Applied Electromagnetic Waves

ECE 3317 Course Notes Summary

University of Houston

Department of Electrical and Computer Engineering

Based on lecture notes by

Prof. David R. Jackson
Prof. Stuart A. Long

Compiled and summarized by

Dr. Yufang Sun

with the assistance of AI (Claude)

February 26, 2026

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Part I

Mathematical Foundations

Chapter 1

Introduction

Electromagnetics is the most fundamental of all electrical and computer engineering disciplines. Every branch of ECE—from circuit design to wireless communications, from power systems to microelectronics—rests upon the laws governing electric and magnetic fields. In this course, we study *applied* electromagnetic waves: the generation, propagation, and reception of electromagnetic energy.

1.1 Why Study Electromagnetics?

Important

All of circuit theory is a *special case* of electromagnetics, valid only at low frequencies where the physical dimensions of the circuit are small relative to a wavelength. When dimensions become comparable to a wavelength, circuit theory fails and the full electromagnetic description is required.

Electromagnetics provides answers to fundamental questions about the physical world:

- What is light? It is an electromagnetic wave.
- How do electric and magnetic fields behave and interact?
- Why can we communicate wirelessly across vast distances?

From a practical standpoint, electromagnetics is essential in the following areas:

- **Antenna design**: cell phone antennas, satellite dishes, radar arrays
- **Microwave engineering**: filters, amplifiers, oscillators at GHz frequencies
- **RF circuit design**: transmitters, receivers, impedance matching networks
- **Power engineering**: transformers, motors, generators, power transmission
- **Micro/nano-electronics**: signal integrity, electromagnetic compatibility (EMC)
- **Photonics and fiber optics**: optical communication systems

1.2 Applications of Electromagnetic Waves

1.2.1 Wireless Communications

Modern wireless systems are ubiquitous: cellular telephones, Bluetooth devices, Wi-Fi routers, cordless handsets, and radio-frequency identification (RFID) tags all rely on the transmission and reception of electromagnetic waves through free space.

1.2.2 Radar and Remote Sensing

Radar (Radio Detection and Ranging) systems transmit electromagnetic pulses and analyze the reflected signals to determine the range, velocity, and characteristics of distant objects. Applications range from air traffic control to weather monitoring and autonomous vehicles.

1.2.3 Microwave Heating

Microwave ovens operate at approximately 2.45 GHz. At this frequency, the electric field of the microwave causes polar water molecules in food to rotate rapidly, converting electromagnetic energy into thermal energy through molecular friction.

Physical Insight

Why does a microwave sometimes heat food unevenly? The standing wave pattern inside the oven cavity creates regions of high and low electric field intensity ("hot spots" and "cold spots"). This is why most microwave ovens include a rotating turntable.

1.2.4 Millimeter-Wave Imaging

Millimeter waves (30–300 GHz) can penetrate clothing and other non-metallic materials but are reflected by the human body and metallic objects. This property is exploited in airport security scanners for full-body imaging.

1.2.5 Medical Imaging

Magnetic Resonance Imaging (MRI) uses strong static magnetic fields and radio-frequency electromagnetic pulses to produce detailed images of internal body structures without ionizing radiation.

1.3 The Electromagnetic Spectrum

Electromagnetic waves span an enormous range of frequencies, from extremely low frequency (ELF) radio waves to gamma rays. Table 1.1 summarizes the major bands.

Table 1.1: The electromagnetic spectrum.

Band	**Frequency Range**	**Applications**
ELF/VLF	3–30 kHz	Submarine communications
AM Radio	520–1610 kHz	AM broadcasting
FM Radio	88–108 MHz	FM broadcasting
VHF TV	55–216 MHz	Television channels 2–13
UHF TV	470–806 MHz	Television channels 14–69
Microwave	1–300 GHz	Radar, satellite, 5G
Infrared	300 GHz–400 THz	Thermal imaging, remotes
Visible	400–750 THz	Human vision
Ultraviolet	750–30000 THz	Sterilization
X-rays	30–30000 PHz	Medical imaging

Physical Insight

Antenna dimensions scale with wavelength. A monopole antenna is ideally about one-quarter wavelength long, while a dipole antenna is about one-half wavelength. At AM radio frequencies ($\lambda \approx 300\,\text{m}$), antennas are enormous towers. At cellular frequencies ($\lambda \approx 15\,\text{cm}$), they fit inside a phone.

1.4 Course Overview

This textbook is organized into five parts:

1. **Mathematical Foundations** (Chapters 1–3): Review of complex numbers, phasors, and vector calculus—the essential mathematical tools.

2. **Electromagnetic Theory** (Chapters 4–5): Maxwell's equations and Poynting's theorem—the fundamental laws governing all electromagnetic phenomena.

3. **Transmission Lines** (Chapters 6–14): A comprehensive treatment of guided-wave propagation along two-conductor systems, including time-domain and frequency-domain analysis, Smith charts, and impedance matching.

4. **Plane Waves** (Chapters 15–18): Propagation of electromagnetic waves in unbounded media, including lossy media, polarization, and reflection/transmission at interfaces.

5. **Waveguides and Antennas** (Chapters 19–22): Waveguiding structures (fiber optics, rectangular waveguides) and antenna fundamentals (radiation, directivity, gain, patterns).

Exercises

1. Calculate the wavelength in free space of the following signals: (a) an FM radio station at 100 MHz, (b) a Wi-Fi signal at 2.4 GHz, (c) a 5G millimeter-wave signal at 28 GHz.

2. A quarter-wave monopole antenna is designed for a frequency of 900 MHz. What is the physical length of the antenna?

3. The speed of light in free space is $c \approx 3 \times 10^8$ m/s. How long does it take for an electromagnetic signal to travel from a geostationary satellite (altitude 35 786 km) to the Earth's surface?

4. Explain why circuit theory breaks down at high frequencies. At what frequency does a 10 cm wire become "electrically long" (i.e., comparable to a wavelength)?

Chapter 2

Review of Complex Numbers and Phasors

Complex numbers and phasors are indispensable tools in electromagnetic wave analysis. This chapter provides a concise review of the essential concepts that will be used throughout the course.

2.1 Complex Numbers

A complex number z can be written in **rectangular form**:

$$z = x + jy, \tag{2.1}$$

where $x = \mathrm{Re}(z)$ is the real part, $y = \mathrm{Im}(z)$ is the imaginary part, and $j = \sqrt{-1}$ is the imaginary unit (engineers use j rather than i to avoid confusion with current).

The **polar form** (or exponential form) is:

$$z = |z|\, e^{j\phi}, \tag{2.2}$$

where the magnitude and phase are:

$$|z| = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{2.3}$$

Euler's Formula

The bridge between rectangular and polar forms is Euler's formula:

$$e^{j\phi} = \cos\phi + j\sin\phi. \tag{2.4}$$

From Euler's formula, the trigonometric functions can be expressed as:

$$\cos\phi = \mathrm{Re}\Big(e^{j\phi}\Big) = \frac{e^{j\phi} + e^{-j\phi}}{2}, \qquad \sin\phi = \mathrm{Im}\Big(e^{j\phi}\Big) = \frac{e^{j\phi} - e^{-j\phi}}{2j}. \tag{2.5}$$

2.1.1 Complex Conjugate

The complex conjugate of $z = x + jy$ is:

$$z^* = x - jy = |z|\, e^{-j\phi}. \tag{2.6}$$

Useful properties:

$$z \cdot z^* = |z|^2, \tag{2.7}$$
$$\mathrm{Re}(z) = \frac{z + z^*}{2}, \tag{2.8}$$
$$\mathrm{Im}(z) = \frac{z - z^*}{2j}. \tag{2.9}$$

2.1.2 Operations in Polar Form

Multiplication and division are most natural in polar form:

$$z_1 z_2 = |z_1||z_2|\, e^{j(\phi_1+\phi_2)}, \tag{2.10}$$
$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\, e^{j(\phi_1-\phi_2)}. \tag{2.11}$$

2.2 Phasors

In electromagnetic wave analysis, we frequently deal with sinusoidal steady-state quantities. The **phasor** representation provides an efficient way to work with such quantities by eliminating the explicit time dependence.

2.2.1 Time-Harmonic Convention

We adopt the $e^{j\omega t}$ time convention. A general time-harmonic scalar quantity is written as:

$$v(t) = \mathrm{Re}\left(\tilde{V}\, e^{j\omega t}\right), \tag{2.12}$$

where $\tilde{V}$ is the **phasor** (a complex number independent of time).

Example

Let $v(t) = V_0 \cos(\omega t + \phi)$. The corresponding phasor is:

$$\tilde{V} = V_0\, e^{j\phi}. \tag{2.13}$$

To verify: $\mathrm{Re}\big(V_0 e^{j\phi} e^{j\omega t}\big) = V_0 \cos(\omega t + \phi) = v(t)$. ✓

Important

The phasor convention requires a *cosine* reference. If a quantity is given in terms of sine, convert to cosine first using $\sin(\alpha) = \cos(\alpha - \pi/2)$:

$$v(t) = A\sin(\omega t + \theta) = A\cos(\omega t + \theta - \pi/2) \quad \Longrightarrow \quad \tilde{V} = A\, e^{j(\theta - \pi/2)}.$$

Be careful with signs: $\cos(\alpha + \pi/2) = -\sin\alpha$, while $\cos(\alpha - \pi/2) = +\sin\alpha$. Confusing these is a common source of sign errors.

2.2.2 Key Property: Time Derivatives Become Multiplication

The most powerful feature of phasors is that differentiation with respect to time becomes multiplication by $j\omega$:

$$\frac{\partial}{\partial t} \longleftrightarrow j\omega. \tag{2.14}$$

Phasor Conversion Rules

Time Domain	Phasor Domain
$v(t) = V_0 \cos(\omega t + \phi)$	$\tilde{V} = V_0\, e^{j\phi}$
$\dfrac{\partial v}{\partial t}$	$j\omega\, \tilde{V}$
$\displaystyle\int v\, dt$	$\dfrac{\tilde{V}}{j\omega}$

2.3 Complex Vectors

A **complex vector** (or phasor vector) has complex-valued components:

$$\tilde{\mathbf{A}} = \tilde{A}_x\, \hat{\mathbf{x}} + \tilde{A}_y\, \hat{\mathbf{y}} + \tilde{A}_z\, \hat{\mathbf{z}}, \tag{2.15}$$

where each component $\tilde{A}_x$, $\tilde{A}_y$, $\tilde{A}_z$ is a complex number (phasor). The corresponding time-domain vector is:

$$\mathbf{A}(t) = \mathrm{Re}\Big(\tilde{\mathbf{A}}\, e^{j\omega t}\Big). \tag{2.16}$$

Important

A complex vector $\tilde{\mathbf{A}}$ does *not* generally point in a fixed direction. As ωt varies, the real vector $\mathbf{A}(t)$ may trace out an ellipse in space. This is the origin of *polarization*, which we will study in Chapter 17.

Physical Insight

To determine the polarization type, evaluate $\mathbf{A}(t) = \mathrm{Re}(\tilde{\mathbf{A}}\, e^{j\omega t})$ at several values of ωt (e.g., $0, \pi/4, \pi/2, 3\pi/4, \ldots$) and plot the tip of the vector:

- If the tip traces a *straight line*, the polarization is **linear**.
- If the tip traces a *circle*, the polarization is **circular**.
- In general, the tip traces an *ellipse*: **elliptical** polarization.

For example, $\tilde{\mathbf{A}} = A_0(\hat{\mathbf{x}} + j\,\hat{\mathbf{y}})$ gives $\mathbf{A}(t) = A_0[\cos(\omega t)\,\hat{\mathbf{x}} - \sin(\omega t)\,\hat{\mathbf{y}}]$, which traces a circle of radius A_0 (right-hand circular polarization).

2.3.1 Dot and Cross Products of Complex Vectors

The dot product and cross product of complex vectors follow the standard rules, but with complex arithmetic:

$$\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}} = \tilde{A}_x \tilde{B}_x + \tilde{A}_y \tilde{B}_y + \tilde{A}_z \tilde{B}_z, \tag{2.17}$$

$$\tilde{\mathbf{A}} \times \tilde{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \tilde{A}_x & \tilde{A}_y & \tilde{A}_z \\ \tilde{B}_x & \tilde{B}_y & \tilde{B}_z \end{vmatrix}. \tag{2.18}$$

Note that $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}$ is a complex scalar, and $\tilde{\mathbf{A}} \times \tilde{\mathbf{B}}$ is a complex vector.

2.4 Time-Average of Products

When computing power or energy in the phasor domain, we need the time-average of products of sinusoidal quantities.

Time-Average Formula

If $a(t) = \mathrm{Re}(\tilde{A}\, e^{j\omega t})$ and $b(t) = \mathrm{Re}(\tilde{B}\, e^{j\omega t})$, then the time-average of their product is:

$$\langle a(t)\, b(t) \rangle = \frac{1}{2} \mathrm{Re}\Big(\tilde{A}\, \tilde{B}^*\Big). \tag{2.19}$$

Derivation. Write $a(t) = \frac{1}{2}(\tilde{A}e^{j\omega t} + \tilde{A}^* e^{-j\omega t})$ and similarly for $b(t)$. The product contains terms at frequencies 2ω and 0. The 2ω terms average to zero over one period, leaving only the DC terms:

$$\langle a \cdot b \rangle = \frac{1}{4}\left(\tilde{A}\, \tilde{B}^* + \tilde{A}^*\, \tilde{B}\right) = \frac{1}{2} \mathrm{Re}\Big(\tilde{A}\, \tilde{B}^*\Big). \quad \square$$

This formula is used extensively in computing time-average power (Poynting vector), stored energy, and dissipated power throughout electromagnetics.

Example

Let $v(t) = 10\cos(\omega t + 30°)$ V and $i(t) = 2\cos(\omega t - 15°)$ A. Find the time-average power.
Solution: The phasors are $\tilde{V} = 10\, e^{j30°}$ and $\tilde{I} = 2\, e^{-j15°}$.

$$P_{\text{avg}} = \frac{1}{2}\mathrm{Re}(\tilde{V}\, \tilde{I}^*) = \frac{1}{2}\mathrm{Re}\Big(10\, e^{j30°} \cdot 2\, e^{j15°}\Big) = \frac{1}{2}\mathrm{Re}\Big(20\, e^{j45°}\Big) = 10\cos 45° \approx 7.07\,\mathrm{W}.$$

Exercises

1. Express the following in both rectangular and polar form: (a) $(3 + j4)(1 - j2)$, (b) $\dfrac{2 + j3}{1 + j}$, (c) $(1 + j)^8$.

2. Given $v(t) = 5\cos\big(2\pi \times 10^9 t - 45°\big)$ V, write the phasor $\tilde{V}$ and compute dv/dt using phasors.

3. Prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1 z_2^*)$.

4. Given $\tilde{\mathbf{E}} = (3 + j4)\,\hat{\mathbf{x}} + (1 - j2)\,\hat{\mathbf{y}}$, find $|\tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^*|$ and $\langle|\mathbf{E}(t)|^2\rangle$.

5. Show that for a complex vector $\tilde{\mathbf{A}} = A_0(\hat{\mathbf{x}} + j\hat{\mathbf{y}})$, the time-domain vector $\mathbf{A}(t) = \mathrm{Re}(\tilde{\mathbf{A}}\, e^{j\omega t})$ traces a circle. What is the radius and direction of rotation?

Chapter 3

Vector Calculus Review

This chapter reviews the vector calculus operations and theorems that form the mathematical language of electromagnetics. Every equation in Maxwell's theory is written using these tools.

3.1 The Del Operator

The **del operator** (or **nabla**) in rectangular coordinates is:

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{3.1}$$

This vector differential operator can act on scalar and vector fields in four distinct ways: gradient, divergence, curl, and Laplacian.

3.2 Gradient

The **gradient** of a scalar field $\phi(x, y, z)$ is:

$$\nabla\phi = \frac{\partial \phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{z}}. \tag{3.2}$$

The gradient is a *vector* that points in the direction of the steepest increase of ϕ, and its magnitude equals the maximum rate of change.

Physical Insight

Think of a topographic map: the gradient of the elevation function points directly uphill, perpendicular to the contour lines (lines of constant elevation). The steeper the hill, the larger $|\nabla\phi|$.

The **directional derivative** of ϕ in the direction of a unit vector $\hat{\mathbf{a}}$ is:

$$\left.\frac{d\phi}{dl}\right|_{\hat{\mathbf{a}}} = \nabla\phi \cdot \hat{\mathbf{a}}. \tag{3.3}$$

3.3 Divergence

The **divergence** of a vector field $\mathbf{A}(x, y, z)$ is:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{3.4}$$

The divergence is a *scalar* that measures the net outward flux per unit volume from an infinitesimal region. In the limiting sense:

$$\nabla \cdot \mathbf{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \mathbf{A} \cdot d\mathbf{S}. \tag{3.5}$$

Bathtub Analogy

Imagine a bathtub filled with tiny pipes continuously injecting water: the divergence is positive (source). If the pipes are instead vacuuming water out, the divergence is negative (sink). Away from these sources/sinks, water flows through without accumulating: the divergence is zero there.

Key insight: Zero divergence does *not* mean the field is zero! Consider water draining from a single point in the bathtub. The water is clearly flowing (nonzero velocity) everywhere, but at any point *away* from the drain, what enters a small cube from one face exits through the opposite face—so the divergence is zero even though the velocity field is nonzero. Only at the drain itself is the divergence negative.

Example

Let $\mathbf{A} = x^2\,\hat{\mathbf{x}} + y^2\,\hat{\mathbf{y}} + z^2\,\hat{\mathbf{z}}$. Then:

$$\nabla \cdot \mathbf{A} = 2x + 2y + 2z.$$

At the origin, the divergence is zero. Away from the origin, the field has nonzero divergence.

