ECE 3317
Applied Electromagnetic Waves

Prof. David R. Jackson
Fall 2023

Notes 20
Rectangular Waveguides
- We assume that the boundary is a perfect electric conductor (PEC).

No $\text{TEM}_z$ mode can exist!
Why is there no TEM\textsubscript{z} mode?

\[ \varepsilon, \mu \]

Rectangular waveguide mode \((m,n)\):

\[ k_z^{(m,n)} = k = \omega \sqrt{\mu \varepsilon} \]

\[ 22, 2 (m,n) \neq (0,0) \]

\[ k_z^{(m,n)} \neq k \]
Rectangular Waveguide (cont.)

Rectangular Waveguide

- Two types of modes can exist independently:

  - $TM_z$: $E_z$ only
  - $TE_z$: $H_z$ only
We analyze the problem to solve for $E_z$ or $H_z$ (all other fields come from these).

- **TM**$^z$: $E_z$ only
- **TE**$^z$: $H_z$ only
\( H_z = 0, \quad E_z \neq 0 \)

\[
\nabla^2 E_z + k^2 E_z = 0 \quad \text{(Helmholtz equation)}
\]

\[ E_z = 0 \quad \text{on boundary} \quad \text{(PEC walls)} \]

Guided-wave assumption: 
\[ E_z(x, y, z) = E_{z0}(x, y) e^{-jk_z z} \]

\[
\left( \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) + k^2 E_z = 0
\]

\[
\left( \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - k_z^2 E_z \right) + k^2 E_z = 0
\]
We solve the above equation by using the method of separation of variables.

\[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(k^2 - k_z^2\right) E_z = 0
\]

Define: \[k_c^2 \equiv k^2 - k_z^2\]

We then have: \[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0
\]

Note that \(k_c\) is an unknown at this point.

Dividing by the \(\exp(-j k_z z)\) term, we have:

\[
\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + k_c^2 E_{z0} = 0
\]

We solve the above equation by using the method of separation of variables.

Please see Appendix A for the solution.
Solution from separation of variables method: \( \text{TM}_{mn} \) mode

\[
E_z(x, y, z) = A_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-jk_z^{(m,n)} z}
\]

\[
k_z^{(m,n)} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2}
\]

\[
k_c^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2
\]

\[
k = \omega \sqrt{\mu \varepsilon} = k_0 \sqrt{\mu_r \varepsilon_r}
\]

\[
k_0 = \omega \sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}
\]

\[
m = 1, 2, \ldots
\]

\[
n = 1, 2, \ldots
\]

**Note:** If either \( m \) or \( n \) is zero, the entire field is zero.
Cutoff Frequency for **Lossless** Waveguide

We start with

\[
k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}
\]

Set \(k_z^{(m,n)} = 0\) This defines the **cutoff frequency**.

\[
k \bigg|_{f = f_c} = 2\pi f_c^{\text{TM}_{m,n}} \sqrt{\mu\varepsilon} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}
\]

\[
f_c^{\text{TM}_{m,n}} = \frac{c_d}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}
\]

Note:
Cutoff frequency only has a clear meaning in the lossless case \((k\text{ is real})\).
Summary of \( \text{TM}_z \) Solution: \( \text{TM}_{mn} \) mode

\[
E_z(x, y, z) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jk_{z}^{(m,n)}z}
\]

\[
k_{z}^{(m,n)} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}
\]

\[
f_{c}^{\text{TM}_{m,n}} = \frac{c_d}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}
\]

Note: If either \( m \) or \( n \) is zero, the entire field is zero.

