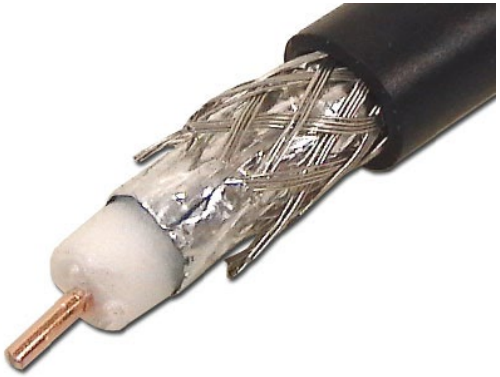
ECE 3317
Applied Electromagnetic Waves

Prof. David R. Jackson

Fall 2023

Notes 6
Transmission Lines
(Time Domain)



Note about Books

Note:

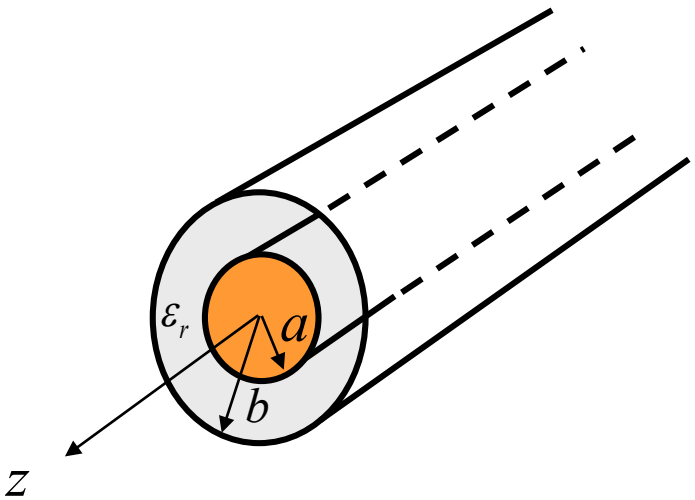
Chapter 10 of the Hayt & Buck book has a thorough discussion of transmission lines in the time domain.

Transmission lines is the subject of Chapter 6 in the Shen & Kong book. However, the subject of wave propagation in the <u>time domain</u> is not treated very thoroughly there.

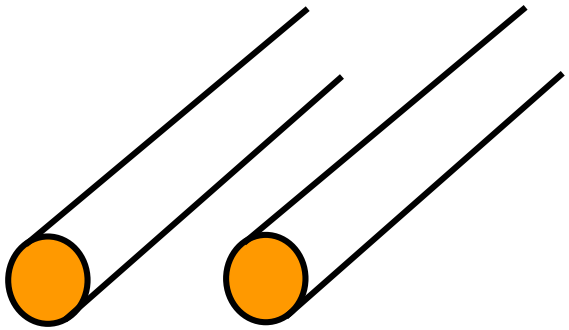
Transmission Lines

A transmission line is a two-conductor system that is used to transmit a signal from one point to another point.

Two common examples:



Coaxial cable

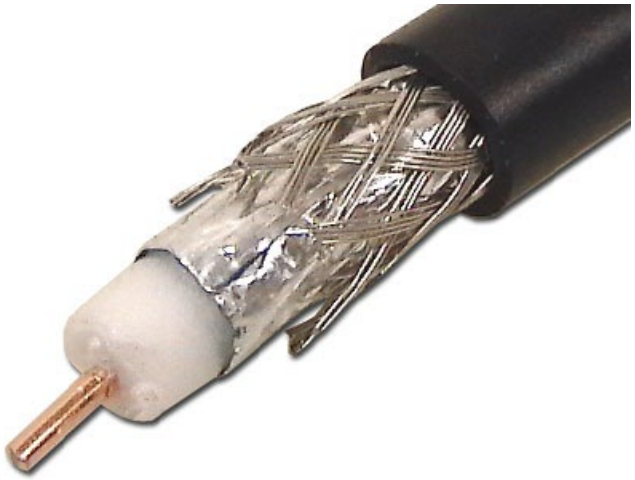


Twin lead

A transmission line is normally used in the <u>balanced</u> (or “differential”) mode, meaning equal and opposite currents (and charges) on the two conductors.

Transmission Lines (cont.)

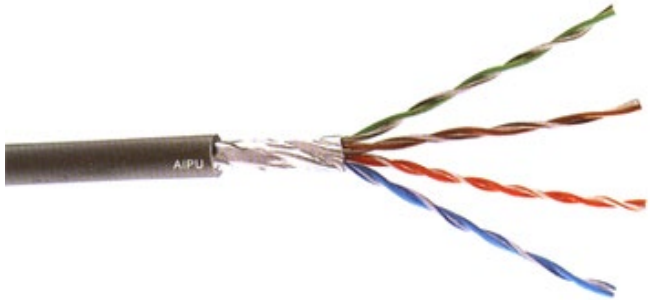
Here's what they look like in real-life.



Coaxial cable

Twin lead

Transmission Lines (cont.)

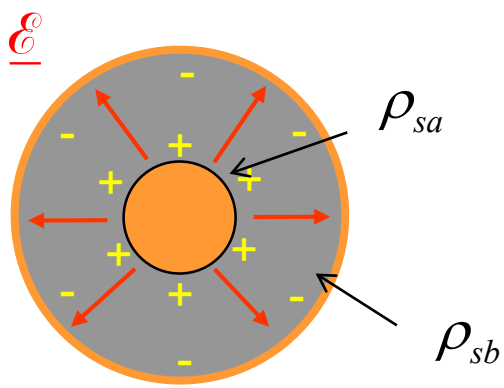


CAT 5 cable
(twisted pair)

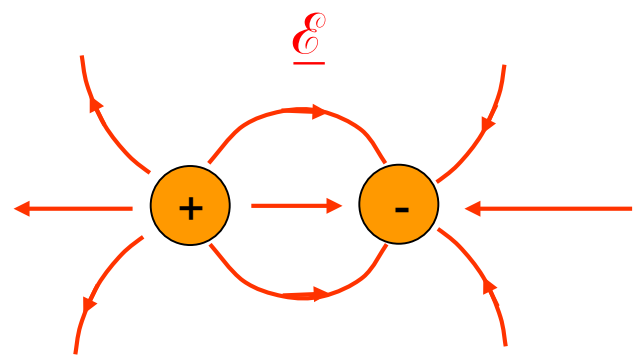
Transmission Lines (cont.)

Some practical notes:

- Coaxial cable is a perfectly shielded system (no interference).
- Twin line is not a shielded system – more susceptible to noise and interference.
- Twin lead may be improved by using a form known as “twisted pair” (e.g., CAT 5 cable). This results in less interference.



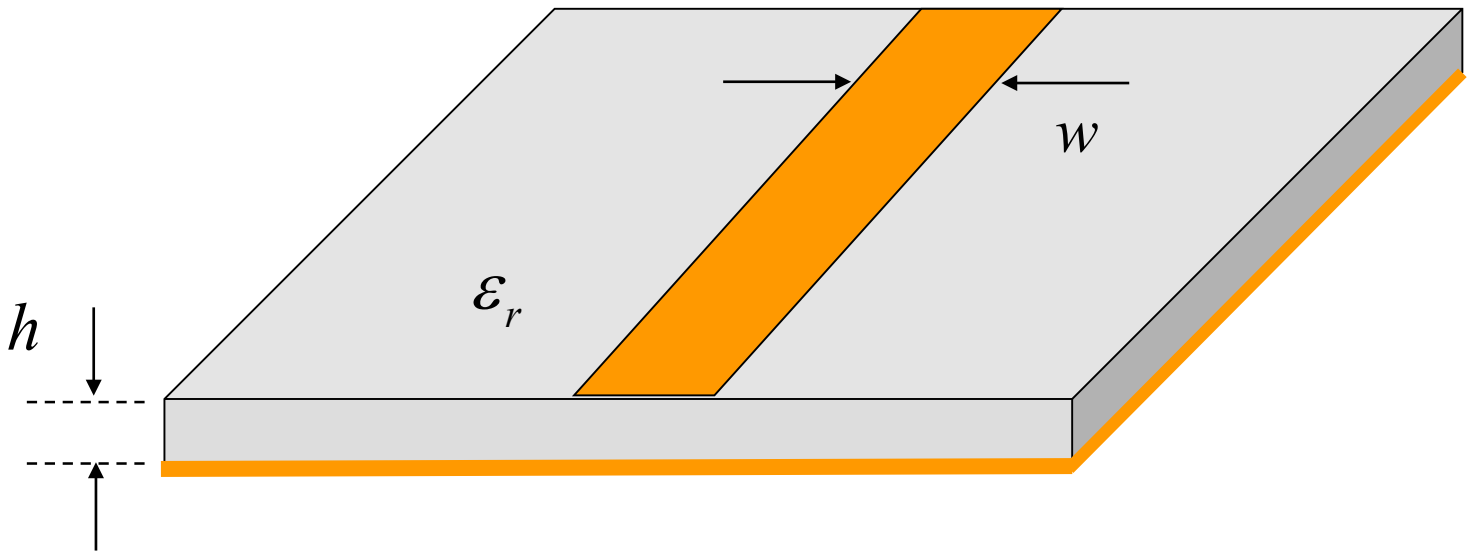
Coax



Twin lead

Transmission Lines (cont.)

A common transmission line for printed circuit boards:

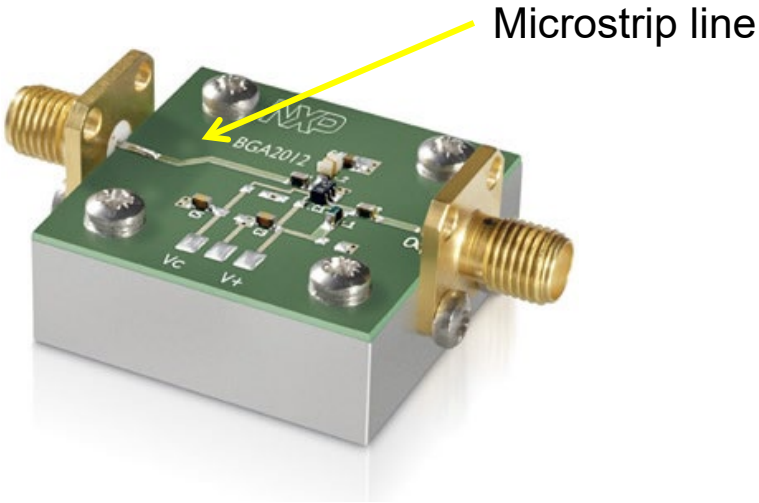


Microstrip line

(a “planar” transmission line)

Transmission Lines (cont.)