3.4 Curl

The **curl** of a vector field $\mathbf{A}$ is:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{3.6}$$

Expanding the determinant:

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}. \tag{3.7}$$

The curl is a *vector* that measures the circulation (or rotation) of the field per unit area. Its direction is the axis of maximum rotation (by the right-hand rule), and its magnitude is the circulation per unit area about that axis.

Paddle Wheel Analogy

Imagine placing a tiny paddle wheel (a "curl meter") in a flowing fluid. If the flow has curl at that point, the paddle wheel will spin. The axis of rotation is the direction of $\nabla \times \mathbf{A}$, and the spin rate is proportional to $|\nabla \times \mathbf{A}|$.

Example: Consider a river where the water velocity increases with height: $\mathbf{v} = v(z)\,\hat{\mathbf{x}}$ with $dv/dz > 0$. Place the paddle wheel with its axis in the x-direction: the upper vane feels a stronger push than the lower vane, causing rotation—so the x-component of curl is nonzero. With the axis in z or y, the vanes are pushed symmetrically and do not spin—those curl components are zero. Indeed, $\nabla \times \mathbf{v} = -\frac{dv}{dz}\,\hat{\mathbf{y}}$ has only a y-component (the minus sign follows from the determinant expansion).

3.5 Laplacian

The **scalar Laplacian** of ϕ is the divergence of the gradient:

$$\nabla^2\phi = \nabla \cdot (\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{3.8}$$

The **vector Laplacian** of $\mathbf{A}$ is defined as:

$$\nabla^2\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}). \tag{3.9}$$

Important

In rectangular coordinates (and *only* in rectangular coordinates), the vector Laplacian reduces to the scalar Laplacian applied to each component:

$$\nabla^2\mathbf{A} = \nabla^2 A_x\,\hat{\mathbf{x}} + \nabla^2 A_y\,\hat{\mathbf{y}} + \nabla^2 A_z\,\hat{\mathbf{z}}. \tag{3.10}$$

This identity does *not* hold in cylindrical or spherical coordinates.

3.6 Important Identities

3.6.1 Zero Identities

Two identities that are zero for any well-behaved field:

Zero Identities

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad \text{(divergence of curl is always zero)}, \tag{3.11}$$

$$\nabla \times (\nabla\phi) = \mathbf{0} \qquad \text{(curl of gradient is always zero)}. \tag{3.12}$$

Equation (3.11) has a profound physical consequence: it implies that the magnetic field $\mathbf{B}$, which can always be written as $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector potential $\mathbf{A}$, is always divergence-free—there are no magnetic monopoles.

3.6.2 Product Identity

The following identity is essential for deriving Poynting's theorem (Chapter 5):

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \tag{3.13}$$

3.7 Divergence Theorem

The **divergence theorem** (also called Gauss's theorem) relates a volume integral of the divergence to a closed surface integral:

Divergence Theorem

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{A})\, dV, \tag{3.14}$$

where S is the closed surface bounding the volume V, and $d\mathbf{S} = \hat{\mathbf{n}}\, dS$ points outward.

Physical interpretation: The total outward flux of $\mathbf{A}$ through a closed surface equals the total "source strength" (divergence) contained within the volume.

3.8 Stokes's Theorem

Stokes's theorem relates a surface integral of the curl to a closed line integral:

Stokes's Theorem

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}, \tag{3.15}$$

where C is the closed contour bounding the open surface S, and the directions of $d\mathbf{l}$ and $d\mathbf{S}$ are related by the right-hand rule.

Cereal Bowl Analogy

Think of a cereal bowl: the rim of the bowl is the contour C, and the bowl itself is the surface S. Stokes's theorem says that the total circulation of $\mathbf{A}$ around the rim equals the total curl "flux" passing through the bowl. The surface S can be *any* surface bounded by C—flat, curved, or bowl-shaped. The direction of $d\mathbf{l}$ around the rim and $d\mathbf{S}$ through the bowl are linked by the right-hand rule: curl the fingers in the direction of traversal, and the thumb points in the direction of $\hat{\mathbf{n}}$.

Stokes's theorem is the mathematical tool used to convert Faraday's law from differential form ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$) to integral form—the basis of how transformers and AC generators work.

3.9 Coordinate Systems

While rectangular coordinates are convenient for theory, many electromagnetic problems have cylindrical or spherical symmetry.

3.9.1 Cylindrical Coordinates (ρ, ϕ, z)

The position vector is $\mathbf{r} = \rho\,\hat{\rho} + z\,\hat{\mathbf{z}}$. The gradient, divergence, and curl are:

$$\nabla\phi = \frac{\partial\phi}{\partial\rho}\,\hat{\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\phi}\,\hat{\phi} + \frac{\partial\phi}{\partial z}\,\hat{\mathbf{z}}, \tag{3.16}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}. \tag{3.17}$$

3.9.2 Spherical Coordinates (r, θ, ϕ)

The position vector is $\mathbf{r} = r\,\hat{\mathbf{r}}$. The gradient is:

$$\nabla\phi = \frac{\partial\phi}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\phi}\,\hat{\phi}, \tag{3.18}$$

and the divergence is:

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}. \tag{3.19}$$

Exercises

1. Verify the identity $\nabla\cdot(\nabla\times\mathbf{A}) = 0$ for $\mathbf{A} = xy\,\hat{\mathbf{x}} + yz\,\hat{\mathbf{y}} + zx\,\hat{\mathbf{z}}$.
2. Compute $\nabla\times\mathbf{A}$ for $\mathbf{A} = -y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$. Interpret the result physically (what kind of flow does this represent?).
3. Use the divergence theorem to evaluate $\oint_S \mathbf{r}\cdot d\mathbf{S}$ over the surface of a sphere of radius R, where $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$.
4. Verify Stokes's theorem for $\mathbf{A} = -y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$ over a circular disk of radius a in the $z = 0$ plane.
5. Prove the product identity (3.13) by expanding both sides in rectangular coordinates.
6. Express the Laplacian $\nabla^2\phi$ in cylindrical coordinates.

Part II

Electromagnetic Theory

Chapter 4

Maxwell's Equations

Maxwell's equations are the four fundamental laws that govern all electromagnetic phenomena. Originally formulated by James Clerk Maxwell in 1865 and later cast into their modern vector form by Oliver Heaviside, these equations unify electricity, magnetism, and optics into a single theoretical framework.

4.1 Electromagnetic Field Quantities

There are four electromagnetic field vectors:

Symbol	**Name**	**Units**	**Type**
E	Electric field intensity	V/m	Physical
H	Magnetic field intensity	A/m	Defined
D	Electric flux density	C/m^2	Defined
B	Magnetic flux density	T (Wb/m^2)	Physical

Physical Insight

The "physical" fields **E** and **B** are the ones that appear in the Lorentz force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, and are directly measurable. The "defined" fields **D** and **H** are mathematical constructs that account for the effects of material polarization and magnetization, making the equations more convenient to use inside materials.

4.2 Maxwell's Equations in Differential Form

Maxwell's Equations (Time Domain)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law)} \tag{4.1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{(Ampère's law)} \tag{4.2}$$

$$\nabla \cdot \mathbf{D} = \rho_v \quad \text{(Electric Gauss law)} \tag{4.3}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Magnetic Gauss law)} \tag{4.4}$$

4.2.1 Physical Interpretation

Faraday's law (4.1): A time-varying magnetic field produces a circulating electric field. This is the principle behind transformers, generators, and electromagnetic induction.

Ampère's law (4.2): Magnetic fields are produced by currents $\mathbf{J}$ and by time-varying electric fields $\partial\mathbf{D}/\partial t$ (Maxwell's displacement current). The displacement current term is Maxwell's key contribution—it predicted the existence of electromagnetic waves.

Displacement Current in a Capacitor

Consider a parallel-plate capacitor being charged by a current I. In the wire, conduction current flows ($\mathbf{J} \neq 0$). But between the plates, there is no conduction current ($\mathbf{J} = 0$)—yet the magnetic field must be continuous. Maxwell resolved this by recognizing that the changing electric field between the plates produces a displacement current $\partial\mathbf{D}/\partial t$ that is exactly equal to the conduction current in the wire. Without this term, Ampère's law would give different answers depending on which surface you choose for the integration—a physical impossibility.

Electric Gauss law (4.3): Electric flux originates from electric charges. Positive charges are sources; negative charges are sinks.

Magnetic Gauss law (4.4): There are no magnetic monopoles. Magnetic field lines always close upon themselves.

4.3 Source Terms

The **volume charge density** ρ_v [C/m^3] describes the distribution of electric charge in space. The **current density vector J** [A/m^2] describes the flow of charge. The current flowing through a surface S is:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}. \tag{4.5}$$

In a conducting material, the current density is related to the electric field by Ohm's law:

$$\mathbf{J} = \sigma \mathbf{E}, \tag{4.6}$$

where σ [S/m] is the electrical conductivity of the material.

4.4 Continuity Equation

Taking the divergence of Ampère's law (4.2) and using the zero identity $\nabla \cdot (\nabla \times \mathbf{H}) = 0$:

$$0 = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t}. \tag{4.7}$$

Continuity Equation (Conservation of Charge)

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}. \tag{4.8}$$

In integral form (using the divergence theorem):

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_{\text{enc}}}{dt} \qquad \Longleftrightarrow \qquad I_{\text{out}} = -\frac{dQ_{\text{enc}}}{dt}. \tag{4.9}$$

Important

Conservation of charge is not an independent assumption—it is a *consequence* of Maxwell's equations. Charge is never created or destroyed; it can only flow from one place to another.

4.5 Integral Forms of Maxwell's Equations

Applying Stokes's theorem to the curl equations and the divergence theorem to the divergence equations:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \qquad \text{(Faraday's law)} \tag{4.10}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S} \qquad \text{(Ampère's law)} \tag{4.11}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}} \qquad \text{(Electric Gauss law)} \tag{4.12}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \text{(Magnetic Gauss law)} \tag{4.13}$$

4.6 Statics

When all time derivatives vanish ($\partial/\partial t = 0$), Maxwell's equations decouple into two independent sets:

$$\text{Electrostatics:} \quad \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{D} = \rho_v; \tag{4.14}$$

$$\text{Magnetostatics:} \quad \nabla \times \mathbf{H} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0. \tag{4.15}$$

In statics, electric and magnetic fields are completely independent of each other. It is the time-varying coupling between **E** and **H** that enables wave propagation.

4.7 Phasor Domain Maxwell's Equations

For time-harmonic (sinusoidal) fields with $e^{j\omega t}$ convention, $\partial/\partial t \to j\omega$:

Maxwell's Equations (Phasor Domain)

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{B}}, \tag{4.16}$$

$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}, \tag{4.17}$$

$$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_v, \tag{4.18}$$

$$\nabla \cdot \tilde{\mathbf{B}} = 0. \tag{4.19}$$

4.8 Constitutive Relations

The constitutive relations connect the defined fields to the physical fields through material properties.

4.8.1 Free Space

In free space (vacuum):

$$\mathbf{D} = \varepsilon_0\mathbf{E}, \qquad \mathbf{B} = \mu_0\mathbf{H}, \tag{4.20}$$

where the fundamental constants are:

$$\varepsilon_0 \approx 8.854 \times 10^{-12} \text{ F/m} \quad \text{(permittivity of free space)}, \tag{4.21}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m} \quad \text{(permeability of free space)}. \tag{4.22}$$

The speed of light in free space is:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \approx 2.998 \times 10^8 \text{ m/s}. \tag{4.23}$$

4.8.2 Linear Isotropic Materials

In a simple (linear, isotropic, homogeneous) material:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H} = \mu_0\mu_r\mathbf{H}, \tag{4.24}$$

where ε_r and μ_r are the relative permittivity and relative permeability.

4.8.3 Physical Origin of ε_r

When an external electric field is applied to a dielectric material, the molecular electric dipoles partially align with the field. This **polarization** effectively increases the electric flux density. The relative permittivity is:

$$\varepsilon_r = 1 + \chi_e, \tag{4.25}$$

where χ_e is the electric susceptibility. Water, with its highly polar molecules that rotate freely, has $\varepsilon_r \approx 81$ at low frequencies.

4.8.4 Physical Origin of μ_r

The magnetic properties arise from the alignment of electron spin magnetic moments. Most materials have $\mu_r \approx 1$. Ferromagnetic materials (iron, nickel, cobalt) have $\mu_r \gg 1$ due to cooperative alignment of magnetic domains.

4.8.5 Material Classifications

- **Homogeneous**: Properties are the same at every point.
- **Isotropic**: Properties are the same in all directions.
- **Linear**: Properties do not depend on field strength.
- **Non-dispersive**: Properties do not depend on frequency.
- **Anisotropic**: Properties depend on direction. The permittivity becomes a tensor:

$$\mathbf{D} = \bar{\bar{\varepsilon}} \cdot \mathbf{E}, \qquad \bar{\bar{\varepsilon}} = \varepsilon_0 \begin{pmatrix} \varepsilon_{rx} & 0 & 0 \\ 0 & \varepsilon_{ry} & 0 \\ 0 & 0 & \varepsilon_{rz} \end{pmatrix}. \tag{4.26}$$

Example

Given $\tilde{\mathbf{E}} = E_0 e^{-j\beta z}\,\hat{\mathbf{x}}$ in free space, find $\tilde{\mathbf{H}}$.
Solution: From Faraday's law: $\nabla \times \tilde{\mathbf{E}} = -j\omega\mu_0\tilde{\mathbf{H}}$.

$$\nabla \times \tilde{\mathbf{E}} = -\frac{\partial E_x}{\partial z}\,\hat{\mathbf{y}} = j\beta E_0 e^{-j\beta z}\,\hat{\mathbf{y}}.$$

Therefore:

$$\tilde{\mathbf{H}} = \frac{j\beta E_0}{-j\omega\mu_0} e^{-j\beta z}\,\hat{\mathbf{y}} = -\frac{\beta}{\omega\mu_0} E_0 e^{-j\beta z}\,\hat{\mathbf{y}} = \frac{E_0}{\eta_0} e^{-j\beta z}\,\hat{\mathbf{y}},$$

where $\eta_0 = \omega\mu_0/\beta = \sqrt{\mu_0/\varepsilon_0} \approx 377\ \Omega$ is the intrinsic impedance of free space (noting $\beta = \omega\sqrt{\mu_0\varepsilon_0}$ and the minus sign resolves with the cross-product direction).

Exercises

1. Starting from Ampère's law in differential form, derive the continuity equation.
2. Show that in a source-free region ($\rho_v = 0$, $\mathbf{J} = \mathbf{0}$), Faraday's law and Ampère's law can be combined to produce the wave equation for $\mathbf{E}$.
3. A region of space has $\varepsilon_r = 4$ and $\mu_r = 1$. What is the speed of an electromagnetic wave in this medium?
4. Given $\tilde{\mathbf{H}} = H_0 \cos(\beta x)\, e^{-j\beta z}\,\hat{\mathbf{y}}$ in free space (no sources), find $\tilde{\mathbf{E}}$ using Faraday's law.
5. Explain physically why the displacement current term $\partial\mathbf{D}/\partial t$ is necessary in Ampère's law. What would go wrong without it? (Hint: consider the continuity equation.)

Chapter 5

Poynting's Theorem

Poynting's theorem is the statement of conservation of electromagnetic energy. It tells us how power flows through space, how energy is stored in fields, and how energy is dissipated in lossy materials.

5.1 Derivation

We derive Poynting's theorem directly from Maxwell's equations.

Step 1: Take the dot product of $\mathbf{H}$ with Faraday's law:

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{5.1}$$

Step 2: Take the dot product of $\mathbf{E}$ with Ampère's law:

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}. \tag{5.2}$$

Step 3: Subtract the second from the first and use the vector identity $\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$:

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{5.3}$$

Step 4: For linear media ($\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$), use the chain rule:

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon |\mathbf{E}|^2\right), \qquad \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu |\mathbf{H}|^2\right). \tag{5.4}$$

Step 5: Using Ohm's law, $\mathbf{E} \cdot \mathbf{J} = \sigma |\mathbf{E}|^2$. Integrate over a volume V bounded by surface S and apply the divergence theorem:

Poynting's Theorem (Time Domain)

$$-\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_V \sigma |\mathbf{E}|^2 \, dV + \frac{\partial}{\partial t} \int_V \frac{1}{2}\mu |\mathbf{H}|^2 \, dV + \frac{\partial}{\partial t} \int_V \frac{1}{2}\varepsilon |\mathbf{E}|^2 \, dV. \tag{5.5}$$

Each term has a clear physical meaning:

- **Left side**: Net power flowing *into* the volume through the surface S.

- **First term**: Power dissipated as heat (Joule heating), $P_d = \int_V \sigma |\mathbf{E}|^2 \, dV$.
- **Second term**: Rate of change of stored magnetic energy, $\frac{dW_m}{dt}$.
- **Third term**: Rate of change of stored electric energy, $\frac{dW_e}{dt}$.

In words: *power in = power dissipated + rate of increase of stored energy.*

5.2 The Poynting Vector

Poynting Vector

The **Poynting vector** is defined as:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \quad [\mathrm{W/m^2}]. \tag{5.6}$$

It represents the instantaneous power flux density: the direction of **S** is the direction of power flow, and $|\mathbf{S}|$ is the power per unit area.