(lossless waveguide)
$E_z = 0, \quad H_z \neq 0$

We now start with

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \left( k^2 - k_z^2 \right) H_z = 0$$

Guided-wave assumption: $H_z(x, y, z) = H_{z0}(x, y)e^{-jk_zz}$

Define: $k_c^2 \equiv k^2 - k_z^2$

$$\frac{\partial^2 H_{z0}}{\partial x^2} + \frac{\partial^2 H_{z0}}{\partial y^2} + k_c^2 H_{z0} = 0$$

Please see Appendix B for the solution.
Summary of TE_\text{z} Solution: TE_{mn} mode

\[ H_z(x, y, z) = A_{mn} \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) e^{-jk_z^{(m,n)}z} \]

\[ k_z^{(m,n)} = \sqrt{k^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \]

\[ m = 0, 1, 2 \ldots \]
\[ n = 0, 1, 2 \ldots \]
\[ (m, n) \neq (0, 0) \]

\[ f_c^{TE_{m,n}} = \frac{c_d}{2\pi} \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} \]

Note: Same formula for cutoff frequency as the TM_\text{z} case!


**Summary for Both Modes**

\[ E_z(x, y, z) = A_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-jk_z^{(m,n)}z} \]

\[ H_z(x, y, z) = A_{mn} \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) e^{-jk_z^{(m,n)}z} \]

\[ k_z^{(m,n)} = \sqrt{k^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \]

\[ f_c^{(m,n)} = \frac{c_d}{2\pi} \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} \]

\[ c_d = \frac{c}{\sqrt{\varepsilon_r}} \]

(lossless waveguide)

\[ m = 1, 2, \ldots \]

\[ n = 1, 2, \ldots \]

\[ m = 0, 1, 2, \ldots \]

\[ n = 0, 1, 2, \ldots \]

\[ (m, n) \neq (0, 0) \]
Field Plots

Color denotes magnitude, arrows show direction of electric field.
Wavenumber

**TM**\(_z\) or **TE**\(_z\) mode: \( k_z = \sqrt{k^2 - k_c^2} \) with \( k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \)

**Note:** The \((m,n)\) notation is suppressed here on \(k_z\).

**Lossless waveguide:**

Above cutoff: \( k_z = \beta \)

\[
\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}
\]

Below cutoff: \( k_z = -j\alpha \)

\[
\alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}
\]

(Recall the general formula for \(k_z\): \( k_z = \beta - j\alpha \))
Wavenumber Plot

\[ k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \]

\[ \beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}, \quad f > f_c \]

\[ \alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}, \quad f < f_c \]

\[ \beta = k = 2\pi f \sqrt{\mu\varepsilon} \]

\[ f_c = f_c^{(m,n)} = \frac{c_d}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \]

\[ c_d = \frac{c}{\sqrt{\varepsilon_r}} \]
Recall: The guided wavelength $\lambda_g$ is the distance $z$ that it takes for the wave to repeat itself.

$$\lambda_g = \frac{2\pi}{\beta}$$

(This assumes that we are above the cutoff frequency – otherwise guided wavelength makes no sense.)

After some algebra (see next slide): 

$$\lambda_g = \frac{\lambda_d}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$\left(\lambda_d = \frac{\lambda_0}{\sqrt{\varepsilon_r}}\right)$$

(lossless waveguide)

(Note: $\lambda_g > \lambda_d$)
Guided Wavelength (cont.)

Derivation of wavelength formula (lossless waveguide):

\[
\begin{align*}
\lambda_g &= \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} = \frac{2\pi}{\sqrt{k^2 - k_c^2}} \\
\end{align*}
\]

\[
\begin{align*}
\lambda_g &= \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{2\pi}{k\sqrt{1 - (k_c / k)^2}} = \frac{2\pi}{\lambda_d \sqrt{1 - (k_c / k)^2}} = \frac{\lambda_d}{\sqrt{1 - (k_c / k)^2}} \\
\end{align*}
\]

\[
\begin{align*}
k &= \omega \sqrt{\mu\varepsilon} = 2\pi f \sqrt{\mu\varepsilon} \\
k_c &= \omega_c \sqrt{\mu\varepsilon} = 2\pi f_c \sqrt{\mu\varepsilon} \\
k_c / k &= f_c / f \\
\end{align*}
\]

\[
\begin{align*}
\lambda_g &= \frac{\lambda_d}{\sqrt{1 - (f_c / f)^2}} \\
\end{align*}
\]
The "dominant" mode is the one with the **lowest cutoff frequency**.

