ECE 3317
Applied Electromagnetic Waves

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Notes 20
Rectangular Waveguides
We assume that the boundary is a perfect electric conductor (PEC).

No $\text{TEM}_z$ mode can exist!
Why is there no TEM\textsubscript{z} mode?

TEM\textsubscript{z} mode:

\[ k_z = k = \omega \sqrt{\mu \varepsilon} \]

Theorem:
The shape of the fields of a TEM\textsubscript{z} mode do not change with frequency (ECE 5317).

Faraday cage effect (ECE 3318):
No static electric field can exist inside of a perfectly closed (shielded) conducting shell.

We need two conductors (a transmission line) to have a static electric field.
Two types of modes can exist independently:

- \( \text{TM}_z \): \( E_z \) only
- \( \text{TE}_z \): \( H_z \) only
We analyze the problem to solve for $E_z$ or $H_z$ (all other fields come from these).

- **TM$_z$:** $E_z$ only
- **TE$_z$:** $H_z$ only
\( H_z = 0, \quad E_z \neq 0 \)

\[ \nabla^2 E_z + k^2 E_z = 0 \quad \text{(Helmholtz equation)} \]

\( E_z = 0 \quad \text{on boundary} \quad \text{(PEC walls)} \)

Guided-wave assumption: \( E_z(x, y, z) = E_{z0}(x, y) e^{-jk_z z} \)

\[
\left( \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) + k^2 E_z = 0
\]

\[
\left( \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - k_z^2 E_z \right) + k^2 E_z = 0
\]
We solve the above equation by using the method of separation of variables.

\[ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (k^2 - k_z^2) E_z = 0 \]

Define: \( k_c^2 \equiv k^2 - k_z^2 \)

We then have: \[ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0 \]

Note that \( k_c \) is an unknown at this point.

Dividing by the \( \exp(-j k_z z) \) term, we have:

\[ \frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + k_c^2 E_{z0} = 0 \]

We solve the above equation by using the method of separation of variables.

Please see Appendix A for the solution.
Solution from separation of variables method: \( \text{TM}_{mn} \) mode

\[
E_z(x, y, z) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jk_z^{(m,n)}z}
\]

\[
k_z^{(m,n)} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}
\]

\[
k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2
\]

\[
k = \omega\sqrt{\mu\varepsilon} = k_0\sqrt{\mu_r\varepsilon_r}
\]

\[
k_0 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}
\]

\[
m = 1, 2, \ldots
\]

\[
n = 1, 2, \ldots
\]

**Note:** If either \( m \) or \( n \) is zero, the entire field is zero.
Cutoff Frequency for **Lossless** Waveguide

We start with

\[ k_{z}^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \]

Set \( k_{z}^{(m,n)} = 0 \) This defines the **cutoff frequency**.

\[ k \bigg|_{f = f_c} = 2\pi f_c^{\text{TM}_{m,n}} \sqrt{\mu\varepsilon} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \]

\[ f_c^{\text{TM}_{m,n}} = \frac{c_d}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad c_d = \frac{c}{\sqrt{\varepsilon_r}} \quad \text{(nonmagnetic material)} \]
Summary of TMₚ Solution: TMₘₙ mode

\[ E_z(x, y, z) = A_{mn} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) e^{-jk_{z(m,n)}^{(m,n)}z} \]

\[ k_{z(m,n)}^{(m,n)} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left( \frac{m \pi}{a} \right)^2 - \left( \frac{n \pi}{b} \right)^2} \]

\[ m = 1, 2, \ldots \]
\[ n = 1, 2, \ldots \]

Note: If either \( m \) or \( n \) is zero, the entire field is zero.

\[ f_c^{TM_{m,n}} = \frac{c_d}{2\pi} \sqrt{\left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2} \]

(lossless waveguide)
We now start with

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \left( k^2 - k_z^2 \right) H_z = 0$$

Guided-wave assumption: \( H_z(x, y, z) = H_{z0}(x, y) e^{-jk_zz} \)