The total power flowing through a surface is:

$$P = \int_S \mathbf{S} \cdot d\mathbf{S} = \int_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S}. \tag{5.7}$$

5.3 Complex Poynting Vector

For time-harmonic fields in the phasor domain, the **complex Poynting vector** is:

Complex Poynting Vector

$$\tilde{\mathbf{S}} = \frac{1}{2} \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \quad [\mathrm{VA/m^2}]. \tag{5.8}$$

- $\mathrm{Re}(\tilde{\mathbf{S}})$ = time-average power density [W/m^2]
- $\mathrm{Im}(\tilde{\mathbf{S}})$ = reactive power density [VAR/m^2]

The time-average power flow through a surface is:

$$P_{\text{avg}} = \int_S \mathrm{Re}(\tilde{\mathbf{S}}) \cdot d\mathbf{S} = \frac{1}{2} \mathrm{Re} \int_S \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \cdot d\mathbf{S}. \tag{5.9}$$

5.4 Complex Poynting Theorem

The phasor-domain version of Poynting's theorem, derived from the phasor Maxwell's equations, is:

$$-\oint_S \tilde{\mathbf{S}} \cdot d\mathbf{S} = P_d + 2j\omega(\langle W_m \rangle - \langle W_e \rangle), \tag{5.10}$$

where:

$$P_d = \frac{1}{2}\int_V \sigma|\tilde{\mathbf{E}}|^2\,dV \quad \text{(time-average dissipated power)}, \tag{5.11}$$

$$\langle W_m\rangle = \frac{1}{4}\int_V \mu|\tilde{\mathbf{H}}|^2\,dV \quad \text{(time-average stored magnetic energy)}, \tag{5.12}$$

$$\langle W_e\rangle = \frac{1}{4}\int_V \varepsilon|\tilde{\mathbf{E}}|^2\,dV \quad \text{(time-average stored electric energy)}. \tag{5.13}$$

5.4.1 Physical Meaning of VARs

Taking the imaginary part of (5.10):

$$\text{VARs}_{\text{in}} = 2\omega\left(\langle W_m\rangle - \langle W_e\rangle\right). \tag{5.14}$$

Physical Insight

VARs (volt-amperes reactive) represent the imbalance between stored magnetic and electric energy, *not* a power loss.

- An inductor has $W_m > W_e$: it *absorbs* VARs.
- A capacitor has $W_e > W_m$: it *generates* (supplies) VARs.
- A resonant circuit has $W_m = W_e$: zero VARs (power factor = 1).

This is the deep physical basis for the concept of reactive power in circuit theory.

5.5 Equivalence with Circuit Theory

As a validation, we can show that the Poynting vector gives the same result as circuit theory for a transmission line. Consider a parallel-plate transmission line with plate separation h and width w. Between the plates, with voltage V across them and current I flowing:

$$\mathbf{E} = \frac{V}{h}\,\hat{\mathbf{y}}, \qquad \mathbf{H} = \frac{I}{w}\,\hat{\mathbf{x}}. \tag{5.15}$$

The Poynting vector is:

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{VI}{hw}\,\hat{\mathbf{z}}. \tag{5.16}$$

Integrating over the cross-section (area $= hw$):

$$P = \int_S \mathbf{S}\cdot d\mathbf{S} = \frac{VI}{hw}\cdot hw = VI. \tag{5.17}$$

In the phasor domain: $P_{\text{avg}} = \frac{1}{2}\,\text{Re}(\tilde{V}\,\tilde{I}^*)$, which is the standard circuit-theory result.

Important

The Poynting vector is the universal method for computing power flow. Circuit theory ($P = VI$) is a special case that applies when the fields are confined to a well-defined cross-section.

Exercises

1. Derive Poynting's theorem starting from Maxwell's equations, filling in all the intermediate steps.

2. For a plane wave $\tilde{\mathbf{E}} = E_0 \, e^{-jkz} \, \hat{\mathbf{x}}$ in free space, compute the complex Poynting vector and the time-average power density.

3. Show that the time-average power dissipated in a conducting region with conductivity σ is $P_d = \frac{1}{2} \int_V \sigma |\tilde{\mathbf{E}}|^2 \, dV$.

4. For a coaxial transmission line with inner radius a and outer radius b, verify that the Poynting vector integral gives $P = \frac{1}{2} \operatorname{Re}(\tilde{V} \, \tilde{I}^*)$.

5. A region of space contains $\langle W_m \rangle = 2 \, \mu\text{J}$ and $\langle W_e \rangle = 5 \, \mu\text{J}$ at $f = 1 \, \text{GHz}$. Calculate the net VARs flowing into the region. Is the region behaving more like an inductor or a capacitor?

Part III

Transmission Lines

Chapter 6

Transmission Lines: Time Domain

A transmission line is a two-conductor system designed to guide electromagnetic energy from one point to another. When the physical dimensions of a circuit become comparable to a wavelength, simple circuit theory fails and we must use transmission line theory.

6.1 Types of Transmission Lines

6.1.1 Coaxial Cable

A coaxial cable consists of an inner conductor of radius a surrounded by an outer conductor of inner radius b, with a dielectric filling the space between them. The coax is a *perfectly shielded* system—all fields are confined between the conductors, so it neither radiates nor picks up interference. Industry standards include $50\,\Omega$ (microwave/RF) and $75\,\Omega$ (television).

6.1.2 Twin Lead

Twin lead consists of two parallel wires separated by a fixed distance. It is an *open* system—fields extend to infinity. Standard twin-lead impedance is $300\,\Omega$, commonly used for TV antenna connections (now largely replaced by coax).

6.1.3 Microstrip

A microstrip line consists of a conducting strip on the top of a dielectric substrate with a ground plane on the bottom. It is the most common transmission line in printed circuit board (PCB) design.

6.1.4 Other Types

Other common types include stripline (strip between two ground planes), coplanar waveguide (CPW), and twisted pair (CAT5/CAT6 Ethernet cables). Twisting reduces radiation and interference by causing far-field cancellation.

6.2 Signal Velocity vs. Electron Drift Velocity

Important

The signal on a transmission line travels at a speed close to the speed of light ($\sim 10^8$ m/s), but the electrons themselves drift very slowly (~ 0.1 mm/s). The signal is carried by the electromagnetic wave between the conductors, not by the physical motion of electrons.

Analogy: Consider a long tube filled with steel balls. Push a ball in at one end and a ball pops out instantly at the other end—the "signal" (push) travels much faster than any individual ball.

6.3 Per-Unit-Length Parameters

A transmission line is characterized by four per-unit-length parameters:

Symbol	Name	Physical Origin	Units
R	Resistance	Ohmic loss in conductors	Ω/m
L	Inductance	Magnetic field energy from currents	H/m
G	Conductance	Leakage current through lossy dielectric	S/m
C	Capacitance	Electric field between conductors	F/m

Important

G is **not** $1/R$! They arise from completely different physical mechanisms. R is due to the finite conductivity of the metal conductors. G is due to the finite conductivity (or loss tangent) of the dielectric filling.

6.3.1 Formulas for Coaxial Cable

For a coax with inner radius a, outer radius b, and dielectric parameters ε_r, σ_d:

Coaxial Cable RLGC Parameters

$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln(b/a)} \quad [\text{F/m}], \tag{6.1}$$

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \quad [\text{H/m}], \tag{6.2}$$

$$G = \frac{2\pi\sigma_d}{\ln(b/a)} \quad [\text{S/m}], \tag{6.3}$$

$$R = \frac{1}{2\pi\sigma_m}\left(\frac{1}{a\delta_a} + \frac{1}{b\delta_b}\right) \quad [\Omega/\text{m}] \text{ (at high frequency)}, \tag{6.4}$$

where $\delta = \sqrt{2/(\omega\mu_m\sigma_m)}$ is the skin depth in the conductors.

6.3.2 The Universal LC Relation

From (6.1) and (6.2), the geometry factors cancel:

$$LC = \mu_0 \varepsilon_0 \varepsilon_r = \mu\varepsilon. \tag{6.5}$$

Universal LC Relation

For *any* two-conductor transmission line filled with a homogeneous dielectric:

$$LC = \mu\varepsilon. \tag{6.6}$$

This leads to the wave speed: $c_d = 1/\sqrt{LC} = 1/\sqrt{\mu\varepsilon} = c/\sqrt{\varepsilon_r}$.

6.4 Telegrapher's Equations

Consider a differential element Δz of the transmission line, modeled as a lumped circuit with series impedance $R\Delta z + L\Delta z\, \partial/\partial t$ and shunt admittance $G\Delta z + C\Delta z\, \partial/\partial t$.

Applying KVL and KCL and taking the limit $\Delta z \to 0$:

Telegrapher's Equations

$$\frac{\partial V}{\partial z} = -RI - L\frac{\partial I}{\partial t}, \tag{6.7}$$

$$\frac{\partial I}{\partial z} = -GV - C\frac{\partial V}{\partial t}. \tag{6.8}$$

These are two coupled first-order PDEs relating the voltage $V(z,t)$ and current $I(z,t)$ on the line.

6.5 Wave Equation and d'Alembert Solution

6.5.1 Lossless Case

For a lossless line ($R = 0$, $G = 0$), differentiating (6.7) with respect to z and substituting (6.8):

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} = \frac{1}{c_d^2}\frac{\partial^2 V}{\partial t^2}, \tag{6.9}$$

where $c_d = 1/\sqrt{LC}$ is the wave velocity on the line.

d'Alembert Solution

The general solution of the wave equation (6.9) is:

$$V(z,t) = f(z - c_d t) + g(z + c_d t), \tag{6.10}$$

where f represents a **forward-traveling wave** (positive z direction) and g represents a **backward-traveling wave** (negative z direction). The functions f and g are arbitrary and determined by initial/boundary conditions.

6.5.2 Lossy Case

For the general lossy case, the wave equation becomes:

$$\frac{\partial^2 V}{\partial z^2} = RGV + (RC + LG)\frac{\partial V}{\partial t} + LC\frac{\partial^2 V}{\partial t^2}. \tag{6.11}$$

This equation has no simple closed-form solution in the time domain. Loss causes both **attenuation** (amplitude reduction) and **distortion** (shape change) of pulses.

6.6 Characteristic Impedance

For a forward-traveling wave $V^+ = f(z - c_d t)$, substituting into the telegrapher's equations yields:

$$I^+(z,t) = \frac{V^+(z,t)}{Z_0}, \tag{6.12}$$

where the **characteristic impedance** is:

Characteristic Impedance (Lossless)

$$Z_0 = \sqrt{\frac{L}{C}} \quad [\Omega]. \tag{6.13}$$

For a forward wave: $V/I = +Z_0$. For a backward wave: $V/I = -Z_0$.

The negative sign for the backward wave arises from the reference direction convention for current, not from any physical sign change.

6.6.1 Z_0 Formulas

Coaxial cable:

$$Z_0 = \frac{\eta_0}{2\pi\sqrt{\varepsilon_r}} \ln\left(\frac{b}{a}\right) = \frac{60}{\sqrt{\varepsilon_r}} \ln\left(\frac{b}{a}\right) \quad [\Omega], \tag{6.14}$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0} \approx 120\pi \approx 377\,\Omega$ is the intrinsic impedance of free space.

Twin lead:

$$Z_0 = \frac{\eta_0}{\pi\sqrt{\varepsilon_r}} \cosh^{-1}\left(\frac{d}{2a}\right) \quad [\Omega], \tag{6.15}$$

where d is the center-to-center spacing and a is the wire radius.

Microstrip (approximate):

$$Z_0 \approx \frac{\eta_0}{2\pi\sqrt{\varepsilon_{r,\text{eff}}}} \ln\left(\frac{8h}{w} + \frac{w}{4h}\right) \quad \text{for } w/h \le 1, \tag{6.16}$$

where h is the substrate thickness, w is the strip width, and $\varepsilon_{r,\text{eff}}$ is an effective relative permittivity that accounts for the fields partially in the dielectric and partially in air.

Exercises

1. Derive the telegrapher's equations from the KVL and KCL equations for the lumped-element model of a Δz section.

2. Verify that $V(z,t) = V_0 \cos(\omega t - \beta z)$ is a solution of the lossless wave equation. What are ω, β, and c_d related by?

3. A coaxial cable has $a = 0.5\,\text{mm}$, $b = 3.0\,\text{mm}$, $\varepsilon_r = 2.25$ (Teflon). Calculate C, L, Z_0, and c_d.

4. A $50\,\Omega$ microstrip line is designed on a substrate with $\varepsilon_r = 4.4$ and $h = 1.6\,\text{mm}$. Estimate the required strip width w.

5. Show that for any TEM transmission line filled with a homogeneous dielectric, $LC = \mu\varepsilon$ regardless of the cross-sectional geometry.

Chapter 7

Pulse Propagation and Reflection

When a transmission line is terminated in a load that does not match the characteristic impedance, reflections occur. This chapter analyzes pulse propagation and reflection on lossless transmission lines in the time domain.

7.1 Time-Delay Notation

It is convenient to rewrite the d'Alembert solution using the **time-delay** form. A forward-traveling wave $f(z-c_d t)$ can equivalently be written as $F(t-z/c_d)$, emphasizing that the waveform at position z is a time-delayed copy of the waveform at $z = 0$:

$$V^+(z,t) = F\left(t - \frac{z}{c_d}\right). \tag{7.1}$$

The delay is $\tau = z/c_d$—the time it takes the wave to travel from $z = 0$ to position z.

7.2 Matched Load

Consider a lossless transmission line of characteristic impedance Z_0 and length ℓ, terminated in a resistive load R_L.

When $R_L = Z_0$ (matched load), the load absorbs the entire incident wave with no reflection. The voltage at the load is simply the delayed generator signal. This is the ideal case for signal transmission.

Physical Insight

This is why a $75\,\Omega$ television input is designed to match the $75\,\Omega$ coaxial cable: to prevent reflections that would cause "ghost images" on the screen.

7.3 Load Reflection Coefficient

When $R_L \neq Z_0$, the load cannot absorb all the incident power, and a reflected wave is generated. At the load, Ohm's law requires:

$$V_L = R_L \cdot I_L = R_L \cdot \frac{V^+ + V^-}{Z_0} \cdot (\text{signs}), \tag{7.2}$$

which leads to:

Load Reflection Coefficient

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}. \tag{7.3}$$

The reflected voltage wave is:

$$V^- = \Gamma_L \cdot V^+ \quad \text{(at the load)}. \tag{7.4}$$

Special cases:

Load	Γ_L	Physical Meaning
$R_L = Z_0$ (matched)	0	No reflection
$R_L = 0$ (short circuit)	-1	Total reflection, voltage inverts
$R_L = \infty$ (open circuit)	$+1$	Total reflection, voltage doubles
$R_L > Z_0$	$0 < \Gamma_L < 1$	Partial positive reflection
$R_L < Z_0$	$-1 < \Gamma_L < 0$	Partial negative reflection

7.4 Reflected Wave on the Line

For a step-function source at $z = 0$, the reflected wave at an arbitrary point z on the line (for $z \leq \ell$) is:

$$V^-(z, t) = \Gamma_L \cdot V^+\left(t - \frac{2\ell - z}{c_d}\right). \tag{7.5}$$

The argument shows a total delay of $(2\ell - z)/c_d$: the forward wave travels from z to the load (distance $\ell - z$), reflects, and travels back to z (distance $\ell - z$ again), for a total path of $2(\ell - z)$, plus the original delay to reach z.

7.5 Generator Effects

Consider a generator with Thévenin voltage $V_G(t)$ and source resistance R_G connected at $z = 0$. By the voltage divider at the input:

$$V^+(0, t) = \frac{Z_0}{Z_0 + R_G} V_G(t) \equiv A \cdot V_G(t), \tag{7.6}$$

where $A = Z_0/(Z_0 + R_G)$ is the **amplitude factor**.

When the reflected wave returns to the generator, it sees the source resistance R_G and may itself be partially reflected:

$$\Gamma_G = \frac{R_G - Z_0}{R_G + Z_0} \quad \text{(generator reflection coefficient)}. \tag{7.7}$$

7.6 Multiple Bounces

After the initial forward wave, a sequence of bounces occurs:

1. Forward wave arrives at load, reflects with factor Γ_L.
2. Reflected wave arrives at generator, re-reflects with factor Γ_G.
3. Re-reflected wave arrives at load, reflects again with Γ_L.
4. Process continues indefinitely, with each successive wave smaller by a factor of $\Gamma_L\Gamma_G$.

The total voltage at any point is the sum of all waves that have passed through that point.

7.6.1 Steady-State Convergence

The voltage at the load converges to a geometric series:

$$V_{\text{load}} = A\,V_0\,(1+\Gamma_L)\sum_{n=0}^{\infty}(\Gamma_L\Gamma_G)^n = \frac{A\,V_0\,(1+\Gamma_L)}{1-\Gamma_L\Gamma_G}. \tag{7.8}$$

Steady-State Result

After all bounces die out, the steady-state voltage at the load is:

$$V_{\text{load}} = V_0 \cdot \frac{R_L}{R_L+R_G}, \tag{7.9}$$

which is exactly the DC circuit-theory result (voltage divider). Transmission line theory reduces to circuit theory at steady state.

Example

A $75\,\Omega$ coaxial cable of length $\ell = 0.3\,\text{m}$ (Teflon, $\varepsilon_r = 2.25$) connects a step-function generator ($V_0 = 4\,\text{V}$, $R_G = 50\,\Omega$) to a $100\,\Omega$ load.
Solution:

$$\begin{aligned}
c_d &= c/\sqrt{\varepsilon_r} = 3\times10^8/1.5 = 2\times10^8\ \text{m/s},\\
T &= \ell/c_d = 0.3/(2\times10^8) = 1.5\,\text{ns},\\
A &= Z_0/(Z_0+R_G) = 75/125 = 0.6,\\
\Gamma_L &= (100-75)/(100+75) = 1/7 \approx 0.143,\\
\Gamma_G &= (50-75)/(50+75) = -1/5 = -0.2.
\end{aligned}$$

Initial forward wave: $V_1^+ = 0.6\times4 = 2.4\,\text{V}$.
First reflection: $V_1^- = \Gamma_L\times2.4 = 0.343\,\text{V}$.
Second forward wave: $V_2^+ = \Gamma_G\times0.343 = -0.069\,\text{V}$.
Steady state: $V_{\text{load}} = 4\times100/150 = 2.667\,\text{V}$.