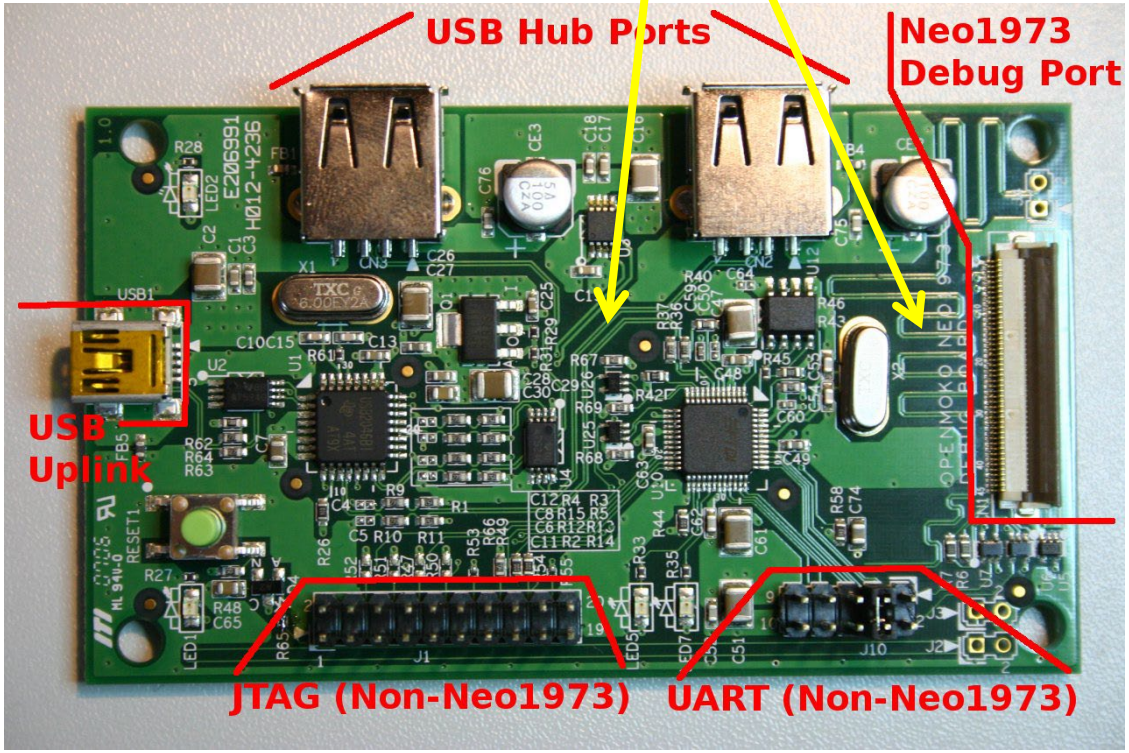
Transmission lines are commonly met on printed-circuit boards.



A microwave integrated circuit (MIC)

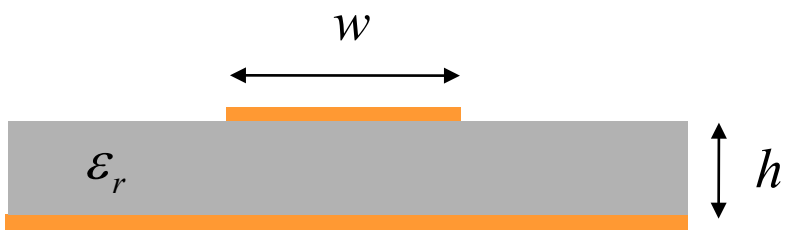
Transmission Lines (cont.)

Microstrip line

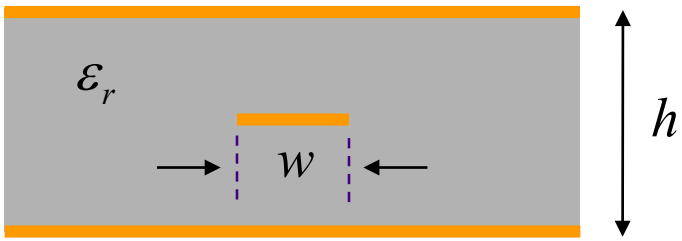


Transmission Lines (cont.)

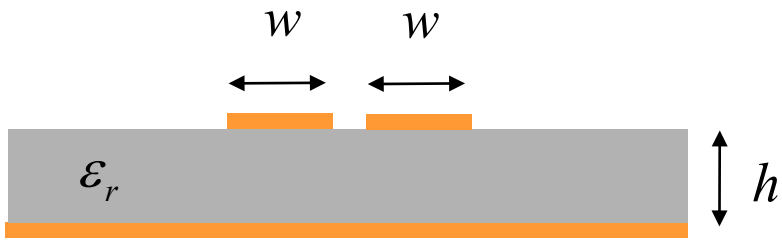
Planar transmission lines commonly met on printed-circuit boards



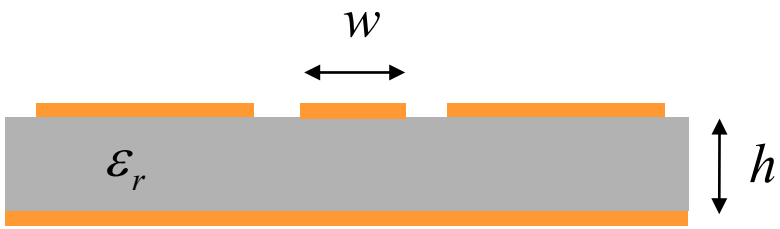
Microstrip



Stripline



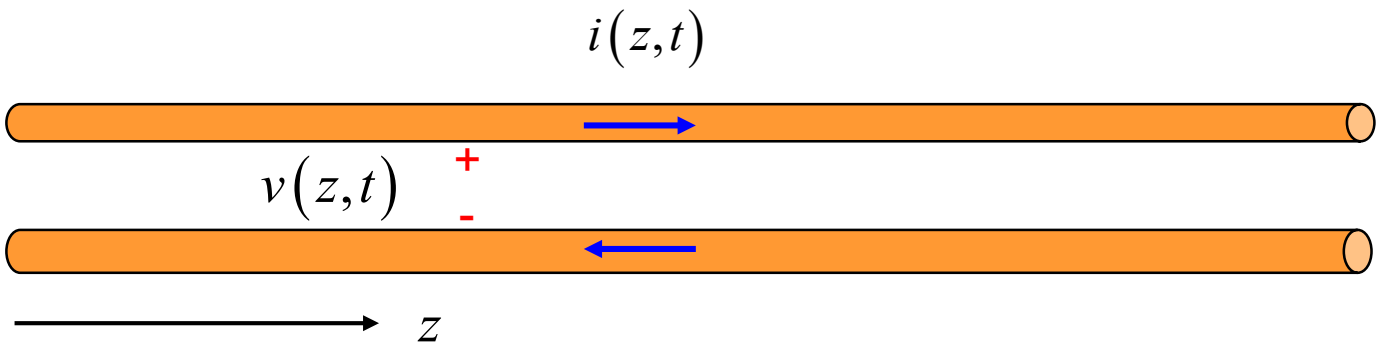
Coplanar strips



Coplanar waveguide (CPW)

Transmission Lines (cont.)

Symbol (schematic) for transmission line:



Note:
We use this schematic to represent a general transmission line, no matter what the actual shape of the conductors (coax, twin lead, etc.).

Note:
The current on the bottom conductor is always assumed to be equal and opposite to the current on the top conductor (but often we do not label the current on the bottom conductor, for simplicity).

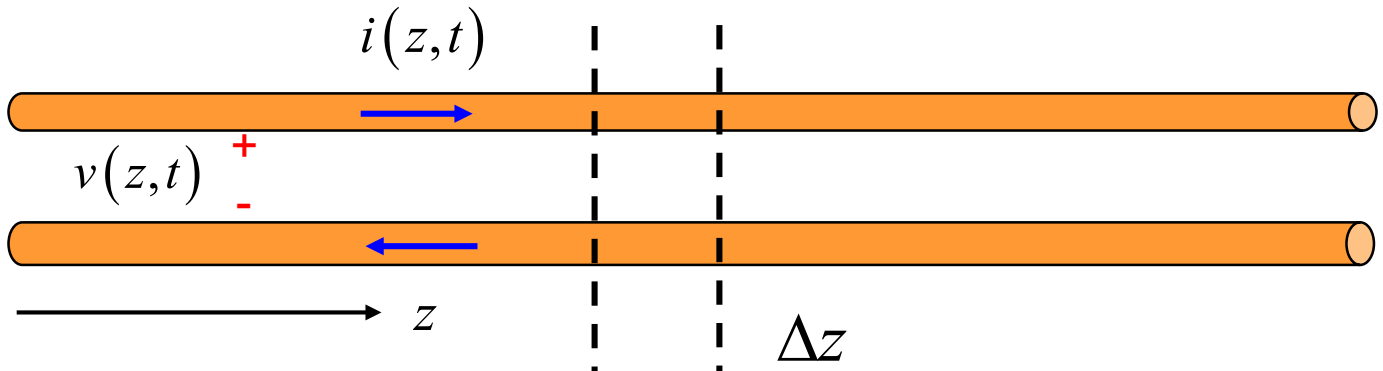
Transmission Lines (cont.)