Assume \( b < a \)

\[
f_c = \frac{c_d}{2\pi} \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2}
\]

\[
c_d = \frac{c}{\sqrt{\varepsilon_r}}
\]

Lowest \( \text{TM}_z \) mode: \( \text{TM}_{11} \)

Lowest \( \text{TE}_z \) mode: \( \text{TE}_{10} \)

The dominant mode is the \( \text{TE}_{10} \) mode.
Dominant Mode (cont.)

Summary (TE\textsubscript{10} Mode)

\[ H_z(x, y, z) = A_{10} \cos\left(\frac{\pi x}{a}\right) e^{-jk_zz} \]

\[ k_z = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} \quad k = k_0 \sqrt{\varepsilon_r} \]

\[ f_c = \frac{c_d}{2a} \quad c_d = \frac{c}{\sqrt{\varepsilon_r}} \]

\[ \beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}, \quad f > f_c \]

\[ \alpha = \sqrt{\left(\frac{\pi}{a}\right)^2 - k^2}, \quad f < f_c \]
Fields of the Dominant TE_{10} Mode

\[ H_z(x, y, z) = A_{10} \cos \left( \frac{\pi x}{a} \right) e^{-jk_z z} \]

Find the other fields from these equations (Appendix A of Notes 19):

\[ E_x = \left( \frac{-j \omega \mu}{k^2 - k_z^2} \right) \frac{\partial H_z}{\partial y} - \left( \frac{jk_z}{k^2 - k_z^2} \right) \frac{\partial E_z}{\partial x} \]

\[ E_y = \left( \frac{j \omega \mu}{k^2 - k_z^2} \right) \frac{\partial H_z}{\partial x} - \left( \frac{jk_z}{k^2 - k_z^2} \right) \frac{\partial E_z}{\partial y} \]

\[ H_x = \left( \frac{j \omega \varepsilon}{k^2 - k_z^2} \right) \frac{\partial E_z}{\partial y} - \left( \frac{jk_z}{k^2 - k_z^2} \right) \frac{\partial H_z}{\partial x} \]

\[ H_y = \left( \frac{-j \omega \varepsilon}{k^2 - k_z^2} \right) \frac{\partial E_z}{\partial x} - \left( \frac{jk_z}{k^2 - k_z^2} \right) \frac{\partial H_z}{\partial y} \]
Summary of fields for TE_{10} mode:

\[ H_z(x, y, z) = A_{10} \cos \left( \frac{\pi x}{a} \right) e^{-jk_z z} \]
\[ E_y(x, y, z) = E_{10} \sin \left( \frac{\pi x}{a} \right) e^{-jk_z z} \]
\[ H_x(x, y, z) = -\left( \frac{k_z}{\omega \mu} \right) E_{10} \sin \left( \frac{\pi x}{a} \right) e^{-jk_z z} \]

where

\[ E_{10} = \left( \frac{j \omega \mu}{k^2 - k_z^2} \right) \left( -\frac{\pi}{a} \right) A_{10} \]
Dominant Mode (cont.)

**TE\textsubscript{10} Mode**

Length of arrows denotes magnitude of field

Color denotes magnitude of field

Spacing between arrows denotes magnitude of field
Dominant Mode (cont.)

$\text{TE}_{10}$ Mode

3D View
What is the mode with the next highest cutoff frequency?

\[
f_c = \frac{c_d}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}
\]

\[
f_c^{(1,0)} = \frac{c_d}{2a}
\]

\[
f_c^{(2,0)} = \frac{c_d}{2\pi} \sqrt{\left(\frac{2\pi}{a}\right)^2} = \frac{c_d}{2} \left(\frac{1}{a/2}\right)
\]

\[
f_c^{(0,1)} = \frac{c_d}{2\pi} \sqrt{\left(\frac{\pi}{b}\right)^2} = \frac{c_d}{2} \left(\frac{1}{b}\right)
\]

Assume \( b < a / 2 \)

Then the next highest is the TE\(_{20}\) mode.

\[
f_c^{(2,0)} = 2 f_c^{(1,0)}
\]

A 2:1 operating band!