7.7 Reactive Loads

The analysis above assumed resistive loads, where Γ_L is a real constant. For **reactive loads** (capacitors or inductors), the reflection coefficient varies with time.

7.7.1 Capacitive Load

When a step function arrives at a capacitive load C_L, the capacitor initially behaves as a short circuit ($\Gamma_L = -1$, uncharged) and eventually as an open circuit ($\Gamma_L = +1$, fully charged). The transmission line appears to the capacitor as a resistor of value Z_0, giving an RC time constant $\tau = Z_0 C_L$. The time-varying reflection coefficient is:

$$\Gamma_L(t) = 1 - 2\,e^{-(t-T)/(Z_0 C_L)}, \qquad t \geq T, \tag{7.10}$$

where $T = \ell/c_d$ is the one-way travel time.

7.7.2 Inductive Load

For an inductive load L_L, the inductor initially acts as an open circuit ($\Gamma_L = +1$) and at steady state as a short circuit ($\Gamma_L = -1$). The RL time constant is $\tau = L_L/Z_0$:

$$\Gamma_L(t) = -1 + 2\,e^{-(t-T)Z_0/L_L}, \qquad t \geq T. \tag{7.11}$$

Physical Insight

Why does the transmission line look like a resistor Z_0 to the load? Because at the instant of reflection, only a forward-traveling wave exists—no reflected wave has returned yet. A semi-infinite line carrying only a forward wave has $V/I = Z_0$, identical to a resistor.

7.8 Time-Domain Reflectometry (TDR)

A **TDR** is a diagnostic instrument that sends a step function down a transmission line and analyzes the reflected signal to locate and characterize faults:

- The **time delay** T_D of the reflected pulse gives the distance to the fault: $d = c_d \cdot T_D/2$.
- A **positive** reflected voltage indicates $R > Z_0$ (partial or complete break in the cable).
- A **negative** reflected voltage indicates $R < Z_0$ (partial or complete short).
- An **exponential** shape in the reflection indicates a reactive fault (capacitive or inductive).

TDR is widely used to locate faults in coaxial cables buried in walls or underground without physical inspection.

Exercises

1. A $50\,\Omega$ transmission line is terminated with a $150\,\Omega$ load. Calculate Γ_L and the percentage of incident power that is reflected.
2. A step-function generator ($V_0 = 10\,\text{V}$, $R_G = Z_0 = 50\,\Omega$) drives a $50\,\Omega$ line terminated in an open circuit. Sketch $V(z = \ell, t)$ for $0 < t < 5T$.
3. Show that the steady-state voltage from the infinite bounce series equals the DC circuit theory result $V_0 R_L/(R_L + R_G)$.

4. A $75\,\Omega$ line is terminated in a short circuit. The generator has $R_G = 75\,\Omega$. After the first reflection returns to the generator, is there a second reflection? Explain.

5. For the example above, compute the voltage at the midpoint of the line ($z = \ell/2$) at time $t = 2.5T$.

Chapter 8

Bounce Diagrams

The bounce diagram is a powerful graphical tool for visualizing wave propagation on transmission lines. It systematically tracks every wavefront as it bounces between the generator and the load.

8.1 Construction of a Bounce Diagram

A bounce diagram is a space-time plot constructed as follows:

1. Draw the **horizontal axis** as position z (from $z = 0$ at the generator to $z = \ell$ at the load).
2. Draw the **vertical axis** as time t (increasing downward), with gridlines at multiples of $T = \ell/c_d$ (one-way travel time).
3. Label Γ_G at the left boundary and Γ_L at the right boundary.
4. Draw **diagonal rays** representing wavefronts. Forward waves slope down-right; backward waves slope down-left.
5. Label each ray with its **wave amplitude** (previous wave multiplied by the appropriate Γ).
6. In each region between rays, write the **running total voltage** (sum of all waves that have passed through).

Important

On the bounce diagram:

- **Rays** represent wavefronts (boundaries between regions).
- **Labels on rays** are individual wave amplitudes.
- **Labels in regions** are the cumulative voltage at that point in space and time.

8.2 Reading the Bounce Diagram

8.2.1 Oscilloscope Trace (Fixed Position)

To find the voltage as a function of time at a fixed position z_0, draw a vertical line at $z = z_0$ and read off the region labels as you move downward in time. This gives the voltage waveform that an oscilloscope at position z_0 would display.

8.2.2 Snapshot (Fixed Time)

To find the voltage as a function of position at a fixed time t_0, draw a horizontal line at $t = t_0$ and read off the region labels as you move from left to right. This gives a "photograph" of the voltage distribution along the entire line at that instant.

Example

A $50\,\Omega$ line of length ℓ connects a generator ($V_0 = 4\,\text{V}$, $R_G = 25\,\Omega$) to a $100\,\Omega$ load.
Parameters:

$$A = 50/75 = 2/3, \quad V_1^+ = (2/3)(4) = 8/3 \text{ V},$$
$$\Gamma_L = (100 - 50)/(100 + 50) = 1/3,$$
$$\Gamma_G = (25 - 50)/(25 + 50) = -1/3.$$

Successive waves:

Bounce	**Wave**	**Amplitude [V]**
1	V_1^+ (forward)	$8/3 = 2.667$
2	$V_1^- = \Gamma_L V_1^+$	$8/9 = 0.889$
3	$V_2^+ = \Gamma_G V_1^-$	$-8/27 = -0.296$
4	$V_2^- = \Gamma_L V_2^+$	$-8/81 = -0.099$
5	$V_3^+ = \Gamma_G V_2^-$	$8/243 = 0.033$

At the load after the first bounce: $V = 8/3 + 8/9 = 32/9 \approx 3.56\,\text{V}$.
Steady state: $V = 4 \times 100/125 = 3.20\,\text{V}$.

8.3 Junction of Two Transmission Lines

When two transmission lines with different characteristic impedances are connected at a junction, a wave incident from one line is partially reflected and partially transmitted.

At a junction between line 1 (Z_{01}) and line 2 (Z_{02}):

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \quad \text{(reflection coefficient)}, \tag{8.1}$$

$$T = 1 + \Gamma = \frac{2Z_{02}}{Z_{02} + Z_{01}} \quad \text{(transmission coefficient)}. \tag{8.2}$$

Transmission Coefficient

At any junction, the transmission coefficient and reflection coefficient are related by:

$$T = 1 + \Gamma. \tag{8.3}$$

Note: T can be greater than 1 (voltage step-up is possible), but the transmitted *power* never exceeds the incident power.

For a cascaded system (line 1 → line 2 → load), the bounce diagram has *two* boundaries where reflections occur: the junction and the load. Rays can bounce between any pair of boundaries.

8.4 Pulse Input Using Superposition

A rectangular pulse of duration τ and amplitude V_0 can be decomposed as:

$$V_{\text{pulse}}(t) = V_0 \left[u(t) - u(t - \tau) \right], \tag{8.4}$$

where $u(t)$ is the unit step function.

By **superposition**, the response to the pulse is the response to the first step minus the response to the delayed step. Graphically, we construct two bounce diagrams (one for each step, offset by τ) and subtract the region labels.

Physical Insight

For a pulse on a mismatched line, the pulse bounces back and forth between the generator and load, diminishing with each bounce by a factor of $|\Gamma_L \Gamma_G|$. Eventually, the line settles to zero volts (for a pulse input, unlike a step input which settles to the DC voltage divider value).

Exercises

1. Construct a bounce diagram for a $75\,\Omega$ line terminated in a short circuit, driven by a step-function generator with $V_0 = 6\,\text{V}$ and $R_G = 75\,\Omega$. Show at least 4 bounces and determine the steady-state voltage.

2. For the junction of a $50\,\Omega$ line and a $75\,\Omega$ line, calculate Γ and T. If a $2\,\text{V}$ step wave is incident from the $50\,\Omega$ line, what are the reflected and transmitted voltages?

3. A $4\,\text{V}$ pulse of width $\tau = T/2$ propagates on a $50\,\Omega$ line ($R_G = 50\,\Omega$) terminated in an open circuit. Use superposition to sketch $V(\ell, t)$ for $0 < t < 4T$.

4. From the bounce diagram, explain why the voltage at the midpoint $z = \ell/2$ changes at times $t = T/2,\ 3T/2,\ 5T/2, \ldots$ but *not* at $t = T, 2T, 3T, \ldots$

5. Two lines are cascaded: $50\,\Omega$ (length ℓ_1) followed by $100\,\Omega$ (length ℓ_2), terminated in $100\,\Omega$. Is the junction transparent? Explain.

Chapter 9

Transmission Lines: Frequency Domain

The frequency-domain (phasor) analysis of transmission lines is essential for two reasons: (1) modern communications use sinusoidal carriers, and (2) the Fourier transform allows any time-domain signal to be analyzed via frequency-domain transfer functions.

9.1 Phasor-Domain Telegrapher's Equations

In the phasor domain ($\partial/\partial t \to j\omega$), the telegrapher's equations become:

$$\frac{d\tilde{V}}{dz} = -(R + j\omega L)\,\tilde{I} = -Z\,\tilde{I}, \tag{9.1}$$

$$\frac{d\tilde{I}}{dz} = -(G + j\omega C)\,\tilde{V} = -Y\,\tilde{V}, \tag{9.2}$$

where $Z = R + j\omega L$ is the series impedance per unit length and $Y = G + j\omega C$ is the shunt admittance per unit length.

9.2 Wave Equation and General Solution

Combining (9.1) and (9.2):

$$\frac{d^2\tilde{V}}{dz^2} = \gamma^2\,\tilde{V}, \tag{9.3}$$

where the **propagation constant** is:

$$\gamma = \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)}. \tag{9.4}$$

General Solution

$$\tilde{V}(z) = A\,e^{-\gamma z} + B\,e^{+\gamma z}, \tag{9.5}$$

$$\tilde{I}(z) = \frac{1}{Z_0}\left(A\,e^{-\gamma z} - B\,e^{+\gamma z}\right), \tag{9.6}$$

where A and B are complex constants determined by boundary conditions.

The term $Ae^{-\gamma z}$ is the forward-traveling wave and $Be^{+\gamma z}$ is the backward-traveling wave.

9.3 Propagation Constant

Writing $\gamma = \alpha + j\beta$:

- α = **attenuation constant** [Np/m]: controls the exponential decay of wave amplitude.
- β = **phase constant** [rad/m]: controls the spatial oscillation (phase variation).

The principal square root is chosen so that $\alpha \geq 0$ (waves decay in the direction of propagation).

Important

Conversion between nepers and decibels:

$$\alpha\ [\text{dB/m}] = 8.6859 \times \alpha\ [\text{Np/m}]. \tag{9.7}$$

9.3.1 Lossless Line

For a lossless line ($R = 0$, $G = 0$):

$$\gamma = j\omega\sqrt{LC} = j\beta, \qquad \alpha = 0, \qquad \beta = \omega\sqrt{LC}. \tag{9.8}$$

There is no attenuation; the wave propagates without loss.

9.4 Wavelength and Phase Velocity

The **wavelength** is the spatial period of the wave:

$$\lambda = \frac{2\pi}{\beta}. \tag{9.9}$$

The **phase velocity** is the speed at which a point of constant phase moves:

$$v_p = \frac{\omega}{\beta}. \tag{9.10}$$

For a lossless line: $v_p = 1/\sqrt{LC} = c/\sqrt{\varepsilon_r}$.
The **group velocity** is the speed at which the envelope of a narrowband signal propagates:

$$v_g = \frac{d\omega}{d\beta}. \tag{9.11}$$

For a lossless, non-dispersive transmission line, $v_p = v_g = c/\sqrt{\varepsilon_r}$.

9.5 Characteristic Impedance

Characteristic Impedance (General)

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}. \tag{9.12}$$

- For a lossy line, Z_0 is *complex* (voltage and current are not in phase for a single traveling wave).
- For a lossless line: $Z_0 = \sqrt{L/C}$ (real).

Example

A coaxial cable has $R = 2\,\Omega/\text{m}$, $L = 0.5\,\mu\text{H/m}$, $G = 0.01\,\text{S/m}$, $C = 100\,\text{pF/m}$. At $f = 100\,\text{MHz}$:

$$\begin{aligned}
Z &= 2 + j2\pi(10^8)(5 \times 10^{-7}) = 2 + j314.2\ \Omega/\text{m},\\
Y &= 0.01 + j2\pi(10^8)(10^{-10}) = 0.01 + j0.0628\ \text{S/m},\\
\gamma &= \sqrt{ZY}, \quad Z_0 = \sqrt{Z/Y}.
\end{aligned}$$

Since $\omega L \gg R$ and $\omega C \gg G$, this line is approximately lossless at 100 MHz: $Z_0 \approx \sqrt{L/C} = \sqrt{5000} \approx 70.7\,\Omega$.

9.6 Loss Tangent

For practical dielectric insulators (e.g., Teflon, polyethylene), the small dielectric conductivity σ_d is often specified indirectly through the **loss tangent**:

$$\tan\delta_d = \frac{\sigma_d}{\omega\varepsilon} = \frac{G}{\omega C}, \tag{9.13}$$

where the last equality holds for *any* transmission line geometry (not just coax). For good insulators, $\tan\delta_d \sim 10^{-3}$ to 10^{-4} and is approximately constant over a wide frequency range.

Important

Do not confuse σ_d (dielectric conductivity, very small, $\sim 10^{-10}$ S/m for Teflon) with σ_m (metal conductivity, very large, $\sim 5.8 \times 10^7$ S/m for copper). These are completely different physical quantities.

The identity $G = \omega C \tan\delta_d$ is useful because C is usually known, and $\tan\delta_d$ is tabulated for common materials.

Physical Insight

The phase velocity on a lossless line is $v_p = c/\sqrt{\varepsilon_r}$. For a lossy line, the group velocity (speed of a pulse peak) may differ from the phase velocity. On a lossless line, $v_p = v_g = c_d$, but on a lossy or dispersive line, these three velocities may all be different. Intuitively, group velocity is "the speed at which the peak of a pulse envelope travels."

9.7 Summary of Key Relationships

Quantity	**General (Lossy)**	**Lossless**
Propagation constant	$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$	$\gamma = j\beta = j\omega\sqrt{LC}$
Char. impedance	$Z_0 = \sqrt{(R + j\omega L)/(G + j\omega C)}$	$Z_0 = \sqrt{L/C}$
Phase velocity	$v_p = \omega/\beta$	$v_p = 1/\sqrt{LC}$
Wavelength	$\lambda = 2\pi/\beta$	$\lambda = v_p/f$

Exercises

1. Derive (9.3) from the phasor telegrapher's equations.
2. For a lossless line with $L = 0.25\,\mu\text{H/m}$ and $C = 100\,\text{pF/m}$, compute Z_0, v_p, β, and λ at $f = 1\,\text{GHz}$.
3. Show that a forward-traveling wave on a lossy line, $\tilde{V}^+(z) = A\,e^{-\alpha z}e^{-j\beta z}$, has an amplitude that decreases by a factor of e^{-1} over a distance of $1/\alpha$.
4. A lossy cable has $\alpha = 0.1\,\text{Np/m}$. Express this in dB/m. Over what distance does the signal drop by 20 dB?
5. Explain physically why Z_0 is complex for a lossy line. What does this mean for the phase relationship between voltage and current?

Chapter 10

Reflection and Input Impedance

This chapter extends the frequency-domain analysis to transmission lines terminated with arbitrary (possibly complex) loads. We derive the fundamental formulas for the reflection coefficient and input impedance that are central to all transmission line design.

10.1 Setup and Coordinate Convention

Consider a transmission line of length ℓ terminated with a load impedance Z_L at $z = 0$. The generator is at $z = -\ell$. We define the distance from the load as $d = -z$ (always positive on the physical line).

Important

In the frequency-domain convention, $z = 0$ is placed at the **load**, not the generator. Points on the line have $z < 0$. The distance variable $d = -z > 0$ measures distance from the load toward the generator.

10.2 Load Reflection Coefficient

The general voltage and current on the line are:

$$\tilde{V}(z) = A\, e^{-\gamma z} + B\, e^{+\gamma z}, \tag{10.1}$$

$$\tilde{I}(z) = \frac{A}{Z_0}\, e^{-\gamma z} - \frac{B}{Z_0}\, e^{+\gamma z}. \tag{10.2}$$

At the load ($z = 0$), Ohm's law requires $\tilde{V}(0) = Z_L\, \tilde{I}(0)$:

$$A + B = Z_L \cdot \frac{A - B}{Z_0}. \tag{10.3}$$

Solving for B/A:

Load Reflection Coefficient (Frequency Domain)

$$\Gamma_L = \frac{B}{A} = \frac{Z_L - Z_0}{Z_L + Z_0}. \tag{10.4}$$

This is the same formula as the time domain, but now Z_L and Z_0 may be **complex**.

The voltage and current can be rewritten as:

$$\tilde{V}(z) = A\left[e^{-\gamma z} + \Gamma_\mathrm{L}\, e^{+\gamma z}\right], \tag{10.5}$$

$$\tilde{I}(z) = \frac{A}{Z_0}\left[e^{-\gamma z} - \Gamma_\mathrm{L}\, e^{+\gamma z}\right]. \tag{10.6}$$

10.3 Input Impedance

The **input impedance** at any point z on the line is:

$$Z_\mathrm{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)}. \tag{10.7}$$

Important

The input impedance depends only on what is to the **right** of the observation point (toward the load). It does not depend on the generator. The constant A cancels in the ratio V/I.