Define: \( k_c^2 \equiv k^2 - k_z^2 \)

$$\frac{\partial^2 H_{z0}}{\partial x^2} + \frac{\partial^2 H_{z0}}{\partial y^2} + k_c^2 H_{z0} = 0$$

Please see Appendix B for the solution.
Summary of TE\textsubscript{z} Solution: TE\textsubscript{mn} mode

\[ H_z(x, y, z) = A_{mn} \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) e^{-jk_z^{(m,n)}z} \]

\[ k_z^{(m,n)} = \sqrt{k^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \]

\[ m = 0, 1, 2\ldots \]
\[ n = 0, 1, 2\ldots \]
\[ (m, n) \neq (0, 0) \]

\[ f_c^{TE_{m,n}} = \frac{c_d}{2\pi} \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} \]

(lossless waveguide)

Note: Same formula for cutoff frequency as the TM\textsubscript{z} case!
Summary for Both Modes

$$E_z(x, y, z) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jk_{z(m,n)}^z}$$

$$H_z(x, y, z) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jk_{z(m,n)}^z}$$

$$k_{z(m,n)}^z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$f_{c(m,n)} = \frac{c_d}{2\pi} \sqrt\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)$$

$$c_d = \frac{c}{\sqrt\varepsilon_r}$$

Same formula for both modes

(lossless waveguide)

$$M = 1, 2, \ldots$$

$$n = 1, 2, \ldots$$

$$TM_z$$

$$M = 0, 1, 2, \ldots$$

$$n = 0, 1, 2, \ldots$$

$$(m, n) \neq (0, 0)$$

$$TE_z$$
Field Plots

Color denotes magnitude, arrows show direction of electric field.
Wavenumber

$Wavenumber$

$\text{TM}_z \text{ or } \text{TE}_z \text{ mode: } k_z = \sqrt{k^2 - k_z^2} \quad \text{with} \quad k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

**Note:** The $(m, n)$ notation is suppressed here on $k_z$.

**Lossless waveguide:**

Above cutoff: $k_z = \beta$

\[
\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}
\]

Below cutoff: $k_z = -j\alpha$

\[
\alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}
\]
\[
k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}
\]

\[
\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}, \quad f > f_c
\]

\[
\alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}, \quad f < f_c
\]

\[
f_c = f_c^{(m,n)} = \frac{c_d}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}
\]

\[
c_d = \frac{c}{\sqrt{\varepsilon_r}}
\]
Recall: The guided wavelength $\lambda_g$ is the distance $z$ that it takes for the wave to repeat itself.

$$\lambda_g = \frac{2\pi}{\beta}$$

(This assumes that we are above the cutoff frequency – otherwise guided wavelength makes no sense.)

After some algebra (see next slide):

$$\lambda_g = \frac{\lambda_d}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$\left(\lambda_d = \frac{\lambda_0}{\sqrt{\varepsilon_r}}\right)$$

(lossless waveguide)

(Note: $\lambda_g > \lambda_d$)
Guide Wavelength

Derivation of wavelength formula (lossless waveguide):

\[
\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}} = \frac{2\pi}{\sqrt{k^2 - k_c^2}}
\]

\[
\lambda_g = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{2\pi}{k\sqrt{1 - (k_c / k)^2}} = \frac{2\pi}{\frac{\lambda_d}{\lambda_d} \sqrt{1 - (k_c / k)^2}} = \frac{\lambda_d}{\sqrt{1 - (k_c / k)^2}}
\]

\[
k = \omega \sqrt{\mu \varepsilon} = 2\pi f \sqrt{\mu \varepsilon}
\]

\[
k_c = \omega_c \sqrt{\mu \varepsilon} = 2\pi f_c \sqrt{\mu \varepsilon}
\]

\[
k_c / k = f_c / f
\]

\[
\lambda_g = \frac{\lambda_d}{\sqrt{1 - (f_c / f)^2}}
\]
The "dominant" mode is the one with the lowest cutoff frequency.