Four fundamental parameters that characterize any transmission line:

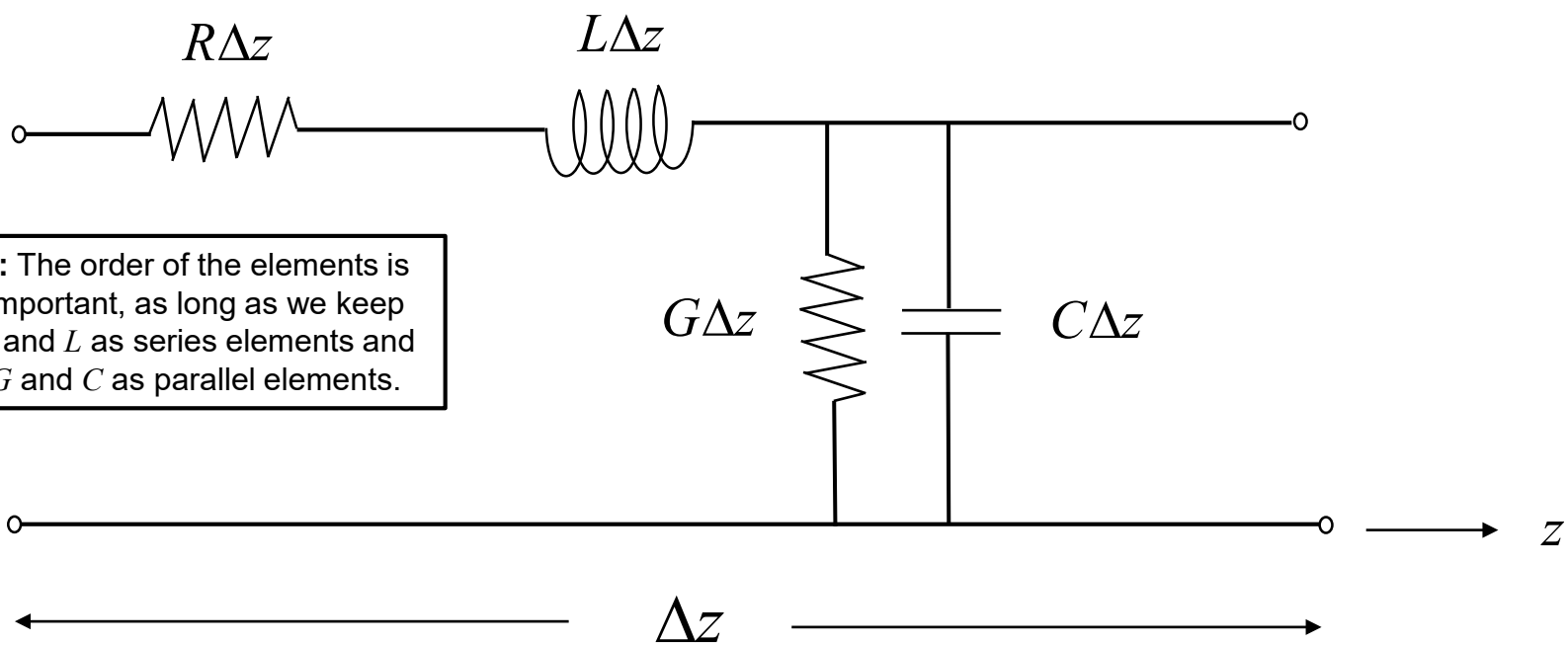
These are "per unit length" parameters.

C = capacitance/length [F/m]	Capacitance between the two wires
L = inductance/length [H/m]	Inductance due to stored magnetic energy
R = resistance/length [Ω/m]	Resistance due to the conductors
G = conductance/length [S/m]	Conductance due to the filling material between the wires

Circuit Model



Circuit Model:



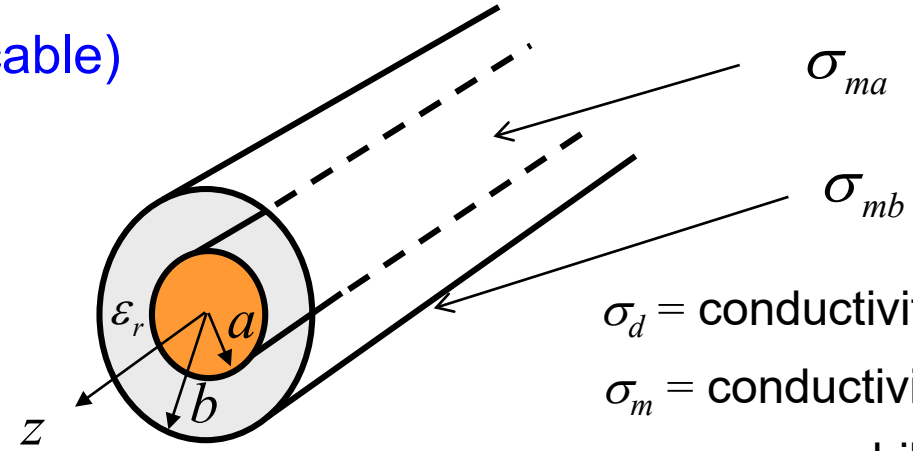
Note: The order of the elements is not important, as long as we keep the R and L as series elements and the G and C as parallel elements.

Coaxial Cable

Example (coaxial cable)

Usually $\mu_m = \mu_0$
(cooper or aluminum)

Usually $\mu_d = \mu_0$
(e.g., Teflon)



σ_d = conductivity of dielectric [S/m]

σ_m = conductivity of metal [S/m]

μ_m = permeability of metal [H/m]

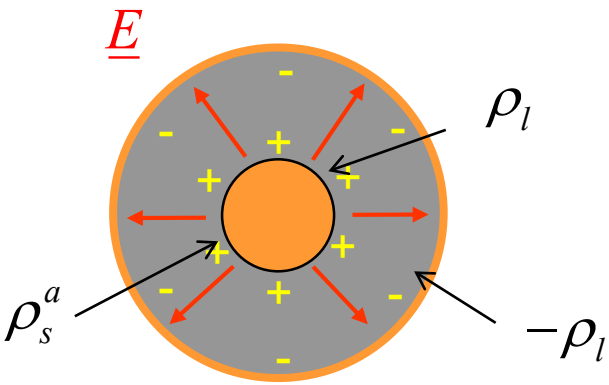
$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\mathrm{F/m}] \qquad G = \frac{2\pi\sigma_d}{\ln\left(\frac{b}{a}\right)} \quad [\mathrm{S/m}]$$

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \quad [\mathrm{H/m}] \qquad R = \left(\frac{1}{2\pi a\delta_a\sigma_{ma}} + \frac{1}{2\pi b\delta_b\sigma_{mb}}\right) \quad [\Omega/\mathrm{m}]$$

$$\delta = \sqrt{\frac{2}{\omega\mu_m\sigma_m}}$$ (skin depth of metal) $$\delta_{a,b} = \sqrt{\frac{2}{\omega\mu_{ma,mb}\sigma_{ma,mb}}}$$

Coaxial Cable (cont.)

Overview of derivation: capacitance per unit length



$$\rho_\ell = \rho_s^a \left(2\pi a\right)$$

Assume static fields

$$C = \frac{Q}{V}\frac{1}{\Delta z} = \frac{\rho_\ell}{V}$$

Gauss's law

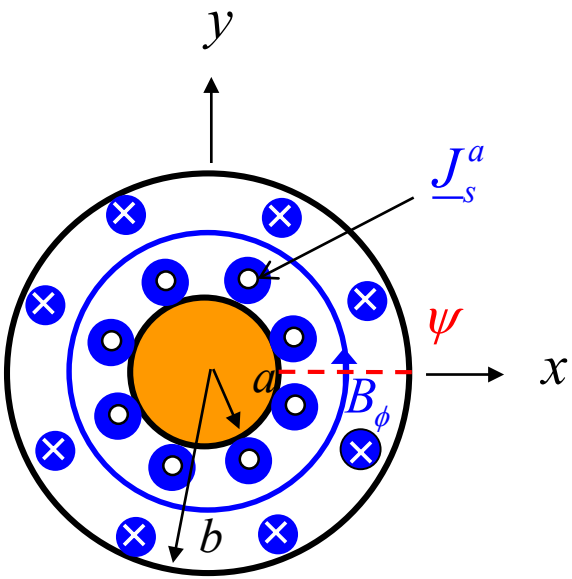
$$V = \int_a^b E_\rho \, d\rho = \int_a^b \left(\frac{\rho_\ell}{2\pi\varepsilon_0\varepsilon_r\rho}\right) d\rho = \frac{\rho_\ell}{2\pi\varepsilon_0\varepsilon_r}\ln\left(\frac{b}{a}\right)$$

$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

More details can be found in ECE 3318, Notes 25.

Coaxial Cable (cont.)

Overview of derivation: inductance per unit length



$$I = J_{sz}^a \left(2\pi a\right)$$

(current flowing in z direction on inner conductor)

Assume static fields

$$L = \frac{\psi}{I}\frac{1}{\Delta z}$$

Ampere's law

$$\psi = \Delta z \int_a^b B_\phi \, d\rho = \Delta z \int_a^b \mu_0 \left(\frac{I}{2\pi\rho}\right) d\rho$$

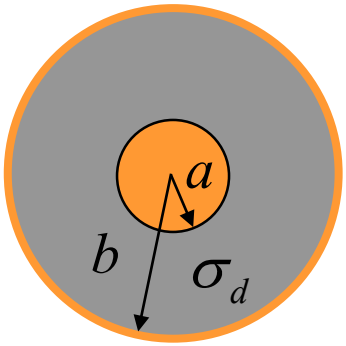
$$= \Delta z \mu_0 \left(\frac{I}{2\pi}\right) \ln\left(\frac{b}{a}\right) \qquad (\mu = \mu_0)$$

$$L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \quad [\text{H/m}]$$

More details can be found in ECE 3318, Notes 31.

Coaxial Cable (cont.)

Overview of derivation: conductance per unit length



RC Analogy (ECE 3318): $\begin{cases} \varepsilon \rightarrow \sigma_d \\ C \rightarrow G \end{cases}$

$$C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}$$

More details can be found in ECE 3318, Notes 27.

$$G = \frac{2\pi\sigma_d}{\ln\left(\frac{b}{a}\right)} \quad [\text{S/m}]$$

Coaxial Cable (cont.)

Relation Between L and C:

$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}] \qquad L = \frac{\mu_0\mu_r}{2\pi}\ln\left(\frac{b}{a}\right) \quad [\text{H/m}]$$

$$LC = \mu_0\mu_r\varepsilon_0\varepsilon_r = \mu\varepsilon$$

Speed of light in dielectric medium:

$$c_d = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{1}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}}$$

$$c \equiv 2.99792458\times 10^8 \quad [\text{m/s}]$$

Hence:

$$LC = \frac{1}{c_d^2}$$

This is true for ALL transmission lines.