10.3.1 Reflection Coefficient Form

Substituting $z = -d$ and simplifying:

$$Z_\mathrm{in}(d) = Z_0\,\frac{1 + \Gamma_\mathrm{L}\, e^{-2\gamma d}}{1 - \Gamma_\mathrm{L}\, e^{-2\gamma d}}. \tag{10.8}$$

10.3.2 Tangent (Hyperbolic) Form

Substituting $\Gamma_\mathrm{L} = (Z_\mathrm{L} - Z_0)/(Z_\mathrm{L} + Z_0)$ and simplifying algebraically:

Input Impedance (General)

$$Z_\mathrm{in}(d) = Z_0\,\frac{Z_\mathrm{L} + Z_0 \tanh(\gamma d)}{Z_0 + Z_\mathrm{L} \tanh(\gamma d)}. \tag{10.9}$$

10.3.3 Lossless Line

For a lossless line, $\gamma = j\beta$ and $\tanh(j\beta d) = j\tan(\beta d)$:

Input Impedance (Lossless)

$$\boxed{Z_\mathrm{in}(d) = Z_0\,\frac{Z_\mathrm{L} + jZ_0 \tan(\beta d)}{Z_0 + jZ_\mathrm{L} \tan(\beta d)}.} \tag{10.10}$$

10.4 Sanity Checks

The input impedance formula can be verified with four limiting cases:

1. **Zero length** ($d \to 0$): $\tanh(0) = 0$, so $Z_{\text{in}} = Z_{\text{L}}$. ✓
2. **Matched load** ($Z_{\text{L}} = Z_0$): $\Gamma_{\text{L}} = 0$, so $Z_{\text{in}} = Z_0$ for all d. ✓
3. **Infinite lossy line** ($d \to \infty$, $\alpha > 0$): $\tanh(\gamma d) \to 1$, so $Z_{\text{in}} \to Z_0$. The reflected wave is completely attenuated. ✓
4. **Low frequency/short line**: For $\beta d \ll 1$, $\tan(\beta d) \approx \beta d$, and $Z_{\text{in}} \to Z_{\text{L}}$, recovering circuit theory. ✓

10.5 Periodicity on Lossless Lines

For a lossless transmission line:

- The input impedance repeats every $\lambda/2$.
- The voltage and current magnitudes repeat every $\lambda/2$.
- The complex voltage and current repeat every λ (and become their negatives after $\lambda/2$).

10.6 Special Load Cases

10.6.1 Short-Circuit Load ($Z_{\text{L}} = 0$)

$$Z_{\text{in}} = jZ_0 \tan(\beta\ell). \tag{10.11}$$

- Pure imaginary (purely reactive) for all ℓ.
- At low frequency ($\beta\ell \ll 1$): $Z_{\text{in}} \approx j\omega L_{\text{total}}$ (behaves as an inductor).
- At $\ell = \lambda/4$: $Z_{\text{in}} \to \infty$ (open circuit). The short is transformed to an open!

10.6.2 Open-Circuit Load ($Z_{\text{L}} \to \infty$)

$$Z_{\text{in}} = -jZ_0 \cot(\beta\ell). \tag{10.12}$$

- Pure imaginary for all ℓ.
- At low frequency: $Z_{\text{in}} \approx 1/(j\omega C_{\text{total}})$ (behaves as a capacitor).
- At $\ell = \lambda/4$: $Z_{\text{in}} \to 0$ (short circuit). The open is transformed to a short!

10.6.3 Quarter-Wave Transformer

When $\ell = \lambda/4$ (i.e., $\beta\ell = \pi/2$), $\tan(\beta\ell) \to \infty$, and:

$$Z_{\text{in}} = \frac{Z_0^2}{Z_{\text{L}}}. \tag{10.13}$$

Quarter-Wave Transformer

A quarter-wavelength section of transmission line with characteristic impedance Z_0 transforms a load Z_{L} to $Z_{\mathrm{in}} = Z_0^2/Z_{\mathrm{L}}$. This is an **impedance inverter**: high impedance → low impedance and vice versa.

Physical Insight

At microwave frequencies, lumped inductors and capacitors become impractical. Short and open-circuited transmission line stubs can synthesize any reactive impedance, enabling the construction of filters, matching networks, and resonators using only transmission lines. This is the basis of **microstrip filter design**.

10.6.4 Half-Wave Line ($\ell = \lambda/2$)

When $\ell = \lambda/2$, $\beta\ell = \pi$ and $\tan(\pi) = 0$:

$$Z_{\mathrm{in}} = Z_{\mathrm{L}}. \tag{10.14}$$

A half-wave line reproduces the load impedance at its input. This is useful for "moving" a load to a more convenient location without changing its impedance.

Exercises

1. Derive (10.9) from (10.8) by substituting the expression for Γ_{L}.
2. A $50\,\Omega$ lossless line of length $\lambda/8$ is terminated in $Z_{\mathrm{L}} = (100 + j50)\,\Omega$. Find Z_{in}.
3. A $75\,\Omega$ line is terminated in a short circuit. At what length (in terms of λ) does the input impedance first equal $j75\,\Omega$?
4. Design a quarter-wave transformer to match a $100\,\Omega$ load to a $50\,\Omega$ line. What characteristic impedance is needed for the transformer section?
5. Show that for a lossless line, $Z_{\mathrm{in}}(d + \lambda/2) = Z_{\mathrm{in}}(d)$ (the impedance repeats every half wavelength).

Chapter 11

Standing Wave Ratio

When a lossless transmission line is terminated in a mismatched load, the superposition of forward and backward traveling waves creates a **standing wave pattern**. The standing wave ratio (SWR) is a key measurable quantity that characterizes the degree of mismatch.

11.1 Voltage Standing Wave Pattern

On a lossless line, the voltage magnitude at position z is:

$$|\tilde{V}(z)| = |A| \left|1 + \Gamma_{\mathrm{L}}\, e^{+2j\beta z}\right| = |A| \left|1 + |\Gamma_{\mathrm{L}}|\, e^{j(\theta_L + 2\beta z)}\right|, \tag{11.1}$$

where $\Gamma_{\mathrm{L}} = |\Gamma_{\mathrm{L}}| e^{j\theta_L}$ in polar form. This is a periodic function of z with period $\lambda/2$.

The maximum and minimum voltages are:

$$V_{\max} = |A|(1 + |\Gamma_{\mathrm{L}}|) \quad \text{(constructive interference)}, \tag{11.2}$$

$$V_{\min} = |A|(1 - |\Gamma_{\mathrm{L}}|) \quad \text{(destructive interference)}. \tag{11.3}$$

11.2 VSWR Definition

Voltage Standing Wave Ratio

$$\mathrm{SWR} = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_{\mathrm{L}}|}{1 - |\Gamma_{\mathrm{L}}|}. \tag{11.4}$$

The SWR ranges from 1 (perfect match, $|\Gamma_{\mathrm{L}}| = 0$) to ∞ (total reflection, $|\Gamma_{\mathrm{L}}| = 1$).

The inverse relationship is:

$$|\Gamma_{\mathrm{L}}| = \frac{\mathrm{SWR} - 1}{\mathrm{SWR} + 1}. \tag{11.5}$$

11.2.1 Current Standing Wave

The current magnitude has the opposite pattern: current is maximum where voltage is minimum, and vice versa. The current SWR equals the voltage SWR.

11.2.2 SWR for Real Loads

For a purely real (resistive) load $Z_L = R_L$:

$$\text{SWR} = \frac{R_L}{Z_0} \quad \text{or} \quad \frac{Z_0}{R_L} \qquad \text{(whichever is } \geq 1\text{)}. \tag{11.6}$$

11.3 Locations of Maxima and Minima

Voltage maxima occur where $e^{j(\theta_L + 2\beta z)} = +1$, i.e.,

$$\theta_L + 2\beta z_{\max} = -2n\pi, \quad n = 0, 1, 2, \ldots \tag{11.7}$$

Voltage minima occur where $e^{j(\theta_L + 2\beta z)} = -1$, i.e.,

$$\theta_L + 2\beta z_{\min} = -(2n+1)\pi. \tag{11.8}$$

Converting to distance from load ($d = -z$):

$$d_{\max} = \frac{\theta_L}{4\pi}\lambda + n\frac{\lambda}{2}, \qquad d_{\min} = d_{\max} + \frac{\lambda}{4}. \tag{11.9}$$

11.4 Determining Unknown Loads from SWR Measurements

A slotted-line measurement provides:

1. The SWR (from $V_{\max}/V_{\min}$), giving $|\Gamma_L|$.
2. The position of a voltage minimum $z_{\min}$, giving the phase θ_L.

With $|\Gamma_L|$ and θ_L known, the load impedance can be calculated:

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L}. \tag{11.10}$$

Example

A $50\,\Omega$ line has SWR $= 3$ and the first voltage minimum is at $d_{\min} = 0.1\lambda$ from the load. Find Z_L.
Solution:

$$|\Gamma_L| = \frac{3-1}{3+1} = 0.5.$$

From $d_{\min}$: $\theta_L = 2\beta d_{\min} - \pi = 2(2\pi/\lambda)(0.1\lambda) - \pi = 0.4\pi - \pi = -0.6\pi$ rad.

$$\Gamma_L = 0.5\, e^{-j0.6\pi} = 0.5\, e^{-j108^\circ}.$$

$$Z_L = 50\,\frac{1 + 0.5e^{-j108^\circ}}{1 - 0.5e^{-j108^\circ}} = 50\,\frac{1 + (-0.155 - j0.476)}{1 - (-0.155 - j0.476)}.$$

Computing: $Z_L \approx (30.8 - j28.1)\ \Omega$.

11.5 Generalized Reflection Coefficient

We define the **generalized reflection coefficient** at any point z_0 on the line:

$$\Gamma(z_0) = \frac{B\,e^{+\gamma z_0}}{A\,e^{-\gamma z_0}} = \Gamma_\mathrm{L}\,e^{+2\gamma z_0}. \tag{11.11}$$

For a lossless line, using $d = -z_0$:

$$\Gamma(d) = \Gamma_\mathrm{L}\,e^{-2j\beta d}. \tag{11.12}$$

Key Result

The generalized reflection coefficient at distance d from the load has the same magnitude as Γ_L but a phase that rotates by $-2\beta d$ (clockwise on the complex plane as we move toward the generator).

The input impedance at z_0 can be written as:

$$Z_\mathrm{in}(z_0) = Z_0\,\frac{1+\Gamma(z_0)}{1-\Gamma(z_0)}. \tag{11.13}$$

This relationship between impedance and reflection coefficient is the foundation of the **Smith chart** (next chapter).

11.6 Crank Diagram

The **crank diagram** is a polar plot of $\Gamma(z)$ in the complex plane. As we move along the line toward the generator (d increases), the point $\Gamma(d) = |\Gamma_\mathrm{L}|e^{j(\theta_L - 2\beta d)}$ traces a circle of radius $|\Gamma_\mathrm{L}|$, rotating **clockwise**. One complete revolution corresponds to $\Delta d = \lambda/2$.

Exercises

1. A lossless $50\,\Omega$ line is terminated in $Z_\mathrm{L} = 150\,\Omega$. Find the SWR and $|\Gamma_\mathrm{L}|$.
2. On a $75\,\Omega$ line with SWR $= 2$, what are the possible real load impedances?
3. Measurements on a $50\,\Omega$ line show SWR $= 4.0$ and the first voltage maximum at $d = 0.15\lambda$. Determine Z_L.
4. Show that on a lossless line, the distance between a voltage maximum and the nearest voltage minimum is always $\lambda/4$.
5. For a $50\,\Omega$ line terminated in $Z_\mathrm{L} = (25 - j25)\ \Omega$, compute $\Gamma(d)$ at $d = \lambda/8$ and find Z_in at that point.

Chapter 12

Smith Chart

The Smith chart is a graphical tool that maps the complex reflection coefficient plane onto impedance (or admittance) circles. Invented by Phillip Smith in 1939 at Bell Telephone Laboratories, it remains one of the most widely used tools in RF and microwave engineering.

12.1 The Smith Chart as the Γ Plane

The Smith chart is a plot of the complex reflection coefficient plane, $\Gamma = x + jy$, restricted to $|\Gamma| \leq 1$ (the unit circle). The relationship between normalized impedance and Γ is:

$$z_N = \frac{Z_{\text{in}}}{Z_0} = \frac{1+\Gamma}{1-\Gamma}, \qquad \Gamma = \frac{z_N - 1}{z_N + 1}. \tag{12.1}$$

Writing $z_N = r_N + jx_N$ (normalized resistance and reactance) and $\Gamma = x + jy$, we can derive the equations for the constant-resistance and constant-reactance circles.

12.2 Constant Resistance Circles

Setting r_N constant and eliminating x_N:

$$\left(x - \frac{r_N}{r_N+1}\right)^2 + y^2 = \left(\frac{1}{r_N+1}\right)^2. \tag{12.2}$$

Resistance Circles

- Center: $\left(\frac{r_N}{r_N+1},\ 0\right)$
- Radius: $\frac{1}{r_N+1}$
- All circles pass through the point $(1, 0)$.
- The $r_N = 0$ circle is the unit circle; as $r_N \to \infty$, the circle shrinks to the point $(1, 0)$.

12.3 Constant Reactance Circles

Setting x_N constant and eliminating r_N:

$$(x-1)^2 + \left(y - \frac{1}{x_N}\right)^2 = \left(\frac{1}{x_N}\right)^2. \tag{12.3}$$

Reactance Circles

- Center: $\left(1,\ \frac{1}{x_N}\right)$
- Radius: $\frac{1}{|x_N|}$
- All arcs pass through the point $(1, 0)$.
- Inductive ($x_N > 0$) arcs are in the upper half; capacitive ($x_N < 0$) arcs are in the lower half.

12.4 Moving Along the Line

As we move along a lossless transmission line from the load toward the generator (increasing d), the generalized reflection coefficient rotates **clockwise**:

$$\Gamma(d) = |\Gamma_{\mathrm{L}}| e^{j(\theta_L - 2\beta d)}. \tag{12.4}$$

We stay on a circle of constant radius $|\Gamma_{\mathrm{L}}|$. The angular change is:

$$\Delta\theta = 2\beta\Delta d = 2\left(\frac{2\pi}{\lambda}\right)\Delta d. \tag{12.5}$$

One complete revolution ($\Delta\theta = 2\pi$) corresponds to $\Delta d = \lambda/2$. The outer scale of the Smith chart is calibrated in “wavelengths toward generator.”

12.5 Key Smith Chart Operations

12.5.1 Reading SWR

The SWR is read from the normalized resistance on the positive real axis where the constant-$|\Gamma|$ circle intersects:

$$\mathrm{SWR} = r_N\big|_{\text{positive real axis}}. \tag{12.6}$$

12.5.2 Impedance to Admittance Conversion

The normalized admittance $y_N = 1/z_N$ is obtained by going exactly **halfway around** the Smith chart ($\lambda/4$ rotation, 180°). This is the **reciprocal property**.

Physical Insight

Why does a 180° rotation give the admittance? If z_N maps to Γ, then $1/z_N$ maps to $-\Gamma$ (verify algebraically). On the Smith chart, $-\Gamma$ is the diametrically opposite point—exactly 180° away. Equivalently, this corresponds to a $\lambda/4$ section of line, which is an impedance inverter: $Z_{\text{in}} = Z_0^2/Z_{\text{L}}$, so the normalized input impedance equals $1/z_L = y_L$.
A practical consequence: a **short circuit** ($z_N = 0$, $\Gamma = -1$) maps to an **open circuit** ($y_N = 0$, $\Gamma = +1$) and vice versa. On the Smith chart, these are at opposite ends of the real axis.

12.5.3 Smith Chart as Admittance Calculator

The same Smith chart can be used for admittance calculations by interpreting the resistance circles as conductance circles (g_N) and the reactance arcs as susceptance arcs (b_N).

Example

Given $Z_{\text{L}} = (100+j50)\ \Omega$ on a $50\,\Omega$ line. Find the SWR and the normalized load admittance.
Solution: Normalized: $z_N = 2 + j1$.

$$\Gamma_{\text{L}} = \frac{(2+j1)-1}{(2+j1)+1} = \frac{1+j1}{3+j1} = \frac{(1+j)(3-j)}{|3+j|^2} = \frac{4+j2}{10} = 0.4 + j0.2.$$

$|\Gamma_{\text{L}}| = \sqrt{0.16+0.04} = \sqrt{0.2} \approx 0.447$. SWR $= (1+0.447)/(1-0.447) \approx 2.62$.
For admittance, rotate 180°: $y_N = 1/(2+j1) = (2-j1)/5 = 0.4 - j0.2$, so $Y_L = (0.4 - j0.2)/50 = (8 - j4)$ mS.

Exercises

1. Derive the constant-resistance circle equation starting from $z_N = (1+\Gamma)/(1-\Gamma)$.
2. On a $50\,\Omega$ line, $Z_{\text{L}} = (25 - j25)\ \Omega$. Plot the load on the Smith chart, find the SWR, and determine Z_{in} at $d = 0.15\lambda$.
3. Show that the $r_N = 1$ circle passes through the center of the Smith chart.
4. Use the Smith chart to find the input impedance of a short-circuited $50\,\Omega$ line of length 0.1λ.
5. Verify the reciprocal property: show analytically that if z_N maps to Γ, then $1/z_N$ maps to $-\Gamma$.

Chapter 13

Impedance Matching

Impedance matching is essential in RF and microwave systems to maximize power transfer and minimize reflections. This chapter covers two fundamental matching techniques: the quarter-wave transformer and single-stub matching.

13.1 Why Match?

When a transmission line is mismatched ($Z_\mathrm{L} \neq Z_0$):

- Reflected power is wasted: $P_\mathrm{reflected}/P_\mathrm{incident} = |\Gamma_\mathrm{L}|^2$.
- Standing waves increase peak voltages, risking dielectric breakdown.
- The reflected signal can cause interference at the source.

The goal of impedance matching is to make the input impedance of the matching network equal to Z_0, so that $\Gamma = 0$ on the feed line.