Assume $b < a$

\[ f_c = \frac{c_d}{2\pi} \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} \]

\[ c_d = \frac{c}{\sqrt{\varepsilon_r}} \]

Lowest TM\(_z\) mode: TM\(_{11}\)

Lowest TE\(_z\) mode: TE\(_{10}\)

The dominant mode is the TE\(_{10}\) mode.
Dominant Mode (cont.)

Summary (TE\textsubscript{10} Mode)

\[ H_z(x, y, z) = A_{10} \cos \left( \frac{\pi x}{a} \right) e^{-jk_z z} \]

\[ k_z = \sqrt{k^2 - \left( \frac{\pi}{a} \right)^2} \]

\[ f_c = \frac{c_d}{2a}, \quad c_d = \frac{c}{\sqrt{\varepsilon_r}} \]

\[ \beta = \sqrt{k^2 - \left( \frac{\pi}{a} \right)^2}, \quad f > f_c \]

\[ \alpha = \sqrt{\left( \frac{\pi}{a} \right)^2 - k^2}, \quad f < f_c \]
Fields of the Dominant $\text{TE}_{10}$ Mode

$$H_z(x, y, z) = A_{10} \cos\left(\frac{\pi x}{a}\right)e^{-jk_z z}$$

Find the other fields from these equations:

$$E_x = \left(-j\omega\mu\right)\frac{k^2 - k_z^2}{k^2 - k_z^2} \frac{\partial H_z}{\partial y} - \left(-j\mu\epsilon\right)\frac{k^2 - k_z^2}{k^2 - k_z^2} \frac{\partial E_y}{\partial y}$$

$$E_y = \left(j\omega\mu\right)\frac{k^2 - k_z^2}{k^2 - k_z^2} \frac{\partial H_z}{\partial x} - \left(j\mu\epsilon\right)\frac{k^2 - k_z^2}{k^2 - k_z^2} \frac{\partial E_x}{\partial y}$$

$$H_x = \left(j\omega\epsilon\right)\frac{k^2 - k_z^2}{k^2 - k_z^2} \frac{\partial E_y}{\partial y} - \left(j\mu\epsilon\right)\frac{k^2 - k_z^2}{k^2 - k_z^2} \frac{\partial H_y}{\partial x}$$

$$H_y = \left(-j\omega\epsilon\right)\frac{k^2 - k_z^2}{k^2 - k_z^2} \frac{\partial E_y}{\partial x} - \left(j\mu\epsilon\right)\frac{k^2 - k_z^2}{k^2 - k_z^2} \frac{\partial H_x}{\partial y}$$
Summary of fields for $\text{TE}_{10}$ mode:

\[
H_z(x, y, z) = A_{10} \cos\left(\frac{\pi x}{a}\right)e^{-jk_z z}
\]

\[
E_y(x, y, z) = E_{10} \sin\left(\frac{\pi x}{a}\right)e^{-jk_z z}
\]

\[
H_x(x, y, z) = -\left(\frac{k_z}{\omega \mu}\right)E_{10} \sin\left(\frac{\pi x}{a}\right)e^{-jk_z z}
\]

where

\[
E_{10} = \left(\frac{j \omega \mu}{k^2 - k_z^2}\right)\left(-\frac{\pi}{a}\right)A_{10}
\]
**Dominant Mode (cont.)**

**TE\textsubscript{10} Mode**

Length of arrows denotes magnitude of field

Color denotes magnitude of field

Spacing between arrows denotes magnitude of field
Dominant Mode (cont.)

$\text{TE}_{10}$ Mode

3D View

Magnetic field

Electric field

$\text{TE mode}$
What is the mode with the next highest cutoff frequency?

\[ f_c = \frac{c_d}{2\pi} \sqrt\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \]

Assume \( b < a / 2 \)

Then the next highest is the TE\(_{20}\) mode.

\[ f_{c}^{(2,0)} = 2 f_{c}^{(1,0)} \]

A 2:1 operating band!