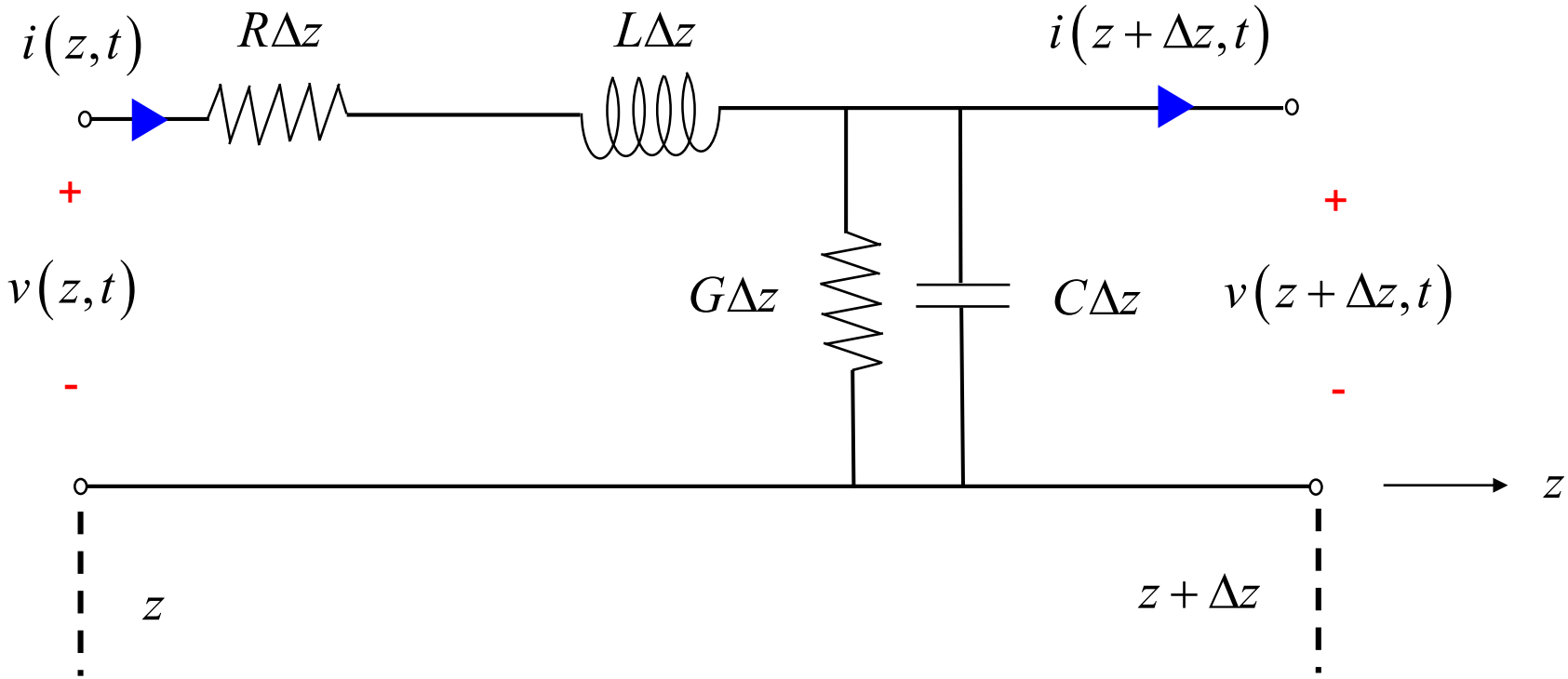
(A proof will be seen later.)

Telegrapher's Equations

- These are a fundamental set of differential equations that describe how voltage and current propagate (travel) on a transmission line.
- The derivation holds for any type of transmission line.
- The equations are exact, showing how any type of signal propagates on a transmission line.
- We only have an exact solution for the telegrapher's equations in the time domain for a <u>lossless</u> transmission line.

Telegrapher's Equations (cont.)

Apply KVL and KCL laws to a small Δz slice of line:



$$\text{KVL:} \quad v(z,t) = v(z+\Delta z,t) + i(z,t)R\Delta z + L\Delta z \frac{\partial i(z,t)}{\partial t}$$

$$\text{KCL:} \quad i(z,t) = i(z+\Delta z,t) + v(z+\Delta z,t)G\Delta z + C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t}$$

Telegrapher's Equations (cont.)

Hence

$$\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z} = -Gv(z+\Delta z,t) - C\frac{\partial v(z+\Delta z,t)}{\partial t}$$

Now let $\Delta z \to 0$: "Telegrapher's Equations (TEs)"

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t}$$

Telegrapher's Equations (cont.)

- Take the derivative of the first TE with respect to z.
- Substitute in from the second TE.

$$\frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t}$$

$$\frac{\partial^2 v}{\partial z^2} = -R\frac{\partial i}{\partial z} - L\frac{\partial}{\partial z}\left(\frac{\partial i}{\partial t}\right)$$

$$= -R\frac{\partial i}{\partial z} - L\frac{\partial}{\partial t}\left(\frac{\partial i}{\partial z}\right)$$

$$= -R\left[-Gv - C\frac{\partial v}{\partial t}\right] - L\left[-G\frac{\partial v}{\partial t} - C\frac{\partial^2 v}{\partial t^2}\right]$$

Telegrapher's Equations (cont.)

$$\frac{\partial^2 v}{\partial z^2} = -R\left[-Gv - C\frac{\partial v}{\partial t}\right] - L\left[-G\frac{\partial v}{\partial t} - C\frac{\partial^2 v}{\partial t^2}\right]$$

Hence, we have:

$$\frac{\partial^2 v}{\partial z^2} - (RG)v - (RC + LG)\frac{\partial v}{\partial t} - LC\left(\frac{\partial^2 v}{\partial t^2}\right) = 0$$

There is no exact solution to this differential equation, except for the <u>lossless</u> case. Hence, we will assume lossless transmission lines in the time domain.

Note: The current satisfies the same differential equation.

Telegrapher's Equations (cont.)

Lossless case: $R = G = 0$

$$\frac{\partial^2 v}{\partial z^2} - (RG)v - (RC + LG)\frac{\partial v}{\partial t} - LC\left(\frac{\partial^2 v}{\partial t^2}\right) = 0$$

$$\Downarrow$$

$$\frac{\partial^2 v}{\partial z^2} - LC\left(\frac{\partial^2 v}{\partial t^2}\right) = 0$$

Recall: $LC = \mu\varepsilon = \dfrac{1}{c_d^2}$

Solution to Telegrapher's Equations

General Solution for the lossless case:

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{c_d^2}\left(\frac{\partial^2 v}{\partial t^2}\right)$$

"wave equation"

Solution:

$$v(z,t) = F(z - c_d t) + G(z + c_d t)$$

or

$$v(z,t) = f(t - z/c_d) + g(t + z/c_d)$$

where (F, G) and (f, g) are <u>arbitrary functions</u>.

This is called the "D'Alembert solution" to the wave equation (the solution is in the form of traveling waves).

Solution to Telegrapher's Equations

General Solution for the lossless case:

$$v(z,t) = F(z - c_d t) + G(z + c_d t) \qquad (1)$$

or

$$v(z,t) = f(t - z/c_d) + g(t + z/c_d) \qquad (2)$$

Form (1): Useful when plotting the voltage vs. distance z for different times.

Form (2): Useful when plotting the voltage vs. time t for different distances.

Traveling Waves

Proof of solution (Form (1)):

(A similar proof applies for form 2.)

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{c_d^2}\left(\frac{\partial^2 v}{\partial t^2}\right)$$

General solution: $v(z,t) = F(z - c_d t) + G(z + c_d t)$

$$\frac{\partial^2 v(z,t)}{\partial z^2} = F''(z - c_d t) + G''(z + c_d t)$$

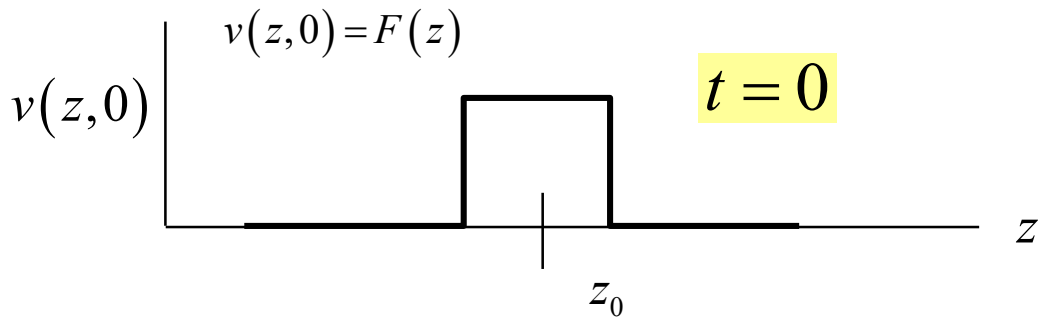
$$\frac{\partial^2 v(z,t)}{\partial t^2} = (-c_d)^2 \; F''(z - c_d t) + c_d^2 \, G''(z + c_d t)$$

It is seen that the differential equation is satisfied by the general solution.

Traveling Waves (cont.)