13.2 Quarter-Wave Transformer

13.2.1 Real Load

For a real load R_L on a lossless line, a quarter-wave section ($\ell = \lambda/4$) with characteristic impedance Z_T transforms the load to:

$$Z_\mathrm{in} = \frac{Z_T^2}{R_L}. \tag{13.1}$$

Setting $Z_\mathrm{in} = Z_0$ for a match:

Quarter-Wave Transformer

$$Z_T = \sqrt{Z_0 \cdot R_L}. \tag{13.2}$$

Example

Match a $100\,\Omega$ load to a $50\,\Omega$ line at $f = 2\,\mathrm{GHz}$.
$Z_T = \sqrt{50 \times 100} = \sqrt{5000} \approx 70.7\,\Omega$. The transformer length is $\lambda/4 = c/(4f) = 3.75\,\mathrm{cm}$ in

free space (adjust for dielectric).

13.2.2 Complex Load

A quarter-wave transformer requires a real load. For a complex Z_L, we must first make it real using one of two methods:

Method (a): Shunt susceptance. Add a shunt reactive element (stub) at the load to cancel the reactive part, leaving a real impedance. Then apply the quarter-wave transformer.

Method (b): Extension line. Add a length of transmission line before the load to rotate to a point on the Smith chart where the impedance is real. Then apply the quarter-wave transformer.

13.3 Single-Stub Matching

Single-stub matching uses a short or open-circuited stub connected in parallel (shunt) at a distance d from the load.

13.3.1 Design Procedure

1. Convert Z_L to normalized admittance $y_L = g_L + jb_L$.

2. On the Smith chart (admittance mode), rotate clockwise from y_L until the $g_N = 1$ circle is reached. The rotation distance is d.

3. At this point, $y_\mathrm{in} = 1 + jb_\mathrm{in}$. The stub must provide $b_s = -b_\mathrm{in}$ to cancel the susceptance.

4. Determine the stub length ℓ_s from the required b_s using the short-circuit ($b = -\cot\beta\ell_s/Z_0$) or open-circuit stub formula.

Single-Stub Matching Condition

At the stub location, the total normalized admittance must be:

$$y_\mathrm{total} = y_\mathrm{in}(d) + y_\mathrm{stub} = 1 + j0 \quad \text{(perfect match)}. \tag{13.3}$$

Example

Match $Z_\mathrm{L} = (100 + j80)\,\Omega$ to a $50\,\Omega$ line using a single shorted stub.
Step 1: $z_L = 2 + j1.6$, $y_L = 1/z_L = (2 - j1.6)/6.56 = 0.305 - j0.244$.
Step 2: Rotate on the Smith chart until $g_N = 1$ circle is reached. From the chart, $d \approx 0.219\lambda$ and $y_\mathrm{in} = 1 + j1.33$.
Step 3: Need $b_s = -1.33$ (normalized). For a shorted stub with $Z_{0s} = Z_0$: $b_s = -\cot(\beta\ell_s)$, so $\cot(\beta\ell_s) = 1.33$, giving $\ell_s \approx 0.101\lambda$.

13.4 Comparison of Methods

Feature	**QW Transformer**	**Single Stub**
Series or shunt	Series section	Shunt stub
Bandwidth	Moderate	Narrower
Requires special Z_0	Yes (Z_T)	No (uses same Z_0)
Complex load directly	No (needs preprocessing)	Yes

Important

In single-stub matching, the $g_N = 1$ circle is generally intersected at **two** points as you rotate from the load, giving two possible stub locations d_1 and d_2. Both solutions are valid. In practice, the shorter stub distance is usually preferred to minimize losses and sensitivity to frequency variation.

Bandwidth Limitations

All matching networks are designed at a single frequency. Away from the design frequency, the match degrades:

- The quarter-wave transformer is no longer exactly $\lambda/4$, so the impedance transformation is imperfect.
- The stub is no longer the correct length, so it does not provide the required susceptance.

The bandwidth of a match is typically defined as the frequency range over which $|\Gamma| <$ some threshold (e.g., SWR < 2). Quarter-wave transformers generally have wider bandwidth than single-stub matches.

Exercises

1. Design a quarter-wave transformer to match a $200\,\Omega$ antenna to a $50\,\Omega$ feed line at 1 GHz. Specify Z_T and the physical length (assuming air dielectric).
2. A load $Z_\mathrm{L} = (75 + j50)\ \Omega$ is on a $50\,\Omega$ line. Use the extension-line method to find the line length needed to make the impedance real, then design the quarter-wave transformer.
3. For the single-stub matching example above, verify the result by computing Z_in analytically.
4. Why does a quarter-wave transformer have limited bandwidth? Sketch $|\Gamma|$ versus frequency for a QW transformer designed at f_0.
5. Design a single open-circuited stub match for $Z_\mathrm{L} = (30 - j40)\ \Omega$ on a $50\,\Omega$ line.

Chapter 14

Discontinuity Effects

Transmission line theory assumes an infinite, uniform two-conductor system. In practice, every real system has discontinuities—bends, junctions, and obstacles—that cause effects not predicted by ideal TL theory: reflections and radiation.

14.1 Two Effects of Discontinuities

At any discontinuity on a transmission line:

1. **Reflections**: Part of the incident wave is reflected back, even if the characteristic impedance is the same on both sides.

2. **Radiation**: Part of the energy escapes as radiation into the surrounding space (for open transmission lines).

Both effects become more pronounced as frequency increases (i.e., as the discontinuity becomes larger relative to a wavelength).

14.2 Coaxial Cable

The coaxial cable is a *perfectly shielded* system: no electromagnetic energy escapes, regardless of frequency. Therefore, discontinuities on a coax cause **reflections only**—never radiation.

Physical Insight

Even though coax does not radiate, bends and connectors still cause reflections due to the local change in cross-sectional geometry. The rule of thumb: keep the bend radius large compared to the cable diameter to minimize reflections.

14.3 Twin Lead

Twin lead is an *open* system: the fields extend to infinity. However, an infinite straight twin lead does **not radiate**, because the transmission line wave is an exact solution to Maxwell's equations on an infinite straight structure.

Discontinuities on twin lead cause **both reflections and radiation**:

- Bends
- Nearby obstacles (pipes, walls)
- Changes in wire spacing or wire diameter

14.3.1 Reducing Discontinuity Effects

- **Reduce wire separation**: Keep $h \ll \lambda_0$.
- **Twist the wires**: Twisted pair (e.g., CAT5 cable) causes far-field cancellation, dramatically reducing both radiation and susceptibility to interference.

14.4 Microstrip

Microstrip lines are partially open structures. At discontinuities (bends, step changes in width, T-junctions), three effects occur:

1. Reflections (modeled as parasitic capacitances or inductances)
2. Radiation into free space
3. Excitation of surface waves in the dielectric substrate

None of these effects are predicted by transmission line theory alone; full-wave electromagnetic analysis is needed for accurate modeling at high frequencies.

Important

Good microstrip design practice includes:

- Using mitered (chamfered) bends instead of right-angle bends.
- Keeping substrate thickness small relative to wavelength.
- Accounting for fringing fields at open-circuit ends (effective length extension).

Exercises

1. Explain why a straight, infinite coaxial cable never radiates, while a straight, infinite twin lead also never radiates, even though twin lead is an open structure.
2. A twin-lead transmission line has wire spacing $h = 1\,\text{cm}$. At what frequency does h become $\lambda/10$ (a rough threshold for significant radiation effects)?
3. Why does twisting the wires in a twisted pair reduce radiation? Explain in terms of far-field cancellation.
4. Compare the advantages and disadvantages of coaxial cable vs. microstrip for a 10 GHz system.

Part IV

Plane Waves

Chapter 15

Plane Waves

A plane wave is the simplest solution to Maxwell's equations for a wave propagating through unbounded space. It serves as a fundamental building block: any electromagnetic wave can be decomposed into a superposition of plane waves.

15.1 The Electromagnetic Spectrum

Electromagnetic waves span an enormous frequency range, from extremely low frequency (ELF) radio waves ($f \sim 1\,\text{Hz}$) to gamma rays ($f > 10^{19}$ Hz). Common bands include AM/FM radio, VHF/UHF television, microwave, infrared, visible light, ultraviolet, and X-rays. All are governed by the same Maxwell's equations and travel at the speed of light in vacuum.

15.2 Wireless vs. Wired Systems

For short distances, wired (transmission line) systems are efficient. However, transmission line attenuation increases with distance (and frequency), while wireless systems using antennas have a path loss that grows as $1/R^2$. Beyond a certain crossover distance, **wireless systems have lower loss than wired systems**.

15.3 Vector Wave Equation

Starting from Maxwell's equations in a source-free region of free space:

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu_0\tilde{\mathbf{H}}, \tag{15.1}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\varepsilon_0\tilde{\mathbf{E}}. \tag{15.2}$$

Taking the curl of Faraday's law and substituting Ampère's law:

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = -j\omega\mu_0(j\omega\varepsilon_0\tilde{\mathbf{E}}) = \omega^2\mu_0\varepsilon_0\tilde{\mathbf{E}}. \tag{15.3}$$

Using the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ and $\nabla \cdot \tilde{\mathbf{E}} = 0$ (source-free):

Vector Helmholtz Equation

$$\nabla^2 \tilde{\mathbf{E}} + k_0^2 \tilde{\mathbf{E}} = 0, \tag{15.4}$$

where $k_0 = \omega\sqrt{\mu_0 \varepsilon_0} = \omega / c$ is the **wavenumber of free space** [rad/m].

In rectangular coordinates, this separates into three scalar Helmholtz equations, one for each component.

15.4 Plane Wave Solution

Assuming $\tilde{\mathbf{E}} = \tilde{E}_x(z)\,\hat{\mathbf{x}}$ (polarized in x, varying only in z):

$$\frac{d^2 \tilde{E}_x}{dz^2} + k_0^2 \tilde{E}_x = 0. \tag{15.5}$$

The solutions are:

$$\tilde{E}_x(z) = E_0^+\, e^{-jk_0 z} + E_0^-\, e^{+jk_0 z}. \tag{15.6}$$

The first term is a wave traveling in the $+z$ direction; the second in $-z$.

15.4.1 Extension to Dielectric Media

For a lossless dielectric with ε_r and $\mu_r = 1$:

$$k = \omega\sqrt{\mu_0 \varepsilon_0 \varepsilon_r} = k_0\sqrt{\varepsilon_r}. \tag{15.7}$$

Physical Insight

The plane wave in a lossless dielectric has the *same mathematical form* as a voltage wave on a lossless transmission line filled with the same material. The wavenumber k of the plane wave equals β of the TL wave. This analogy between plane waves and transmission lines is extremely powerful and will be exploited repeatedly.

15.5 Magnetic Field and Intrinsic Impedance

From Faraday's law, the magnetic field of a $+z$-traveling plane wave is:

$$\tilde{\mathbf{H}} = \frac{E_0}{\eta}\, e^{-jkz}\, \hat{\mathbf{y}}, \tag{15.8}$$

where the **intrinsic impedance** of the medium is:

Intrinsic Impedance

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_0}{\sqrt{\varepsilon_r}}, \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi \approx 377\,\Omega. \tag{15.9}$$

Note the analogy: η for plane waves corresponds to Z_0 for transmission lines.

15.6 Phase Velocity and Wavelength

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}}, \tag{15.10}$$

$$\lambda = \frac{2\pi}{k} = \frac{v_p}{f} = \frac{\lambda_0}{\sqrt{\varepsilon_r}}. \tag{15.11}$$

All plane waves in a lossless medium travel at the same speed regardless of frequency—there is no dispersion, and hence no signal distortion.

15.7 Poynting Vector

For a plane wave in a lossless medium:

$$\tilde{\mathbf{S}} = \frac{1}{2}\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* = \frac{|E_0|^2}{2\eta}\,\hat{\mathbf{z}}. \tag{15.12}$$

The power flows in the direction of propagation, and $\mathrm{Im}(\tilde{\mathbf{S}}) = 0$ (no reactive power in a plane wave in a lossless medium).

For a general plane wave with both forward and backward components (e.g., standing wave), $\tilde{E}_x = E_0^+ e^{-jkz} + E_0^- e^{+jkz}$ and $\tilde{H}_y = (E_0^+ e^{-jkz} - E_0^- e^{+jkz})/\eta$:

$$\tilde{S}_z = \frac{1}{2\eta}\left(|E_0^+|^2 - |E_0^-|^2\right) + \frac{j}{\eta}\,\mathrm{Im}\Big(E_0^+ {E_0^-}^* e^{-2jkz}\Big)\,. \tag{15.13}$$

The real part is the net power flow (forward minus backward), while the imaginary part represents reactive power oscillating between electric and magnetic stored energy in the standing wave.

15.8 Lossy Medium

In a medium with conductivity σ, we define the **complex permittivity**:

$$\varepsilon_c = \varepsilon' - j\varepsilon'' - j\frac{\sigma}{\omega} = \varepsilon'\left(1 - j\tan\delta\right), \tag{15.14}$$

where the **loss tangent** is $\tan\delta = (\sigma + \omega\varepsilon'')/(\omega\varepsilon')$.

The wavenumber becomes complex:

$$k_c = \omega\sqrt{\mu\varepsilon_c} = \beta - j\alpha, \tag{15.15}$$

and the wave decays as $e^{-\alpha z}$ while propagating.

15.8.1 Depth of Penetration

The **depth of penetration** (or skin depth for good conductors) is:

$$d_p = \frac{1}{\alpha}. \tag{15.16}$$

At this distance, the field amplitude drops to $1/e \approx 37\%$ of its surface value.

Ocean Water

Ocean water has $\varepsilon_r \approx 81$ and $\sigma \approx 4\,\text{S/m}$.

Frequency	d_p
1 Hz	252 m
1 kHz	8.0 m
1 MHz	0.26 m
1 GHz	0.013 m

This explains why submarines use very low frequencies (ELF/VLF) for communication, and why microwave ovens heat only the surface of water-containing food.

15.8.2 Low-Loss Limit

When $\tan\delta \ll 1$, approximate formulas simplify the analysis:

$$\alpha \approx \frac{k\tan\delta}{2}, \qquad \beta \approx k = \omega\sqrt{\mu\varepsilon'}. \tag{15.17}$$

Exercises

1. Derive the vector Helmholtz equation from Maxwell's equations in a source-free region.
2. A plane wave at 1 GHz propagates in a lossless dielectric with $\varepsilon_r = 9$. Find k, λ, v_p, and η.
3. Compute the depth of penetration in copper ($\sigma = 5.8 \times 10^7\,\text{S/m}$) at 1 GHz.
4. Show that the time-average Poynting vector of a plane wave in a lossless medium has no imaginary (reactive) part.
5. For ocean water at 100 kHz, compute $\tan\delta$, α, β, d_p, and η.

Chapter 16

Plane Waves in Good Conductors

In a good conductor ($\sigma \gg \omega\varepsilon$), electromagnetic waves are rapidly attenuated. The depth of penetration is called the **skin depth**, and the **surface impedance** concept provides a convenient way to compute losses.

16.1 Good Conductor Approximation

A material is a "good conductor" when $\sigma \gg \omega\varepsilon$, i.e., conduction current dominates displacement current. For copper at 1 GHz: $\sigma = 5.8 \times 10^7$ S/m vs. $\omega\varepsilon_0 \approx 0.056$ S/m, so $\sigma/(\omega\varepsilon_0) \approx 10^9$.

Under this approximation:

$$k_c \approx \omega\sqrt{\mu \cdot \frac{\sigma}{j\omega}} = \sqrt{j\omega\mu\sigma} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}. \tag{16.1}$$

16.2 Skin Depth

Skin Depth

$$\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}} \quad [\mathrm{m}]. \tag{16.2}$$

The fields decay as e^{-z/δ_s} and oscillate as e^{-jz/δ_s}. The attenuation and phase constants are equal: $\alpha = \beta = 1/\delta_s$.

Skin Depth of Copper

At various frequencies:

Frequency	δ_s (copper)
60 Hz	8.5 mm
1 MHz	66 µm
1 GHz	2.1 µm
10 GHz	0.66 µm

At microwave frequencies, essentially all current flows within a few micrometers of the surface.

16.3 Surface Impedance

The current in a good conductor is concentrated near the surface. We model the 3D volume current as an equivalent 2D surface current $\mathbf{J}_s$ [A/m].

The **surface impedance** is defined as:

$$Z_s = \frac{\tilde{E}_{\tan}}{\tilde{J}_s} = \frac{1+j}{\sigma\delta_s} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}. \tag{16.3}$$

Separating into real and imaginary parts:

$$Z_s = R_s + jX_s, \qquad R_s = X_s = \frac{1}{\sigma\delta_s}. \tag{16.4}$$

Surface Resistance

$$R_s = \frac{1}{\sigma\delta_s} = \sqrt{\frac{\omega\mu}{2\sigma}} \quad [\Omega/\text{sq}]. \tag{16.5}$$

The surface resistance equals the surface reactance. Both increase as $\sqrt{f}$.

16.4 Impedance of a Wire

For a cylindrical wire of radius a and length ℓ at high frequency, current flows on the outer surface within a skin depth:

$$Z_{\text{wire}} = \frac{Z_s \cdot \ell}{2\pi a} = \frac{(1+j)\ell}{2\pi a\sigma\delta_s}. \tag{16.6}$$

The resistance at high frequency is:

$$R_{\text{HF}} = \frac{R_s\ell}{2\pi a} = \frac{\ell}{2\pi a\sigma\delta_s}. \tag{16.7}$$

Compare with the DC resistance: $R_{\text{DC}} = \ell/(\sigma\pi a^2)$. At high frequency, $R_{\text{HF}}/R_{\text{DC}} = a/(2\delta_s)$, which can be very large.