Useful operating region

TE\(_{10}\) \hspace{2cm} TE\(_{20}\) \hspace{2cm} \( f_c \)
What is the mode with the next highest cutoff frequency?

\[ f_c^{(2,0)} = \frac{c_d}{2\pi} \sqrt{\left(\frac{2\pi}{a}\right)^2} = \frac{c_d}{2} \left(\frac{1}{a/2}\right) \]

\[ f_c^{(0,1)} = \frac{c_d}{2\pi} \sqrt{\left(\frac{\pi}{b}\right)^2} = \frac{c_d}{2} \left(\frac{1}{b}\right) \]

Assume \( b > a/2 \)

Then the next highest is the TE\(_{01}\) mode.

The useable bandwidth is now lower than before.

Useful operating region

\( \varepsilon, \mu \)

(lossless waveguide)
Power flow in lossless waveguide ($f > f_c$):

$$E_y(x, y, z) = E_{10} \sin\left(\frac{\pi x}{a}\right) e^{-jk_z z}$$

$$P_z = \left(\frac{ab}{4\omega\mu}\right) \beta |E_{10}|^2 \ [\text{W}]$$  \hspace{1cm} \text{(watts flowing down the waveguide)}$$

(The derivation is omitted, but please see the formula box above.)

**Note:** Above cutoff, there is only watts flowing (no vars). Below cutoff there is no watts flowing (only vars).

- Make $b$ larger to get more power flow for a given value of $a$.
- Keep $b$ smaller than $a/2$ to get maximum bandwidth.

The optimum dimension for $b$ is $a/2$ (gives maximum power flow without sacrificing bandwidth).
Plane wave interpretation of TE$_{10}$ mode

\[ E_y(x, y, z) = E_{10} \sin\left(\frac{\pi x}{a}\right) e^{-jk_z z} = E_{10} \sin(k_x x) e^{-jk_z z} \quad \left(k_x \equiv \frac{\pi}{a}\right) \]

\[ = E_{10} \left(\frac{e^{jk_x x} - e^{-jk_x x}}{2j}\right) e^{-jk_z z} \]

Note: \( \sin z = \left(\frac{e^{jz} - e^{-jz}}{2j}\right) \)

\[ E_y(x, y, z) = E'_10 e^{-jk_x x} e^{-jk_z z} + E''_10 e^{jk_x x} e^{-jk_z z} \]

\( E'_10 \equiv -E_{10} / (2j) \)
\( E''_10 \equiv +E_{10} / (2j) \)

PW #1

PW #2

\[ \tan \theta = \frac{k_x}{k_z} = \frac{\pi / a}{\sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}} \]
Losses in Waveguide ($f > f_c$)

\[ \alpha \approx \alpha_d + \alpha_c \]

**Dielectric loss:**

\[
k_z = \sqrt{k_0^2 \varepsilon_r (1 - j \tan \delta_d) - \left( \frac{\pi}{a} \right)^2} = \beta - j \alpha_d \quad (\varepsilon_r \text{ denotes } \varepsilon_r')
\]

\[ \alpha_d = - \text{Im} \sqrt{k_0^2 \varepsilon_r (1 - j \tan \delta_d) - \left( \frac{\pi}{a} \right)^2} \]

**Conductor loss:**

\[
\alpha_c = \frac{R_s}{b \eta_0} \frac{\sqrt{\varepsilon_r}}{\sqrt{1 - \left( \frac{f_c}{f} \right)^2}} \left( 1 + \frac{2b}{a} \left( \frac{f_c}{f} \right)^2 \right) \quad [\text{np/m}]
\]

(This is derived in ECE 5317.)
Example

Find the single-mode operating frequency region for air-filled X-band waveguide.

Standard X-band* waveguide:

\[ a = 0.900 \text{ inches (2.286 cm)} \]
\[ b = 0.400 \text{ inches (1.016 cm)} \]

Note: \( b < a / 2 \)

Use

\[ f_{c}^{(1,0)} = \frac{c}{2a} \]

Hence, we have:

\[ f_{c}^{(1,0)} = 6.56 \text{ [GHz]} \]
\[ f_{c}^{(2,0)} = 13.11 \text{ [GHz]} \]

\[ 6.56 < f < 13.11 \text{ [GHz]} \]

* X-band: from 8.0 to 12 GHz.
Example (cont.)