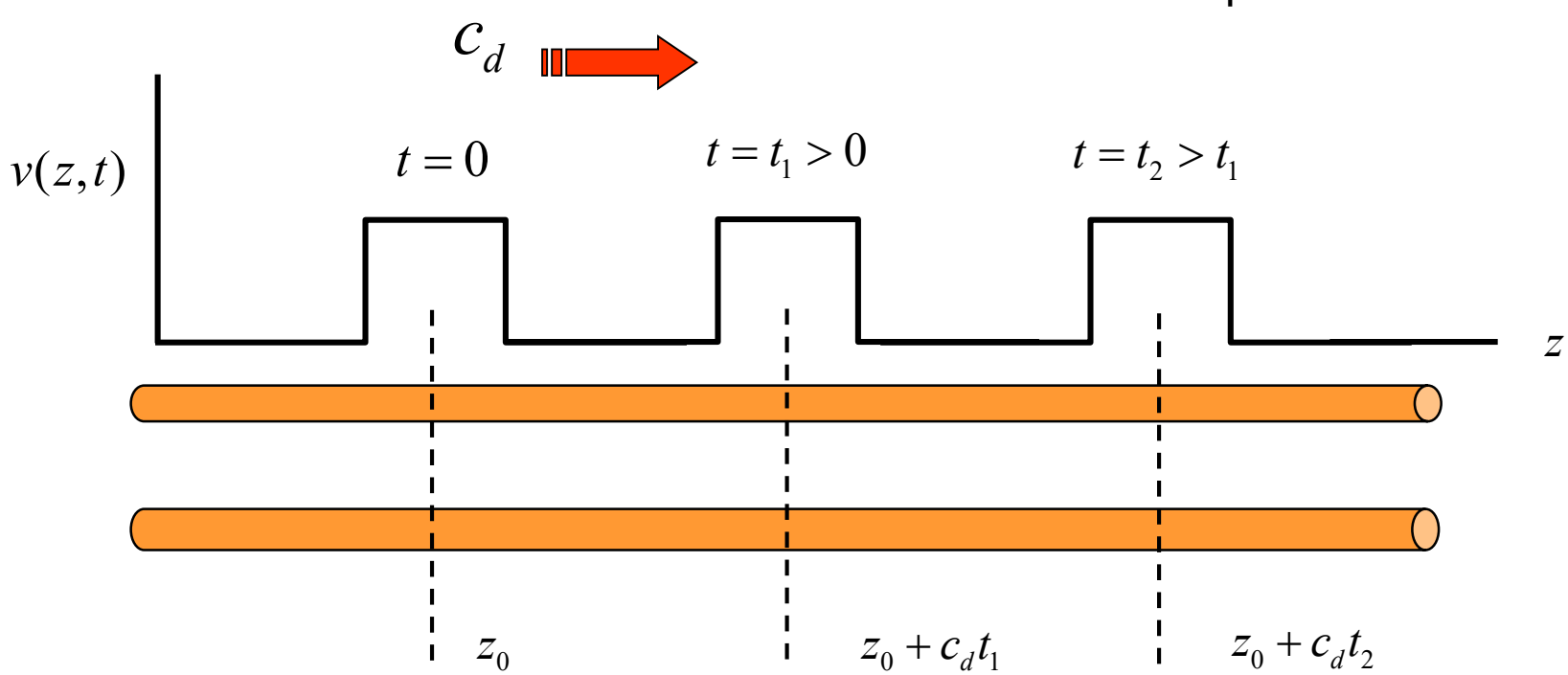
Example
(rectangular pulse):

$$v(z,t) = F(z - c_d t)$$



The waveform is shifted to the right by $\Delta z = c_d t$

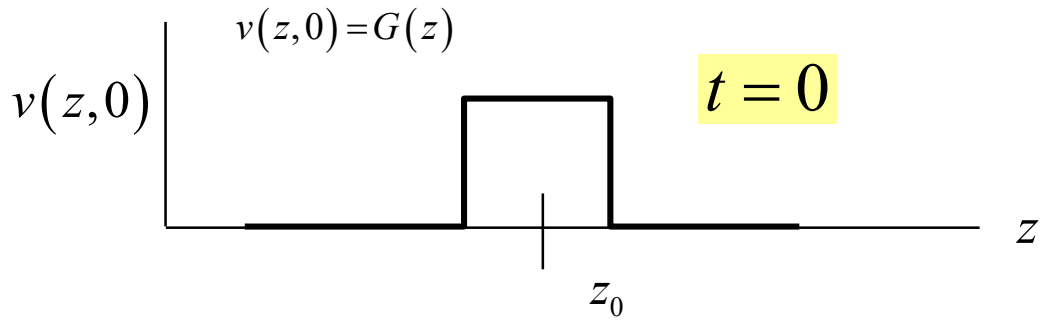
"snapshots of the wave"



Traveling Waves (cont.)

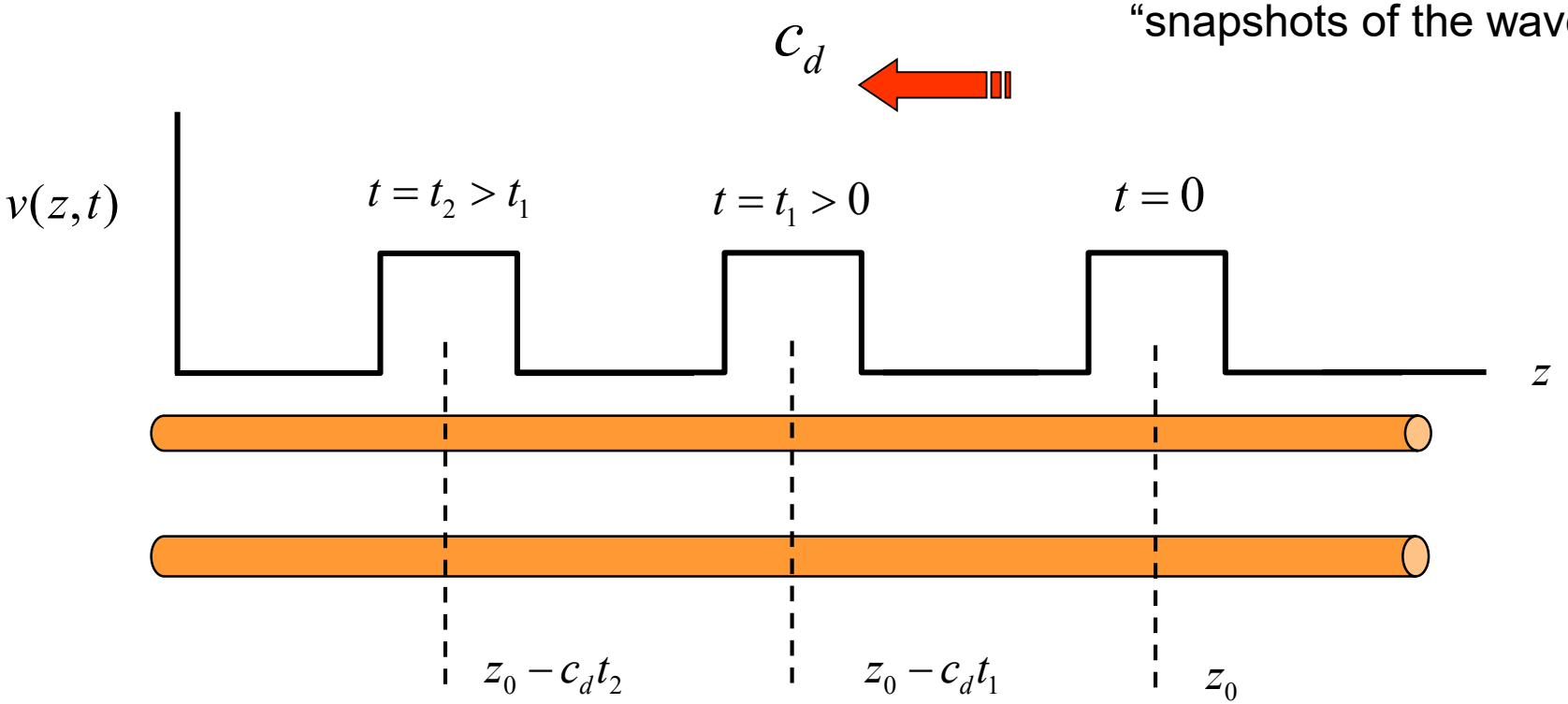
Example
(square pulse):

$$v(z,t) = G(z + c_d t)$$



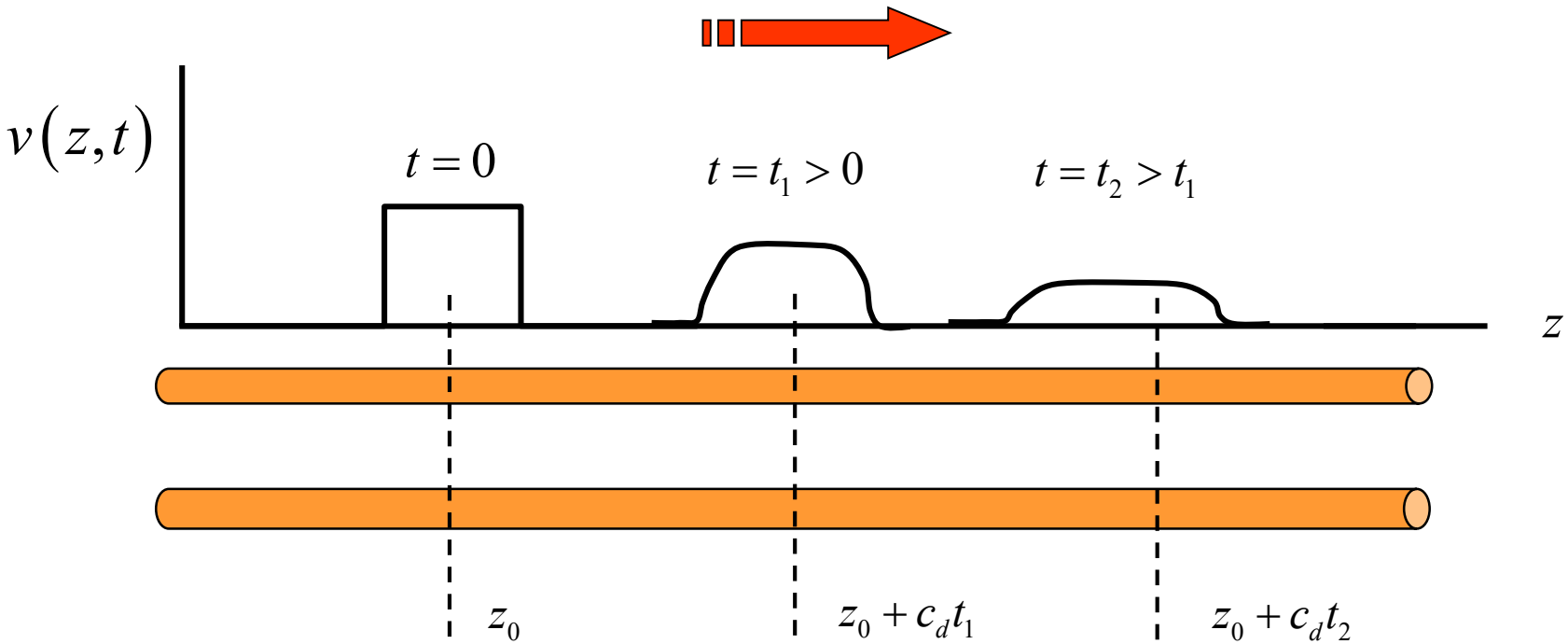
The waveform is shifted to the left by $|\Delta z| = c_d t$

"snapshots of the wave"



Effects of Loss

Loss causes an attenuation in the signal level, and it also causes distortion (the pulse changes shape and usually gets broader).