Internal Inductance

The reactive part of the wire impedance, $X_s = R_s$, corresponds to an **internal inductance** associated with the magnetic field energy stored *inside* the conductor. The internal inductance per unit length is:

$$L_{\text{int}} = \frac{X_s}{2\pi a\omega} = \frac{R_s}{2\pi a\omega} = \frac{1}{2\pi a\sigma\delta_s\omega}. \tag{16.8}$$

At low frequency (where current fills the entire cross-section), $L_{\text{int}} = \mu_0/(8\pi)$ per unit length—independent of wire radius. At high frequency, L_{int} decreases as $1/\sqrt{f}$ because the current is squeezed into a thinner skin layer, storing less internal magnetic energy.

16.5 Application to Coaxial Cable

The resistance per unit length of a coaxial cable, accounting for skin effect on both conductors:

$$R = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \quad [\Omega/\text{m}]. \tag{16.9}$$

Since $R_s \propto \sqrt{f}$, the attenuation of a coax increases as $\sqrt{f}$.

Physical Insight

Fiber-optic cables have dramatically lower attenuation than coaxial cable:

- Single-mode fiber: $\sim 0.3\,\text{dB/km}$
- Multimode fiber: $\sim 3\,\text{dB/km}$
- RG59 coax at 1 GHz: $\sim 700\,\text{dB/km}$

This enormous advantage drives the use of fiber optics for long-distance communications.

Exercises

1. Calculate the skin depth in aluminum ($\sigma = 3.5 \times 10^7\,\text{S/m}$) at 60 Hz and 1 GHz.
2. A copper wire has radius $a = 1\,\text{mm}$. At what frequency does the high-frequency resistance become 10 times the DC resistance?
3. For RG59 coax ($a = 0.32\,\text{mm}$, $b = 2.2\,\text{mm}$, copper conductors), compute R per meter at 1 GHz.
4. Show that $R_s = X_s$ for a good conductor. What is the physical meaning of the inductive (reactive) part of the surface impedance?
5. Why does the outer conductor of a coax contribute less to R than the inner conductor?

Chapter 17

Polarization of Plane Waves

The **polarization** of a plane wave refers to the behavior of the electric field vector in the time domain as observed at a fixed point. Depending on the amplitudes and phase relationship of the field components, three types of polarization arise.

17.1 General Plane Wave with Two Components

Consider a plane wave traveling in $+z$ with both x and y components:

$$\tilde{\mathbf{E}} = \left(a\,\hat{\mathbf{x}} + b\,e^{j\delta}\,\hat{\mathbf{y}}\right) E_0\,e^{-jkz}, \tag{17.1}$$

where a and b are real amplitudes and δ is the relative phase between the components. In the time domain at $z = 0$:

$$E_x(t) = aE_0 \cos(\omega t), \tag{17.2}$$
$$E_y(t) = bE_0 \cos(\omega t + \delta). \tag{17.3}$$

17.2 Linear Polarization

When $\delta = 0$ or $\delta = \pi$:

$$\mathbf{E}(t) = E_0(a\,\hat{\mathbf{x}} \pm b\,\hat{\mathbf{y}}) \cos(\omega t). \tag{17.4}$$

The electric field oscillates along a fixed line at angle $\psi = \tan^{-1}(\pm b/a)$ from the x-axis. This is a "tilted" linearly polarized wave.

17.3 Circular Polarization

When $a = b$ and $\delta = \pm\pi/2$:

$$E_x(t) = aE_0 \cos(\omega t), \tag{17.5}$$
$$E_y(t) = \mp aE_0 \sin(\omega t). \tag{17.6}$$

The tip of $\mathbf{E}$ traces a **circle** in the xy plane, rotating at angular frequency ω.

IEEE Convention for Circular Polarization

Using the right-hand rule with the thumb pointing in the direction of propagation ($+z$):

- **RHCP** (Right-Hand Circular): $\delta = -\pi/2$ — fingers curl in the direction of rotation.
- **LHCP** (Left-Hand Circular): $\delta = +\pi/2$ — rotation is opposite to RHCP.

Important

The rotation in *space* (snapshot at fixed time) and the rotation in *time* (at a fixed point) are in **opposite** senses, due to the minus sign in $e^{j(\omega t-kz)}$.
The IEEE convention used here defines handedness by the right-hand rule with the *thumb pointing in the direction of propagation*. Some optics textbooks use the opposite convention (thumb pointing toward the observer), so RHCP in IEEE corresponds to LHCP in the optics convention. Always check which convention is being used.

17.3.1 Applications of Circular Polarization

Circular polarization is widely used in:

- **Satellite communications**: Avoids signal loss due to Faraday rotation in the ionosphere.
- **GPS**: All GPS satellites transmit RHCP signals.
- **WLAN**: Reduces multipath fading caused by reflections off buildings.

A linearly polarized antenna receiving a CP wave loses 3 dB (half the power), but receives a signal regardless of its rotational orientation.

17.3.2 Generating Circular Polarization

Method 1: Use two identical orthogonal antennas fed 90° out of phase.
Method 2: Use an inherently CP antenna, such as a helical antenna in axial mode.

17.4 Elliptical Polarization

For general a, b, and δ, the tip of $\mathbf{E}$ traces an **ellipse**. Linear and circular polarizations are special cases of elliptical polarization.

The proof that the locus is an ellipse follows from eliminating ωt from the parametric equations for $E_x(t)$ and $E_y(t)$, yielding a quadratic form $AE_x^2 + BE_xE_y + CE_y^2 = 1$ with discriminant $B^2 - 4AC < 0$.

17.4.1 Rotation Rule

Rotation Rule

In time, the electric field vector rotates from the **leading** component axis to the **lagging** component axis.

If E_x leads E_y (i.e., E_y lags), the field rotates from $\hat{\mathbf{x}}$ toward $\hat{\mathbf{y}}$, and the handedness follows from the right-hand rule with the propagation direction.

17.5 Axial Ratio and Tilt Angle

The polarization ellipse is characterized by:

- **Axial Ratio** (AR): the ratio of the major axis to the minor axis of the ellipse (AR ≥ 1; AR $= 1$ for circular, AR $= \infty$ for linear).
- **Tilt angle** (τ): the angle of the major axis measured from the x-axis.

For $\tilde{\mathbf{E}} = a\,\hat{\mathbf{x}} + be^{j\delta}\,\hat{\mathbf{y}}$ propagating in $+z$:

$$\tan(2\tau) = \frac{2ab\cos\delta}{a^2 - b^2}. \tag{17.7}$$

The axial ratio is computed from the auxiliary angle:

$$\sin(2\chi) = \frac{2ab\sin\delta}{a^2 + b^2}, \qquad \text{AR} = \left|\frac{1}{\tan\chi}\right| \text{ or } |\tan\chi|^{-1}. \tag{17.8}$$

Example

A plane wave has $\tilde{\mathbf{E}} = (3\,\hat{\mathbf{x}} + 2e^{j60^\circ}\,\hat{\mathbf{y}})\,e^{-jkz}$. Find the polarization type, handedness, tilt angle, and axial ratio.
Solution: $a = 3$, $b = 2$, $\delta = 60^\circ$.

$$\tan(2\tau) = \frac{2(3)(2)\cos 60^\circ}{9-4} = \frac{6}{5} = 1.2, \quad \tau = \frac{1}{2}\tan^{-1}(1.2) \approx 25.1^\circ.$$

$$\sin(2\chi) = \frac{2(3)(2)\sin 60^\circ}{9+4} = \frac{12 \times 0.866}{13} \approx 0.800, \quad 2\chi \approx 53.1^\circ, \quad \chi \approx 26.6^\circ.$$

AR $= 1/|\tan(26.6^\circ)| = 1/0.500 = 2.0$. Since $\delta > 0$, E_y leads E_x, and the rotation is from $\hat{\mathbf{y}}$ to $\hat{\mathbf{x}}$, giving **LHEP** (left-hand elliptical polarization).

Exercises

1. Determine the polarization (type and handedness) of $\tilde{\mathbf{E}} = E_0(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-jkz}$.
2. A RHCP wave is incident on a linearly polarized receive antenna aligned with $\hat{\mathbf{x}}$. What fraction of the incident power is received?
3. Prove that the tip of $\mathbf{E}(t)$ traces an ellipse for a general plane wave with two orthogonal components.
4. Find the tilt angle and axial ratio for $\tilde{\mathbf{E}} = (1+j)\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}}$ (propagating in $+z$).
5. Two crossed dipoles are fed with equal amplitude and 90° phase difference. What polarization do they radiate in the $+z$ direction? What about the $-z$ direction?

Chapter 18

Reflection and Transmission of Plane Waves

When a plane wave encounters an interface between two different media, part of the wave is reflected and part is transmitted. The reflection and transmission coefficients depend on the angle of incidence and the polarization. This chapter derives the Fresnel equations and explores key phenomena including Snell's law, total internal reflection, and the Brewster angle.

18.1 General Plane Wave and Wavevector

A general plane wave in a medium with wavenumber k has the spatial variation $e^{-j\mathbf{k}\cdot\mathbf{r}}$, where the **wavevector k** points in the direction of propagation and satisfies:

$$|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2\mu\varepsilon. \tag{18.1}$$

18.2 Boundary Conditions

At an interface between two media (with no surface charges or currents):

- Tangential **E** is continuous: $E_{t1} = E_{t2}$.
- Tangential **H** is continuous: $H_{t1} = H_{t2}$.

18.3 Phase Matching and Snell's Law

The fields must match *everywhere* along the interface ($z = 0$). This requires:

$$k_{x,\text{inc}} = k_{x,\text{refl}} = k_{x,\text{trans}}. \tag{18.2}$$

This **phase matching condition** leads to two fundamental results:

Law of Reflection

$$\theta_r = \theta_i. \tag{18.3}$$

The angle of reflection equals the angle of incidence.

Snell's Law

$$n_1 \sin\theta_i = n_2 \sin\theta_t, \tag{18.4}$$

where $n = \sqrt{\varepsilon_r \mu_r}$ is the **index of refraction**. The wave bends toward the normal when entering a denser medium ($n_2 > n_1$).

18.4 Critical Angle and Total Internal Reflection

When a wave travels from a denser medium to a less dense medium ($n_1 > n_2$), there exists a **critical angle**:

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right). \tag{18.5}$$

For $\theta_i > \theta_c$, no power is transmitted—all incident power is reflected. This is **total internal reflection**.

Fish-Eye Effect

For water ($n \approx 1.33$) to air ($n = 1$): $\theta_c = \sin^{-1}(1/1.33) \approx 48.8°$. A fish looking up can see the entire world above the water surface compressed into a cone of half-angle 48.8°!

Beyond the critical angle, the field in the less-dense medium decays exponentially (evanescent wave) with no average power flow across the interface.

Transmission Line Analogy

Total internal reflection has a direct analogy with transmission lines. In the TL analogy (Section 18.7), each medium corresponds to a transmission line with equivalent impedance Z_{TE} or Z_{TM}. Beyond the critical angle, $\cos\theta_t$ becomes *purely imaginary* (since $\sin\theta_t > 1$), making the equivalent impedance in medium 2 purely reactive (like a capacitor or inductor). A purely reactive load on a transmission line gives $|\Gamma| = 1$—total reflection with zero power transfer. This is exactly what happens at total internal reflection.

18.5 TE$_z$ (Perpendicular) Polarization

For a TE$_z$ wave, **E** is perpendicular to the plane of incidence (the xz plane). The reflection and transmission coefficients are:

TE Fresnel Equations

$$\Gamma_{\text{TE}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}, \tag{18.6}$$

$$T_{\text{TE}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = 1 + \Gamma_{\text{TE}}. \tag{18.7}$$

18.6 TM$_z$ (Parallel) Polarization

For a TM$_z$ wave, **E** lies in the plane of incidence. The coefficients are:

TM Fresnel Equations

$$\Gamma_{\text{TM}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}, \tag{18.8}$$

$$T_{\text{TM}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}. \tag{18.9}$$

18.7 Transmission Line Analogy

Both TE and TM reflection problems can be modeled as transmission line problems with equivalent characteristic impedances:

$$Z_{\text{TE}} = \frac{\eta}{\cos\theta}, \tag{18.10}$$

$$Z_{\text{TM}} = \eta \cos\theta. \tag{18.11}$$

The reflection coefficient is then $\Gamma = (Z_2 - Z_1)/(Z_2 + Z_1)$, exactly as for a transmission line junction.

18.8 Power Reflection and Transmission

The fraction of incident power reflected and transmitted:

$$\frac{P_r}{P_i} = |\Gamma|^2, \qquad \frac{P_t}{P_i} = 1 - |\Gamma|^2. \tag{18.12}$$

Beyond the critical angle, $|\Gamma| = 1$ and all power is reflected.

18.9 Brewster Angle

The **Brewster angle** is the angle of incidence at which $\Gamma_{\text{TM}} = 0$ (no reflection for TM polarization). For non-magnetic media ($\mu_1 = \mu_2 = \mu_0$):

Brewster Angle

$$\tan\theta_B = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r1}}}. \tag{18.13}$$

At the Brewster angle, reflected light is purely TE-polarized.

Polaroid Sunglasses

Light reflected from a horizontal surface (road, water) near the Brewster angle is predominantly TE-polarized (horizontal **E**). Polaroid sunglasses block this horizontal polarization, reducing glare while passing the vertically-polarized component.

Example

Light travels from air ($n_1 = 1$) into glass ($n_2 = 1.5$) at $\theta_i = 45°$.
Snell's law: $\sin\theta_t = \sin 45°/1.5 = 0.471$, so $\theta_t = 28.1°$.
Brewster angle: $\theta_B = \tan^{-1}(1.5) = 56.3°$.
TM reflection: $\eta_1 = \eta_0$, $\eta_2 = \eta_0/1.5$.

$$\Gamma_{\text{TM}} = \frac{(\eta_0/1.5)\cos 28.1° - \eta_0 \cos 45°}{(\eta_0/1.5)\cos 28.1° + \eta_0 \cos 45°} = \frac{0.588 - 0.707}{0.588 + 0.707} = -0.092.$$

Power reflected: $|\Gamma_{\text{TM}}|^2 = 0.85\%$.

Exercises

1. Derive Snell's law from the phase-matching condition at an interface.
2. Light goes from glass ($n = 1.5$) to air. Find the critical angle.
3. Compute Γ_{TE} and Γ_{TM} for a wave going from air to water ($n = 1.33$) at $\theta_i = 30°$.
4. Show that at the Brewster angle, $\theta_i + \theta_t = 90°$.
5. A plane wave is incident on a lossy half-space ($\varepsilon_r = 81$, $\sigma = 4\,\text{S/m}$) at $\theta_i = 30°$ and $f = 1\,\text{GHz}$. Compute Γ_{TM} using the wavevector approach (avoid complex angles).

Part V

Waveguides and Antennas

Chapter 19

Waveguiding Structures

A waveguiding structure carries electromagnetic energy from one point to another. This chapter compares three common types: transmission lines, fiber-optic guides, and rectangular waveguides.

19.1 Overview of Waveguide Types

Property	**TL**	**Fiber Optic**	**Rect. WG**
Conductors	2	0	1 (hollow)
Cutoff frequency	None	None*	Yes
Mode type	TEM_z	Hybrid	TE_z/TM_z
Loss	Moderate	Very low	Low
Power handling	Moderate	Low	High
Interference immunity	Varies	Excellent	Excellent

*For practical purposes; single-mode fiber has an effective cutoff.

19.2 Transmission Lines

Transmission lines support TEM_z (transverse electromagnetic) waves: neither E_z nor H_z is present. Key properties:

- Propagation at any frequency (no cutoff).
- $k_z = k = \omega\sqrt{\mu\varepsilon}$ (same as a plane wave in the filling medium).
- Phase velocity equals the speed of light in the medium: $v_p = 1/\sqrt{\mu\varepsilon}$.
- Characteristic impedance $Z_0 = \sqrt{L/C}$ depends only on geometry and material.

19.3 Fiber-Optic Guides

19.3.1 Multi-Mode Fiber

A multi-mode fiber has a core diameter large compared to the wavelength. Light propagates by **total internal reflection** at the core-cladding interface. The **numerical aperture** (NA) determines the maximum acceptance angle:

$$\mathrm{NA} = \sin\theta_{\max} = \sqrt{n_{\text{core}}^2 - n_{\text{clad}}^2}. \tag{19.1}$$

19.3.2 Single-Mode Fiber

A single-mode fiber has a core diameter comparable to the wavelength ($\sim 9\,\mu\text{m}$). Only the fundamental mode propagates. Single-mode fiber has extremely low attenuation ($\sim 0.3\,\text{dB/km}$) and minimal signal distortion, making it the standard for long-distance telecommunications.

19.4 Rectangular Waveguide

A rectangular waveguide is a hollow metallic pipe of width a and height b ($a > b$). Unlike transmission lines, it has a **cutoff frequency** below which waves cannot propagate.

19.4.1 Waveguide Modes

The modes are classified as TE_{mn} (no E_z) or TM_{mn} (no H_z), where m and n count the half-wave variations in the x and y directions.

The **cutoff frequency** of the TE_{mn} or TM_{mn} mode is:

$$f_{c,mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}. \tag{19.2}$$

Dominant Mode TE_{10}

The lowest cutoff frequency (dominant mode) is TE_{10}:

$$f_{c,10} = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{c}{2a} \quad \text{(for air-filled)}. \tag{19.3}$$

The waveguide width a must be at least $\lambda/2$ for the dominant mode to propagate.

19.4.2 Propagation Wavenumber

The axial wavenumber in the waveguide is:

$$k_z = \sqrt{k^2 - k_c^2}, \tag{19.4}$$

where $k_c = \pi/a$ for the TE_{10} mode.

- $f > f_c$: k_z is real $\to$ **propagation**.
- $f < f_c$: k_z is imaginary $\to$ **evanescent decay**.

