{H}(t) \]

\[ \langle U_H(t) \rangle = \frac{1}{4} \text{Re} \left( \mathbf{B} \cdot \mathbf{H}^* \right) \]

“This Stored magnetic energy density”

\[ \mathbf{S}(t) \equiv \mathbf{E}(t) \times \mathbf{H}(t) \]

\[ \langle \mathbf{S}(t) \rangle = \frac{1}{2} \text{Re} \left( \mathbf{E} \times \mathbf{H}^* \right) \]

“This Poynting vector”