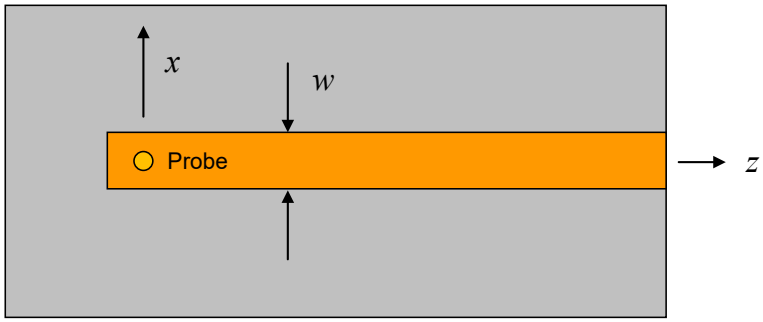


(These effects can be studied numerically.)

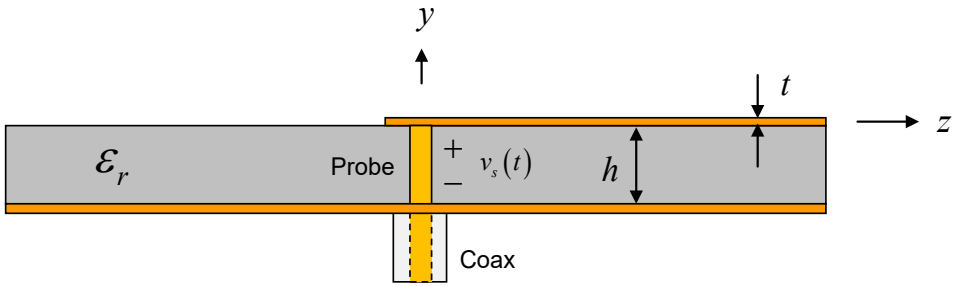
Effects of Loss (cont.)

Example: Propagation on a lossy microstrip line

(From ECE 5317)



TOP VIEW



SIDE VIEW

$\varepsilon_r = 2.33$

$\tan\delta = 0.001$

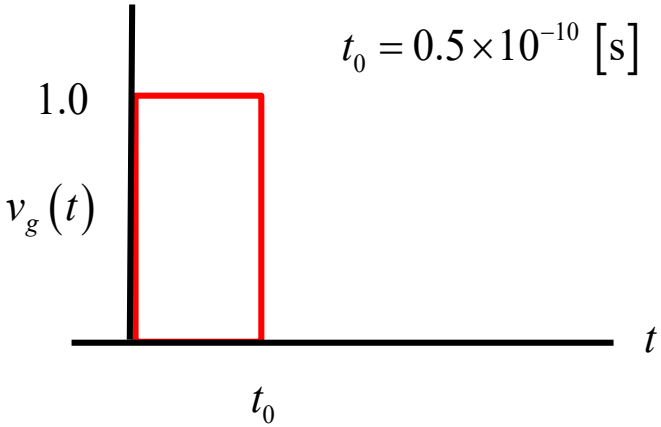
$h = 0.787\ [\text{mm}]\ (31\,\text{mils})$

$w = 2.35\ [\text{mm}]$

$t = 0.0175\,[\text{mm}]$ ("half oz" copper cladding)

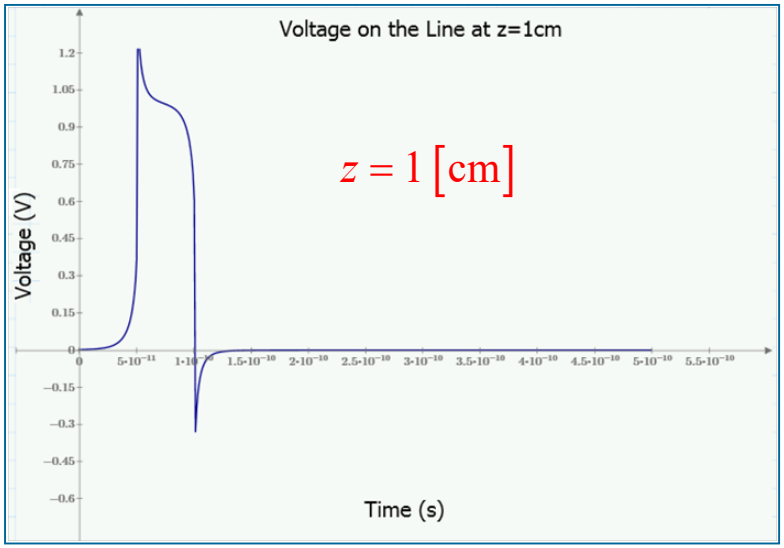
$\sigma_m = 3.0\times10^{7}\ [\text{S/m}]$

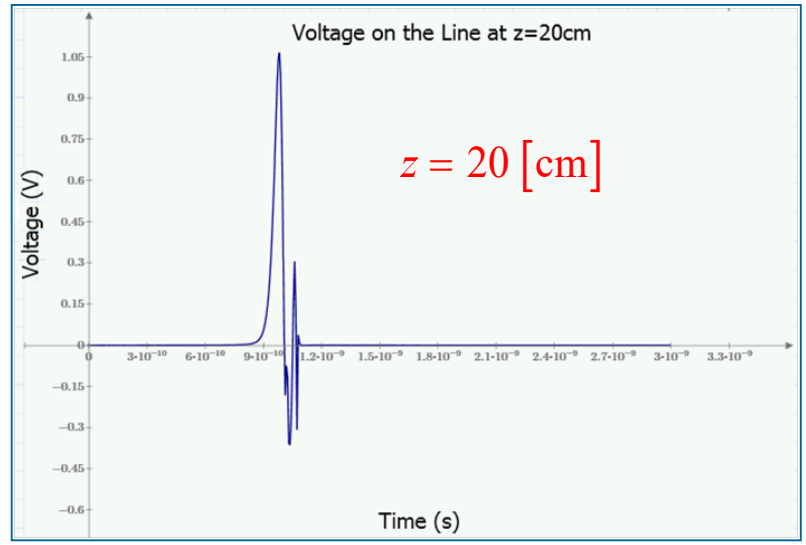
Input signal:



Effects of Loss (cont.)

Example: Propagation on a microstrip line





$\varepsilon_r = 2.33$

$\tan\delta = 0.001$

$h = 0.787\,[\mathrm{mm}]\,(31\,\mathrm{mils})$

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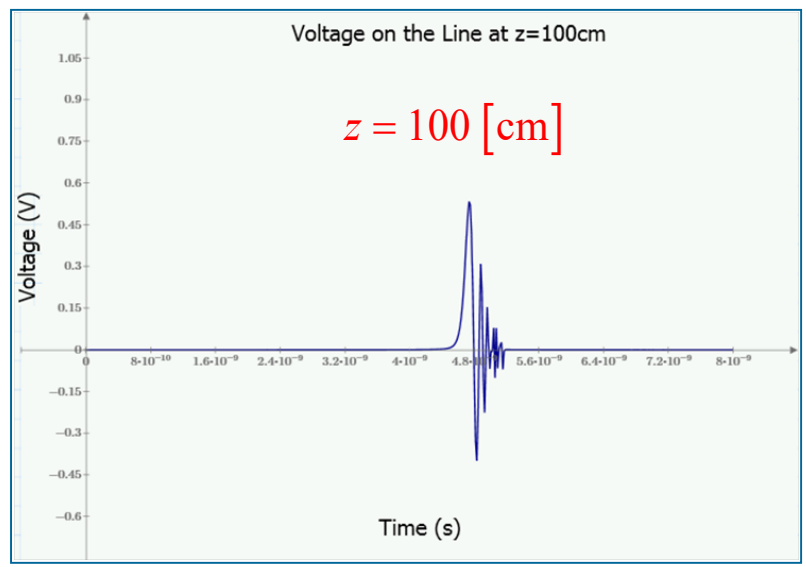
$t = 0.0175\,[\mathrm{mm}]$ ("half oz" copper cladding)

$\sigma_m = 3.0\times10^7\,[\mathrm{S/m}]$



Current

Our goal is to now solve for the <u>current</u> on a lossless line.

(First Telegrapher's equation)

Lossless

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t} \quad \Rightarrow \quad \frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

Assume the following forms:

$$v(z,t) = F(z - c_d t) + G(z + c_d t)$$

$$i(z,t) = U(z - c_d t) + V(z + c_d t)$$

The derivatives are:

$$\frac{\partial v(z,t)}{\partial z} = F'(z - c_d t) + G'(z + c_d t)$$

$$\frac{\partial i(z,t)}{\partial t} = -c_d U'(z - c_d t) + c_d V'(z + c_d t)$$

Current (cont.)

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

This becomes

$$F'(z - c_d t) + G'(z + c_d t) = -L\left[-c_d U'(z - c_d t) + c_d V'(z + c_d t)\right]$$

Equating like terms, we have:

$$F'(z - c_d t) = -L\left[-c_d U'(z - c_d t)\right]$$

$$G'(z + c_d t) = -L\left[c_d V'(z + c_d t)\right]$$

Integrating both sides, we have:

$$U(z - c_d t) = \frac{1}{Lc_d} F(z - c_d t)$$

$$V(z + c_d t) = -\frac{1}{Lc_d} G(z + c_d t)$$

Note:
There may be a constant of integration, but this would correspond to a DC current, which is ignored here.

Current (cont.)

Observation about term:

$$Lc_d = L\left(\frac{1}{\sqrt{\mu\varepsilon}}\right) = L\left(\frac{1}{\sqrt{LC}}\right) = \sqrt{\frac{L}{C}}$$

Define the characteristic impedance Z_0 of the line:

$$\boxed{Z_0 = \sqrt{\frac{L}{C}}}$$

The units of Z_0 are Ohms.

Then we have:

$$U\left(z - c_d t\right) = \frac{1}{Z_0} F\left(z - c_d t\right)$$

$$V\left(z + c_d t\right) = -\frac{1}{Z_0} G\left(z + c_d t\right)$$

Current (cont.)

Recall that

$$i(z,t) = U(z - c_d t) + V(z + c_d t)$$

From the last slide:

$$U(z - c_d t) = \frac{1}{Z_0} F(z - c_d t)$$

$$V(z + c_d t) = -\frac{1}{Z_0} G(z + c_d t)$$

Hence, we have the current as

$$i(z,t) = \frac{1}{Z_0}\left[F(z - c_d t) - G(z + c_d t)\right]$$

Current (cont.)