Useful operating region

(lossless waveguide)
What is the mode with the next highest cutoff frequency?

\[
f_c^{(2,0)} = \frac{c_d}{2\pi} \sqrt{\left(\frac{2\pi}{a}\right)^2} = \frac{c_d}{2} \left(\frac{1}{a/2}\right) \quad f_c^{(0,1)} = \frac{c_d}{2\pi} \sqrt{\left(\frac{\pi}{b}\right)^2} = \frac{c_d}{2} \left(\frac{1}{b}\right)
\]

Assume \( b > a/2 \)

Then the next highest is the TE\(_{01}\) mode.

The useable bandwidth is now lower than before.

Useful operating region

(lossless waveguide)
Power flow in lossless waveguide ($f > f_c$):

$$E_y(x, y, z) = E_{10} \sin\left(\frac{\pi x}{a}\right)e^{-jk_zz}$$

$$P_z = \left(\frac{ab}{4\omega\mu}\right)\beta|E_{10}|^2 \text{ [W]} \quad \text{(watts flowing down the waveguide)}$$

(The derivation is omitted, but please see the formula box above.)

**Note:** Above cutoff, there is only watts flowing (no vars). Below cutoff there is no watts flowing (only vars).

- Make $b$ larger to get more power flow for a given value of $a$.
- Keep $b$ smaller than $a/2$ to get maximum bandwidth.

The optimum dimension for $b$ is $a/2$ (gives maximum power flow without sacrificing bandwidth).
Plane wave interpretation of TE_{10} mode

\[ E_y(x, y, z) = E_{10} \sin\left(\frac{\pi x}{a}\right)e^{-jk_z z} = E_{10} \sin(k_x x) e^{-jk_z z} \quad \left( k_x \equiv \frac{\pi}{a} \right) \]

\[ = E_{10} \left( \frac{e^{jk_x x} - e^{-jk_x x}}{2j} \right) e^{-jk_z z} \]

**Note:** \( \sin z = \left( \frac{e^{jz} - e^{-jz}}{2j} \right) \)

\[ E_y(x, y, z) = E'_10 e^{-jk_x x} e^{-jk_z z} + E''_{10} e^{jk_x x} e^{-jk_z z} \]

\[ \begin{align*}
E'_10 & \equiv -E_{10} / (2j) \\
E''_{10} & \equiv +E_{10} / (2j)
\end{align*} \]

\[ \tan \theta = \frac{k_x}{k_z} = \frac{\pi / a}{\sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}} \]
Losses in Waveguide ($f > f_c$)

**Dielectric loss:**

$$k_z = \sqrt{k_0^2 \varepsilon_r (1 - j \tan \delta_d) - \left(\frac{\pi}{a}\right)^2} = \beta - j\alpha_d$$

Recall: $\varepsilon_{rc} = \varepsilon_r (1 - j \tan \delta)$

**Conductor loss:**

$$\alpha_d = -\text{Im} \sqrt{k_0^2 \varepsilon_r (1 - j \tan \delta_d) - \left(\frac{\pi}{a}\right)^2}$$

**Note:**

If we are below cutoff, attenuation is mainly due to evanescence, so we don’t worry about conductor and dielectric loss then.

**Dielectric loss:**

$$\alpha_c = \frac{R_s}{b\eta_0} \frac{\sqrt{\varepsilon_r}}{\sqrt{1 - (f_c / f)^2}} \left(1 + \frac{2b}{a} \left(\frac{f_c}{f}\right)^2\right) \text{ [np/m]}$$

(This is derived in ECE 5317.)
Find the single-mode operating frequency region for air-filled X-band waveguide.

Standard X-band* waveguide:

\[ a = 0.900 \text{ inches (2.286 cm)} \]
\[ b = 0.400 \text{ inches (1.016 cm)} \]

Note: \( b < a / 2 \)

Use

\[ f_c^{(1,0)} = \frac{c}{2a} \]

Hence, we have:

\[ f_c^{(1,0)} = 6.56 \text{ [GHz]} \]
\[ f_c^{(2,0)} = 13.11 \text{ [GHz]} \]

\[ 6.56 < f < 13.11 \text{ [GHz]} \]

* X-band: from 8.0 to 12 GHz.
Example (cont.)