- Find the phase constant of the TE\(_{10}\) mode at 9.00 GHz.
- Find the attenuation in dB/m at 5.00 GHz

Recall:  \( f_c^{(1,0)} = 6.56 \) [GHz]

\[
\begin{align*}
\beta &= \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}, \quad f > f_c \\
\alpha &= \sqrt{\left(\frac{\pi}{a}\right)^2 - k^2}, \quad f < f_c \\
\end{align*}
\]

\[
k = k_0 = \omega \sqrt{\mu_0 \varepsilon_0} = 2\pi f / c = 2\pi / \lambda_0
\]

At 9.0 GHz:  \( k = 188.62 \) [rad/m]
At 5.0 GHz:  \( k = 104.79 \) [rad/m]

\[
k_c = \pi / a = 137.43 \) [rad/m]

At 9.00 GHz:  \( \beta = 129.13 \) [rad/m]
At 5.00 GHz:  \( \alpha = 88.91 \) [nepers/m]
At 5.0 GHz:

\[ \alpha = 88.91 \text{ [nepers/m]} \]

Recall:

\[ \text{dB/m} = 8.68589 \alpha \]

**Attenuation** = 772 dB/m

This is a very rapid attenuation!
Waveguide Components

- Straight sections
- Flexible waveguides
- Waveguide bends
- Waveguide adapters
- Waveguide couplers
- Waveguide terminations

https://www.pasternack.com
A transmission line normally operates in the TEM\(_z\) mode, where the two conductors have equal and opposite currents.

At high frequencies, waveguide modes can also propagate on transmission lines.

This is undesirable, and limits the high-frequency range of operation for the transmission line.
Dominant waveguide mode in coax (derivation omitted):

**TE\(_{11}\) mode:**

\[ f_c^{TE_{11}} \approx \frac{c}{a \sqrt{\varepsilon_r}} \left( \frac{1}{\pi} \right) \left( \frac{1}{1 + b/a} \right) \]

**Example: RG 142 coax**

\[
\begin{align*}
& a = 0.035 \text{ inches} = 8.89 \times 10^{-4} \text{ [m]} \\
& b = 0.116 \text{ inches} = 29.46 \times 10^{-4} \text{ [m]} \\
& \varepsilon_r = 2.2 \\
\end{align*}
\]

\[ b/a = 3.31 \implies Z_0 = 48.4 [\Omega] \]

\[ f_c^{TE_{11}} \approx 16.8 \text{ [GHz]} \]

This coax cannot be used above 16.8 [GHz]
Appendix A: TM\(_z\) Modes

We want to solve:

\[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(k^2 - k_z^2 \right) E_z = 0
\]

Define:

\[
k_c^2 \equiv k^2 - k_z^2
\]

We then have:

\[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0
\]

Note that \(k_c\) is an unknown at this point.

Dividing by the \(\exp(-j k_z z)\) term, we have:

\[
\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + k_c^2 E_{z0} = 0
\]

We solve the above equation by using the method of separation of variables.

We assume: \(E_{z0}(x, y) = X(x) Y(y)\)
Hence \( E_{z0}(x, y) = X(x) \, Y(y) \)

Hence \( X''Y + XY'' = -k_c^2 XY \)

Divide by \( XY \):
\[
\frac{X''}{X} + \frac{Y''}{Y} = -k_c^2
\]

Hence \( \frac{X''}{X} = -k_c^2 - \frac{Y''}{Y} \)

This has the form \( F(x) = G(y) \) Both sides of the equation must be a constant!
Denote \( \frac{X''}{X} = -k_c^2 - \frac{Y''}{Y} = \text{constant} \)

General solution: \( X(x) = A \sin(k_x x) + B \cos(k_x x) \)

Boundary conditions:

\[
\begin{aligned}
X(0) &= 0 \quad (1) \\
X(a) &= 0 \quad (2)
\end{aligned}
\]

(1) \( \implies B = 0 \quad \Rightarrow \quad X(x) = A \sin(k_x x) \)

(2) \( \implies \sin(k_x a) = 0 \)
From the last slide:

\[ \sin(k_x a) = 0 \]