Summary of the general solution for a lossless line:

$$c_d = \frac{1}{\sqrt{LC}} \quad [\text{m/s}]$$

$$Z_0 = \sqrt{\frac{L}{C}} \quad [\Omega]$$

$$v(z,t) = F(z - c_d t) + G(z + c_d t)$$

$$i(z,t) = \frac{1}{Z_0}\left[F(z - c_d t) - G(z + c_d t)\right]$$

- For a forward wave, the current waveform is the same as the voltage, but reduced in amplitude by a factor of Z_0.
- For a backward traveling wave, there is a minus sign as well.

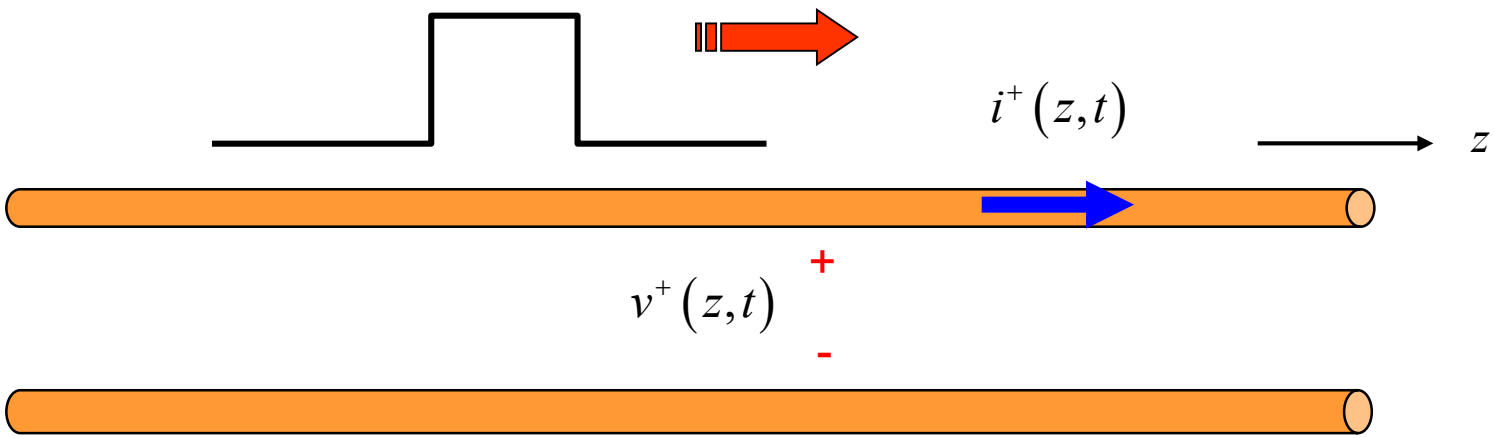
Current (cont.)

Picture for a forward-traveling wave:

$$v^+(z,t) = F(z - c_d t)$$

$$i^+(z,t) = \frac{1}{Z_0} F(z - c_d t)$$

Forward-traveling wave



$$\frac{v^+(z,t)}{i^+(z,t)} = Z_0$$

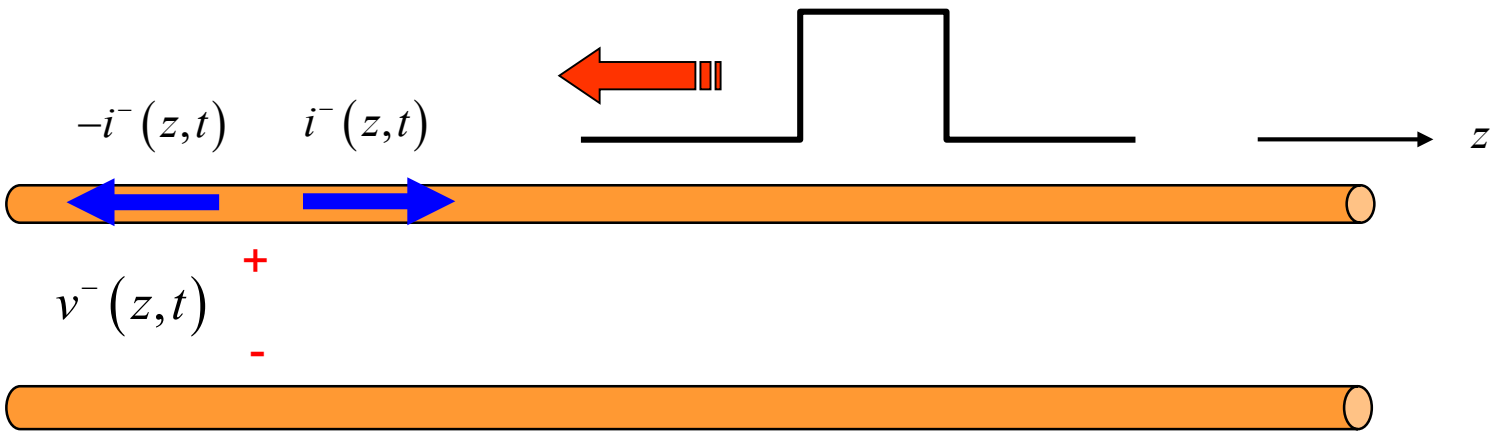
Current (cont.)

Physical interpretation of minus sign for the backward-traveling wave:

$$v^-(z,t) = G(z + c_d t)$$

$$i^-(z,t) = -\frac{1}{Z_0} G(z + c_d t)$$

Backward-traveling wave

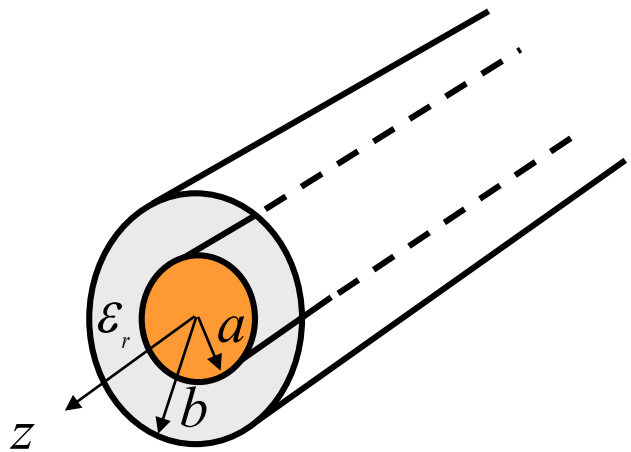


The minus sign arises from the reference direction for the current.

$$\frac{v^-(z,t)}{-i^-(z,t)} = Z_0 \quad \Rightarrow \quad \frac{v^-(z,t)}{i^-(z,t)} = -Z_0$$

Coaxial Cable

Example: Find the characteristic impedance of a coax.



$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

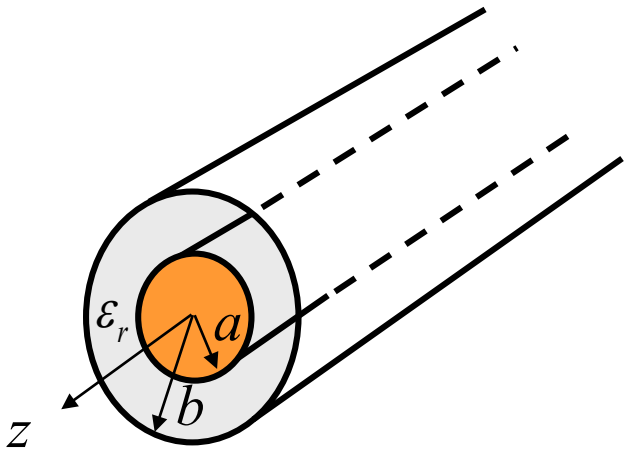
$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \quad [\text{H/m}]$$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)}{\frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)}}}$$

or

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{\sqrt{\varepsilon_r}}\ln\left(\frac{b}{a}\right)$$

Coaxial Cable (cont.)



$$Z_0 = \frac{1}{2\pi}\eta_0 \frac{1}{\sqrt{\varepsilon_r}} \ln\left(\frac{b}{a}\right) \quad [\Omega]$$

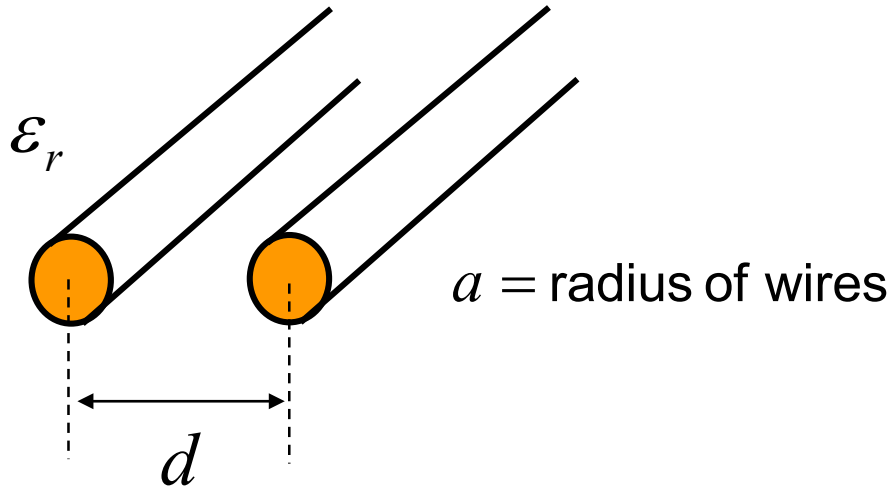
$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$ (intrinsic impedance of free space)

$$\varepsilon_0 \doteq 8.8541878 \times 10^{-12} \ [\text{F/m}]$$

$$\mu_0 = 4\pi \times 10^{-7} \ [\text{H/m}] \ \text{(exact)}$$

$$\eta_0 \doteq 376.7303 \ [\Omega]$$

Twin Lead



$$C = \frac{\pi \varepsilon_0 \varepsilon_r}{\cosh^{-1}\left(\frac{d}{2a}\right)} \quad [\text{F/m}]$$

$$L = \frac{\mu_0}{\pi} \cosh^{-1}\left(\frac{d}{2a}\right) \quad [\text{H/m}]$$

$$Z_0 = \frac{1}{\pi} \eta_0 \frac{1}{\sqrt{\varepsilon_r}} \cosh^{-1}\left(\frac{d}{2a}\right) \quad [\Omega]$$

$$Z_0 \approx \frac{1}{\pi} \eta_0 \frac{1}{\sqrt{\varepsilon_r}} \ln\left(\frac{d}{a}\right) \quad [\Omega]$$

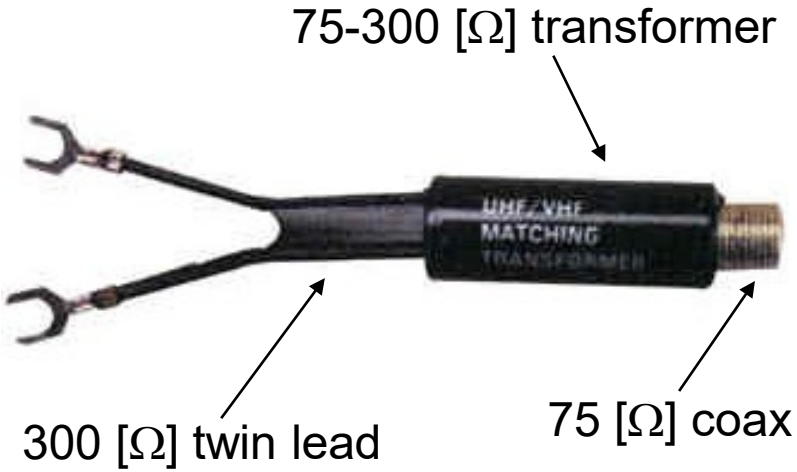
$$a \ll d$$

Twin Line (cont.)