- Find the phase constant of the TE_{10} mode at 9.00 GHz.
- Find the attenuation in dB/m at 5.00 GHz

Recall: $f_{c}^{(1,0)} = 6.56$ [GHz]

\[
\beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}, \quad f > f_c
\]

\[
\alpha = \sqrt{\left(\frac{\pi}{a}\right)^2 - k^2}, \quad f < f_c
\]

\[
k = k_0 = \omega \sqrt{\mu_0 \varepsilon_0} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda_0}
\]

At 9.0 GHz: $k = 188.62$ [rad/m]
At 5.0 GHz: $k = 104.79$ [rad/m]

\[
k_c = \frac{\pi}{a} = 137.43$ [rad/m]

At 9.00 GHz: $\beta = 129.13$ [rad/m]
At 5.00 GHz: $\alpha = 88.91$ [nepers/m]
At 5.0 GHz:

\[ \alpha = 88.91 \text{ [nepers/m]} \]

Recall:

\[ \text{dB/m} = 8.68589 \alpha \]

**Attenuation** = 772 dB/m

This is a very rapid attenuation!
Waveguide Components

- Straight sections
- Flexible waveguides
- Waveguide bends
- Waveguide adapters
- Waveguide couplers
- Waveguide terminations

https://www.pasternack.com
A transmission line normally operates in the TEM_0 mode, where the two conductors have equal and opposite currents.

At high frequencies, waveguide modes can also propagate on transmission lines.

This is undesirable, and it limits the high-frequency range of operation for the transmission line.
Dominant waveguide mode in coax (derivation omitted):

\[
f_c^{TE_{11}} \approx \frac{c}{a \sqrt{\varepsilon_r}} \left( \frac{1}{\pi} \right) \left( \frac{1}{1 + b/a} \right)
\]

**Example: RG 142 coax**

\[
\frac{a}{b} = 3.31 \implies Z_0 = 48.4[\Omega]
\]

\[
f_c^{TE_{11}} \approx 16.8 \text{ [GHz]}
\]

This coax cannot be used above 16.8 [GHz]
We want to solve:
\[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(k^2 - k_z^2\right)E_z = 0
\]

Define:
\[
k_c^2 \equiv k^2 - k_z^2
\]

We then have:
\[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0
\]

Note that \(k_c\) is an unknown at this point.

Dividing by the \(\exp(-jk_z z)\) term, we have:
\[
\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + k_c^2 E_{z0} = 0
\]

We solve the above equation by using the method of separation of variables.

We assume:
\[
E_{z0}(x, y) = X(x) Y(y)
\]
\[ \frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + k_c^2 E_{z0} = 0 \]

\[ E_{z0}(x, y) = X(x) Y(y) \]

Hence \[ X'' Y + X Y'' = -k_c^2 XY \]

Divide by \( XY \) : \[ \frac{X''}{X} + \frac{Y''}{Y} = -k_c^2 \]

Hence \[ \frac{X''}{X} = -k_c^2 - \frac{Y''}{Y} \]

This has the form \[ F(x) = G(y) \]

Both sides of the equation must be a constant!
\[ \frac{X''}{X} = -k_c^2 - \frac{Y''}{Y} = \text{constant} \]

Denote \[ \frac{X''}{X} = -k_x^2 = \text{constant} \]

General solution: \[ X(x) = A \sin(k_x x) + B \cos(k_x x) \]

Boundary conditions:

\[
\begin{align*}
X(0) &= 0 \quad (1) \\
X(a) &= 0 \quad (2)
\end{align*}
\]

\( (1) \quad \Rightarrow \quad B = 0 \quad \Rightarrow \quad X(x) = A \sin(k_x x) \)

\( (2) \quad \Rightarrow \quad \sin(k_x a) = 0 \)
From the last slide:

\[ \sin(k_x a) = 0 \]