This gives us the following result:

\[ k_x a = m\pi, \; m = 1, 2 \ldots \]

Hence

\[ X(x) = A \sin\left(\frac{m\pi x}{a}\right) \]

Now we turn our attention to the \( Y(y) \) function.
We have

\[
\frac{X''}{X} = -k_c^2 - \frac{Y''}{Y} = -k_x^2
\]

Hence

\[
\frac{Y''}{Y} = k_x^2 - k_c^2
\]

Denote

\[k_y^2 = k_c^2 - k_x^2\]

Then we have

\[
\frac{Y''}{Y} = -k_y^2
\]

General solution: \[Y(y) = C \sin(k_y y) + D \cos(k_y y)\]
\[ Y(y) = C \sin(k_y y) + D \cos(k_y y) \]

Boundary conditions:
\[
\begin{align*}
Y(0) &= 0 \quad (3) \\
Y(b) &= 0 \quad (4)
\end{align*}
\]

(3) \quad \Rightarrow \quad D = 0 \quad \Rightarrow \quad Y(y) = C \sin(k_y y)

(4) \quad \Rightarrow \quad \sin(k_y b) = 0

Equation (4) gives us the following result:
\[ k_y b = n\pi, \quad n = 1, 2 \ldots \]
The $Y(y)$ function is then

$$Y(y) = C \sin\left(\frac{n\pi y}{b}\right)$$

Therefore, we have

$$E_{z0}(x, y) = X(x)Y(y) = AC \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

New notation:

$$E_{z0}(x, y) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

The $E_z$ field inside the waveguide thus has the following form:

$$E_z(x, y, z) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jk_z^{(m,n)}z}$$
Recall that

\[ k_y^2 = k_c^2 - k_x^2 \]

Hence

\[ k_c^2 = k_x^2 + k_y^2 \]

Therefore the solution for \( k_c \) is given by

\[ k_c^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \]

Next, recall that

\[ k_c^2 = k^2 - k_z^2 \]

Hence

\[ k_z^2 = k^2 - k_c^2 \]

\[ k_z = k_z^{(m,n)} = \sqrt{k^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \]
Appendix B: TE<sub>z</sub> Modes

\[ E_z = 0, \quad H_z \neq 0 \]

We now start with

\[
\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \left( k^2 - k_z^2 \right) H_z = 0
\]

Using the separation of variables method again, we have

\[ H_{z0}(x, y) = X(x) Y(y) \]

where

\[ X(x) = A \sin(k_x x) + B \cos(k_x x) \]

\[ Y(y) = C \sin(k_y y) + D \cos(k_y y) \]

and

\[ k_c^2 = k_x^2 + k_y^2 \quad k_z^2 = k^2 - k_c^2 \]
Boundary conditions:

\[ E_x(x, 0) = 0 \quad E_y(0, y) = 0 \]
\[ E_x(x, b) = 0 \quad E_y(a, y) = 0 \]

The result is

\[ H_{z0}(x, y) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \]

This can be shown by using the following equations:

\[ E_x = \left( -j\omega \mu \right) \left( \frac{\partial H_z}{\partial y} \right) - \left( jk_z \right) \left( \frac{\partial E_z}{\partial x} \right) \]
\[ E_y = \left( j\omega \mu \right) \left( \frac{\partial H_z}{\partial x} \right) - \left( jk_z \right) \left( \frac{\partial E_z}{\partial y} \right) \]
The $H_z$ field inside the waveguide thus has the following form:

$$H_z(x, y, z) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jk_z^{(m,n)}z}$$

$$k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Same formula for cutoff frequency as the TE$_z$ case!

$m = 0, 1, 2 \ldots$  \hspace{1cm} $(m, n) \neq (0, 0)$

$n = 0, 1, 2 \ldots$

**Note:** The $(0,0)$ TE$_z$ mode is not valid, since it violates the magnetic Gauss law:

$$\mathbf{H}(x, y, z) = \hat{z} A_{00} e^{-jkz} \quad \nabla \cdot \mathbf{H}(x, y, z) \neq 0$$