These are the common values used for TV.

Coaxial cable

$$Z_0 = 75 \quad [\Omega]$$

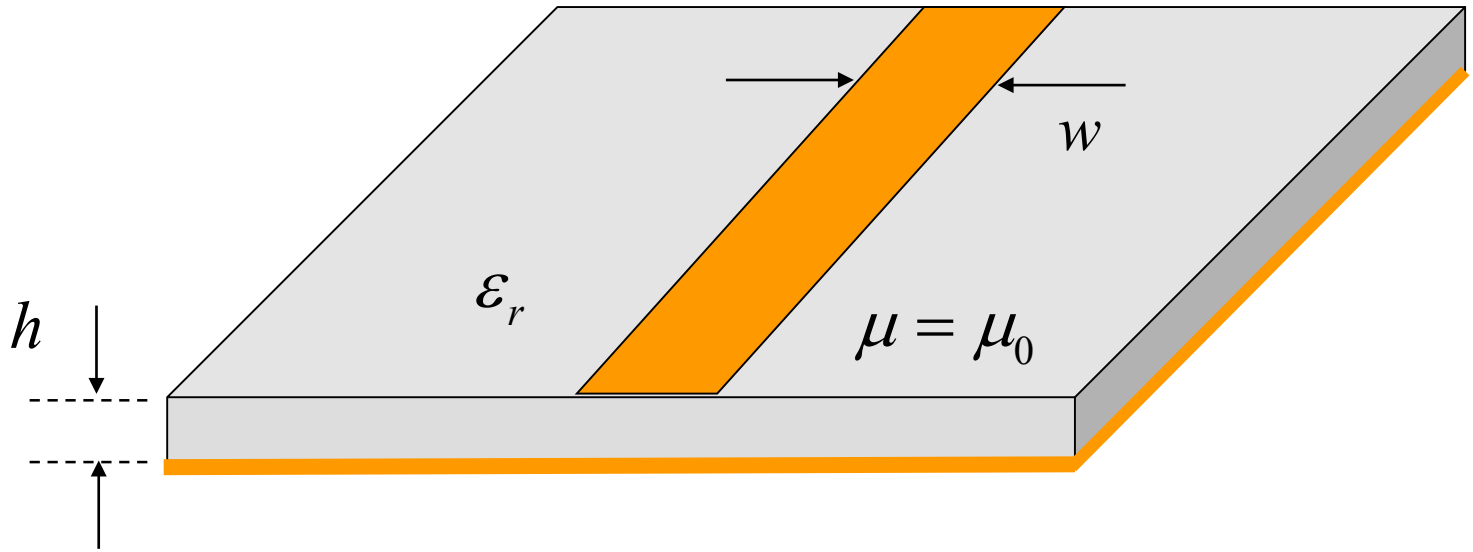


Twin lead

$$Z_0 = 300 \quad [\Omega]$$

Note: In <u>microwave</u> work, the most common value is $Z_0 = 50$ [Ω].

Microstrip Line



Parallel-plate formulas:

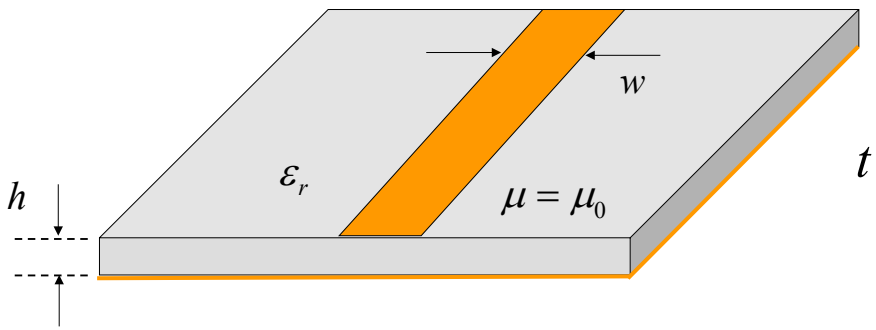
$$C \approx \varepsilon_0 \varepsilon_r \frac{w}{h}, \quad w \gg h$$

$$L \approx \mu_0 \frac{h}{w}, \quad w \gg h$$

$$Z_0 \approx \eta_0 \frac{1}{\sqrt{\varepsilon_r}} \frac{h}{w}, \quad w \gg h$$

Recall: $LC = \mu\varepsilon = \dfrac{1}{c_d^2}$

Microstrip Line (cont.)



$t =$ strip thickness

More accurate CAD formulas (from ECE 5317):

$$Z_0 = \frac{120\pi}{\sqrt{\varepsilon_r^{eff}}\left[\left(w'/h\right)+1.393+0.667\ln\left(\left(w'/h\right)+1.444\right)\right]} \qquad (w/h \geq 1)$$

$$\varepsilon_r^{eff} = \frac{\varepsilon_r+1}{2}+\left(\frac{\varepsilon_r-1}{2}\right)\left(\frac{1}{\sqrt{1+12\left(h/w\right)}}\right)-\left(\frac{\varepsilon_r-1}{4.6}\right)\left(\frac{t/h}{\sqrt{w/h}}\right) \qquad (w/h \geq 1)$$

$$w' = w+\frac{t}{\pi}\left(1+\ln\left(\frac{2h}{t}\right)\right)$$

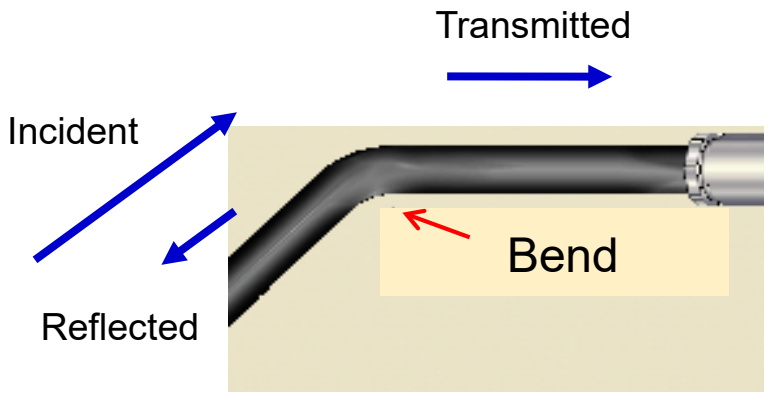
Note: The effective relative permittivity accounts for the fact that some of the fields are outside of the substrate, in the air region. The effective width w' accounts for the strip thickness.

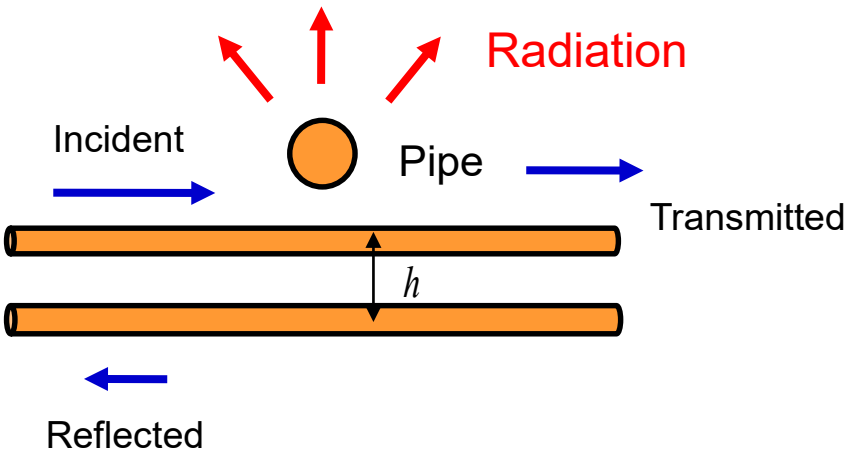
Some Comments

- Transmission-line theory is valid at any frequency, and for any type of waveform (assuming an ideal straight length of transmission line).
- Transmission-line theory is perfectly consistent with Maxwell's equations (although we work with voltage and current, rather than electric and magnetic fields).
- Circuit theory does not view two wires as a “transmission line”: it cannot predict effects such as signal propagation, reflection, distortion, etc.

Some Comments

- One thing that transmission-line theory ignores is the effects of discontinuities (e.g., bends or nearby obstacles). These may cause reflections and possibly also radiation at high frequencies, depending on the type of line.





Coax

(cannot radiate)

Twin Lead

(can radiate*)

*Twisted pair minimizes this radiation.

Summary Page

Lossless Line

$$v(z,t) = F(z - c_d t) + G(z + c_d t)$$

$$i(z,t) = \frac{1}{Z_0}\left[F(z - c_d t) - G(z + c_d t)\right]$$

$$c_d = \frac{1}{\sqrt{LC}} \quad [\text{m/s}]$$

$$Z_0 = \sqrt{\frac{L}{C}} \quad [\Omega]$$

or

$$v(z,t) = f(t - z/c_d) + g(t + z/c_d)$$

$$i(z,t) = \frac{1}{Z_0}\left[f(t - z/c_d) - g(t + z/c_d)\right]$$

F or f = wave going in $+z$ direction.

G or g = wave going in $-z$ direction.