This gives us the following result:

\[ k_x a = m\pi, \quad m = 1, 2 \ldots \]

\[ k_x = \frac{m\pi}{a} \]

Hence

\[ X(x) = A\sin\left(\frac{m\pi x}{a}\right) \]

Now we turn our attention to the \( Y(y) \) function.
We have

\[ \frac{X''}{X} = -k_c^2 - \frac{Y''}{Y} = -k_x^2 \]

Hence

\[ \frac{Y''}{Y} = k_x^2 - k_c^2 \]

Denote

\[ k_y^2 = k_c^2 - k_x^2 \]

Then we have

\[ \frac{Y''}{Y} = -k_y^2 \]

General solution:

\[ Y(y) = C \sin(k_y y) + D \cos(k_y y) \]
\[ Y(y) = C \sin(k_y y) + D \cos(k_y y) \]

Boundary conditions:
\[
\begin{align*}
Y(0) &= 0 \quad \text{(3)} \\
Y(b) &= 0 \quad \text{(4)}
\end{align*}
\]

\( (3) \Rightarrow D = 0 \Rightarrow Y(y) = C \sin(k_y y) \)

\( (4) \Rightarrow \sin(k_y b) = 0 \)

Equation (4) gives us the following result:
\[ k_y b = n\pi, \quad n = 1, 2\ldots \]
The \( Y(y) \) function is then
\[
Y(y) = C \sin \left( \frac{n\pi y}{b} \right)
\]

Therefore, we have
\[
E_{z0}(x, y) = X(x) Y(y) = AC \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)
\]

New notation:
\[
E_{z0}(x, y) = A_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)
\]

The \( E_z \) field inside the waveguide thus has the following form:
\[
E_z(x, y, z) = A_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-jk_z^{(m,n)}z}
\]
Recall that \( k_y^2 = k_c^2 - k_x^2 \)

Hence, \( k_c^2 = k_x^2 + k_y^2 \)

Therefore, the solution for \( k_c \) is given by

\[
k_c^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2
\]

Next, recall that \( k_c^2 = k^2 - k_z^2 \)

Hence \( k_z^2 = k^2 - k_c^2 \)

\[
k_z = k_z^{(m,n)} = \sqrt{k^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2}
\]
Appendix B: TE\(_z\) Modes

\[ E_z = 0, \quad H_z \neq 0 \]

We now start with

\[
\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \left( k_x^2 - k_z^2 \right) H_z = 0
\]

Using the separation of variables method again, we have

\[ H_{z0}(x, y) = X(x)Y(y) \]

where

\[ X(x) = A \sin(k_x x) + B \cos(k_x x) \]

\[ Y(y) = C \sin(k_y y) + D \cos(k_y y) \]

and

\[ k_c^2 = k_x^2 + k_y^2 \quad k_z^2 = k^2 - k_c^2 \]
Boundary conditions:

\[ E_x(x, 0) = 0 \quad E_y(0, y) = 0 \]
\[ E_x(x, b) = 0 \quad E_y(a, y) = 0 \]

The result is

\[ H_{z0}(x, y) = A_{mn} \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) \]

This can be shown by using the following equations:

\[ E_x = \left( -\frac{j\omega \mu}{k^2 - k_z^2} \right) \frac{\partial H_z}{\partial y} - \left( \frac{jk_z}{k^2 - k_z^2} \right) \frac{\partial E_z}{\partial x} \]
\[ E_y = \left( \frac{j\omega \mu}{k^2 - k_z^2} \right) \frac{\partial H_z}{\partial x} - \left( \frac{jk_z}{k^2 - k_z^2} \right) \frac{\partial E_z}{\partial y} \]
The $H_z$ field inside the waveguide thus has the following form:

$$H_z(x, y, z) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jk_z^{(m,n)} z}$$

$$k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$m = 0, 1, 2 \ldots$

$n = 0, 1, 2 \ldots \quad (m,n) \neq (0,0)$

**Note:** The $(0,0)$ TE$_z$ mode is not valid, since it violates the magnetic Gauss law:

$$\nabla \cdot \mathbf{H}(x, y, z) \neq 0$$