The **waveguide wavelength** (the spatial period along z) is:

$$\lambda_g = \frac{2\pi}{k_z} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}}, \tag{19.5}$$

where $\lambda = c/f$ is the free-space wavelength. Note that $\lambda_g > \lambda$ always—the guided wavelength is longer than the free-space wavelength.

19.4.3 Single-Mode Operation

For practical waveguide systems, the operating frequency is chosen so that only the dominant mode (TE_{10}) propagates. The next higher mode is TE_{20} (with $f_{c,20} = c/a$) or TE_{01} (with $f_{c,01} = c/(2b)$). The **single-mode bandwidth** is:

$$f_{c,10} < f < \min(f_{c,20},\ f_{c,01}). \tag{19.6}$$

For the standard aspect ratio $a = 2b$, both TE_{20} and TE_{01} have the same cutoff at $2f_{c,10}$, giving an octave of single-mode bandwidth.

X-Band Waveguide (WR-90)

WR-90 has $a = 2.286\,\text{cm}$, $b = 1.016\,\text{cm}$.

$$f_{c,10} = \frac{c}{2a} = \frac{3 \times 10^{10}}{2(2.286)} = 6.56\,\text{GHz},$$
$$f_{c,20} = \frac{c}{a} = 13.12\,\text{GHz},$$
$$f_{c,01} = \frac{c}{2b} = 14.76\,\text{GHz}.$$

Single-mode range: 6.56–13.12 GHz. The standard X-band operating range is 8.2–12.4 GHz, safely within this window.

19.4.4 Phase and Group Velocity

In a waveguide:

$$v_p = \frac{\omega}{k_z} > c \quad \text{(superluminal phase velocity)}, \tag{19.7}$$
$$v_g = \frac{d\omega}{dk_z} < c \quad \text{(subluminal group velocity)}. \tag{19.8}$$

Important

The phase velocity exceeding c does *not* violate special relativity. No information or energy travels at v_p. The energy (and information) travels at the group velocity $v_g < c$. The two velocities satisfy $v_p \cdot v_g = c^2$.

19.5 Guided-Wave Theorem

For any guided wave ($e^{-jk_z z}$ dependence), all six field components can be expressed in terms of just E_z and H_z:

- **TEM**$_z$ modes: $E_z = H_z = 0$. Requires $k_z = k$ (no cutoff). Exists only on structures with two or more conductors.
- **TE**$_z$ modes: $E_z = 0$, $H_z \neq 0$. Has cutoff.
- **TM**$_z$ modes: $H_z = 0$, $E_z \neq 0$. Has cutoff.

Exercises

1. A rectangular waveguide has $a = 2.286\,\text{cm}$ and $b = 1.016\,\text{cm}$ (WR-90). Find the cutoff frequency of the TE_{10} mode and the usable frequency range.
2. A multi-mode fiber has $n_{\text{core}} = 1.48$ and $n_{\text{clad}} = 1.46$. Find the numerical aperture and the maximum acceptance angle in air.
3. Show that for the TE_{10} mode in a rectangular waveguide, $v_p \cdot v_g = c^2$.
4. Explain why TEM waves cannot exist in a hollow (single-conductor) waveguide.
5. Compare the attenuation of RG59 coax and single-mode fiber at a link distance of 10 km. Which is preferable and by how many dB?

Chapter 20

Introduction to Antennas

An antenna is a device that converts guided electromagnetic waves (on a transmission line) into free-space electromagnetic waves, and vice versa. It is the essential interface between the wired and wireless worlds.

20.1 What Is an Antenna?

An antenna serves as a transducer between a transmission line and free space. In transmission mode, it converts the guided wave on a feed line into a radiated wave. In reception mode (by reciprocity), it converts an incident wave into a guided signal.

20.2 Near Field and Far Field

The field around an antenna is divided into regions:

- **Reactive near field**: Very close to the antenna; dominated by stored (reactive) energy.
- **Radiating near field** (Fresnel region): Fields are radiating but the pattern depends on distance.
- **Far field** (Fraunhofer region): The radiation pattern is independent of distance. The boundary is approximately:

$$r > \frac{2D^2}{\lambda}, \tag{20.1}$$

where D is the largest dimension of the antenna.

In the far field, the wave locally resembles a plane wave, with E and H perpendicular to each other and to the direction of propagation, and decaying as $1/r$.

Why $1/r$ Decay?

Conservation of energy requires the total power through any sphere surrounding the antenna to be constant (in a lossless medium). Since the surface area of a sphere grows as r^2, the power density must decrease as $1/r^2$. Since power density is proportional to $|E|^2$, the electric field must decrease as $1/r$.

20.3 Hertzian (Short) Dipole

The simplest antenna model is the **Hertzian dipole**: an infinitesimally short current element of length $d\ell$ carrying uniform current I_0.

20.3.1 Far-Field Expressions

In the far field ($r \gg \lambda$):

$$\tilde{E}_\theta = j\eta_0 \frac{k_0 I_0\, d\ell}{4\pi} \frac{\sin\theta}{r} e^{-jk_0 r}, \tag{20.2}$$

$$\tilde{H}_\phi = \frac{\tilde{E}_\theta}{\eta_0} = j\frac{k_0 I_0\, d\ell}{4\pi} \frac{\sin\theta}{r} e^{-jk_0 r}. \tag{20.3}$$

Important

Key features of the Hertzian dipole:

- No radiation along the axis of the dipole ($\theta = 0, \pi$): $\sin\theta = 0$.
- Maximum radiation in the broadside direction ($\theta = \pi/2$).
- The radiation pattern has a "donut" shape.
- Fields decay as $1/r$ (power decays as $1/r^2$).

20.3.2 Radiated Power

The time-average radiated power is obtained by integrating the Poynting vector over a sphere:

$$P_{\text{rad}} = \frac{\eta_0}{12\pi}(k_0 I_0\, d\ell)^2 = \frac{\eta_0 \pi}{3}\left(\frac{I_0\, d\ell}{\lambda}\right)^2. \tag{20.4}$$

20.3.3 Radiation Resistance

The **radiation resistance** is defined so that $P_{\text{rad}} = \frac{1}{2}R_{\text{rad}}|I_0|^2$:

$$R_{\text{rad}} = \frac{2\pi\eta_0}{3}\left(\frac{d\ell}{\lambda}\right)^2 = 80\pi^2\left(\frac{d\ell}{\lambda}\right)^2 \quad [\Omega]. \tag{20.5}$$

For a short dipole ($d\ell \ll \lambda$), the radiation resistance is very small (e.g., for $d\ell = \lambda/50$: $R_{\text{rad}} \approx 0.32\,\Omega$). This makes short dipoles very inefficient—most of the input power is lost to ohmic resistance in the antenna conductor.

20.4 Small Loop Antenna (Magnetic Dipole)

A small loop of area A carrying current I_0 radiates like a **magnetic dipole**. Its radiation pattern is identical to that of a Hertzian dipole (the $\sin\theta$ pattern), but with E and H roles swapped. The radiation resistance is:

$$R_{\text{rad}} = \frac{8\pi^3\eta_0}{3}\left(\frac{A}{\lambda^2}\right)^2 = 320\pi^4\left(\frac{A}{\lambda^2}\right)^2 \quad [\Omega]. \tag{20.6}$$

20.5 Receiving Antenna Equivalent Circuit

A receiving antenna in an incident field can be modeled as a Thévenin equivalent circuit: a voltage source V_{oc} (open-circuit voltage proportional to the incident field) in series with the antenna impedance Z_A. The power delivered to a matched load $Z_L = Z_A^*$ is:

$$P_r = \frac{|V_{\text{oc}}|^2}{8R_A}. \tag{20.7}$$

This model is the bridge between field theory (incident plane wave) and circuit theory (power delivered to a receiver).

20.6 Reciprocity

The **reciprocity theorem** states that an antenna has the same radiation pattern in transmission mode and reception mode. This means:

- Directions of maximum transmission are also directions of maximum reception.
- The effective area and directivity are the same for transmit and receive.

Exercises

1. A Hertzian dipole of length $d\ell = 1\,\text{cm}$ carries a current of $I_0 = 1\,\text{A}$ at $f = 1\,\text{GHz}$. Compute the radiation resistance and the total radiated power.
2. At what distance from a 1 m dipole antenna operating at 300 MHz does the far-field region begin?
3. Show that the Hertzian dipole has zero radiation along its axis and maximum radiation in the broadside direction.
4. A small loop antenna has radius $a = 5\,\text{cm}$ and operates at 100 MHz. Compare its radiation resistance with that of a Hertzian dipole of the same electrical size ($d\ell = 2\pi a$).
5. Using reciprocity, explain why a TV receiving antenna must be pointed in the same direction that it would need to be pointed if it were transmitting.

Chapter 21

Antenna Properties

This chapter defines the key parameters used to characterize antenna performance: directivity, gain, effective area, and the Friis transmission formula for wireless link analysis.

21.1 Radiation Intensity

The **radiation intensity** $U(\theta, \phi)$ [W/sr] is the power radiated per unit solid angle:

$$U(\theta, \phi) = r^2 S_r(r, \theta, \phi), \tag{21.1}$$

where S_r is the radial component of the time-average Poynting vector. In the far field, U is independent of r.

The total radiated power is:

$$P_{\text{rad}} = \oint U(\theta, \phi)\, d\Omega = \int_0^{2\pi} \int_0^{\pi} U(\theta, \phi) \sin\theta \, d\theta \, d\phi. \tag{21.2}$$

21.2 Directivity

Directivity

$$D = \frac{U_{\max}}{U_{\text{avg}}} = \frac{4\pi\, U_{\max}}{P_{\text{rad}}}. \tag{21.3}$$

Directivity measures how much more intensely the antenna radiates in its peak direction compared to an isotropic antenna radiating the same total power.

- **Isotropic antenna**: $D = 1$ (0 dBi). A theoretical reference; it radiates equally in all directions.
- **Hertzian dipole**: $D = 1.5$ (1.76 dBi).
- **Half-wave dipole**: $D \approx 1.64$ (2.15 dBi).

The **beam solid angle** Ω_A is defined by:

$$D = \frac{4\pi}{\Omega_A}, \qquad \Omega_A = \int\int \frac{U(\theta, \phi)}{U_{\max}}\, d\Omega. \tag{21.4}$$

21.3 Antenna Gain

The **gain** accounts for both directivity and ohmic losses:

$$G = e_r \cdot D, \tag{21.5}$$

where e_r is the **radiation efficiency**:

$$e_r = \frac{P_{\text{rad}}}{P_{\text{input}}} = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{loss}}}. \tag{21.6}$$

For a lossless antenna, $G = D$. In practice, $G < D$ due to ohmic losses.

21.4 Effective Area (Aperture)

The **effective area** A_e of a receiving antenna is the equivalent area that captures power from an incident plane wave:

$$P_{\text{received}} = A_e \cdot S_{\text{inc}}, \tag{21.7}$$

where S_{inc} [W/m^2] is the incident power density.

Effective Area

$$A_e = \frac{G\lambda^2}{4\pi}. \tag{21.8}$$

This remarkable relation connects the gain (a transmit property) to the effective area (a receive property).

For a parabolic dish of physical area A_{phys}, the effective area is $A_e = e_{\text{ap}} A_{\text{phys}}$, where e_{ap} is the **aperture efficiency** (typically 0.5–0.7 for practical dishes). Thus the gain of a dish antenna is:

$$G = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi e_{\text{ap}} A_{\text{phys}}}{\lambda^2}. \tag{21.9}$$

Example

A half-wave dipole at 1 GHz ($\lambda = 30$ cm): $G \approx 1.64$.

$$A_e = \frac{1.64 \times (0.3)^2}{4\pi} = \frac{0.1476}{12.57} \approx 0.012\,\text{m}^2 = 120\,\text{cm}^2.$$

21.5 Friis Transmission Formula

The Friis formula gives the received power in a free-space wireless link:

Friis Transmission Formula

$$\frac{P_r}{P_t} = G_t\,G_r \left(\frac{\lambda}{4\pi R}\right)^2, \tag{21.10}$$

where P_t and P_r are the transmitted and received powers, G_t and G_r are the transmit and receive antenna gains, and R is the distance between antennas.

The factor $(\lambda/(4\pi R))^2$ is the **free-space path loss** (FSPL). In decibels:

$$\text{FSPL [dB]} = 20\log_{10}\left(\frac{4\pi R}{\lambda}\right). \tag{21.11}$$

21.6 EIRP and the Decibel-Milliwatt (dBm)

The **Effective Isotropic Radiated Power** is:

$$\text{EIRP} = P_t \cdot G_t. \tag{21.12}$$

It represents the power that an isotropic antenna would need to radiate to produce the same power density in the peak direction of the actual antenna.

Power levels in communications are commonly expressed in **dBm** (decibels relative to 1 mW):

$$P\text{ [dBm]} = 10\log_{10}\left(\frac{P}{1\text{ mW}}\right). \tag{21.13}$$

For example, 1 W = 30 dBm, 100 mW = 20 dBm, and 1 µW = −30 dBm.

The Friis formula (21.10) in dB form becomes a simple sum:

$$P_r\text{ [dBm]} = P_t\text{ [dBm]} + G_t\text{ [dBi]} + G_r\text{ [dBi]} - \text{FSPL [dB]}. \tag{21.14}$$

21.7 Antenna Impedance

The input impedance of an antenna is:

$$Z_A = R_A + jX_A, \qquad R_A = R_{\text{rad}} + R_{\text{loss}}. \tag{21.15}$$

For maximum power transfer, the feed line impedance should be the complex conjugate of Z_A. In practice, the antenna is designed so that $X_A \approx 0$ and $R_A \approx Z_0$ of the feed line.

Exercises

1. Compute the directivity of the Hertzian dipole by integrating its radiation pattern.
2. A satellite at 36 000 km altitude transmits 10 W at 12 GHz through a dish antenna with $G_t = 30$ dBi. The ground station has $G_r = 40$ dBi. Find the received power.
3. An antenna has $R_{\text{rad}} = 73\,\Omega$ and $R_{\text{loss}} = 7\,\Omega$. If $D = 1.64$, find G and e_r.
4. Show that doubling the frequency increases the free-space path loss by 6 dB (for fixed distance).
5. Design a wireless link at 2.4 GHz over 100 m. Both antennas have $G = 2$ dBi and the transmitter power is 100 mW. What is the received power in dBm?

Chapter 22

Antenna Patterns

This chapter covers the radiation patterns of practical antennas, pattern parameters, the half-wave dipole, and an introduction to antenna arrays.

22.1 Radiation Pattern

The **radiation pattern** is a graphical representation of the radiation intensity (or field strength) as a function of direction. Patterns are typically plotted in:

- **Polar plots**: Useful for visualizing the 3D pattern shape.
- **Rectangular (Cartesian) plots**: Better for reading precise values, especially sidelobe levels.
- **dB scale**: Most common, as it reveals low-level features (sidelobes, nulls).

Important

When reading a dB polar plot, remember that the *center* represents $-\infty$ dB (a null) and the *outermost circle* represents 0 dB (the peak). Concentric circles mark fixed dB intervals (e.g., every 10 dB). A sidelobe at the -20 dB circle means its power is 100 times weaker than the main beam peak.

22.1.1 Principal Planes

- **E-plane**: The plane containing the electric field vector and the direction of maximum radiation.
- **H-plane**: The plane containing the magnetic field vector and the direction of maximum radiation.

22.1.2 Pattern Parameters

- **Half-power beamwidth (HPBW)**: The angular width between the -3 dB points of the main beam.
- **First null beamwidth (FNBW)**: The angular width between the first nulls on either side of the main beam.

- **Sidelobe level (SLL)**: The ratio (in dB) of the peak sidelobe to the main beam peak.
- **Front-to-back ratio**: The ratio of radiation in the forward direction to the backward direction.

22.2 Half-Wave Dipole

The half-wave dipole ($\ell = \lambda/2$) is the most common reference antenna in practice.

22.2.1 Current Distribution

The current on a thin half-wave dipole is approximately sinusoidal:

$$I(z) = I_0 \cos(k_0 z), \qquad -\lambda/4 \leq z \leq \lambda/4. \tag{22.1}$$

22.2.2 Far-Field Pattern

The normalized far-field pattern of the half-wave dipole is:

$$F(\theta) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}. \tag{22.2}$$

This is similar to the Hertzian dipole pattern ($\sin\theta$) but slightly more directive—the half-wave dipole has a narrower beam.

22.2.3 Key Parameters

Half-Wave Dipole Properties

- Directivity: $D \approx 1.64$ (2.15 dBi)
- Radiation resistance: $R_{\text{rad}} \approx 73\,\Omega$
- HPBW: $\approx 78°$
- Input impedance: $Z_A \approx 73 + j42.5\ \Omega$ (at exact $\lambda/2$ length)

Physical Insight

The radiation resistance of $73\,\Omega$ is close to common transmission line impedances ($50\,\Omega$ or $75\,\Omega$), making the half-wave dipole easy to match. This is one reason it is so widely used.

22.3 Antenna Arrays

An **antenna array** consists of multiple identical antenna elements arranged in a specific geometric pattern. By controlling the amplitude and phase of the signal fed to each element, the radiation pattern can be shaped and steered.

22.3.1 Array Factor

For a linear array of N isotropic elements with uniform spacing d:

$$AF(\theta) = \sum_{n=0}^{N-1} a_n \, e^{jnk_0 d \cos\theta}, \tag{22.3}$$

where a_n is the complex weight (amplitude and phase) of the n-th element.

For uniform amplitude and progressive phase shift α ($a_n = e^{jn\alpha}$):

$$AF(\theta) = \frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}, \qquad \psi = k_0 d \cos\theta + \alpha. \tag{22.4}$$

22.3.2 Pattern Multiplication

Pattern Multiplication Principle

The total radiation pattern of an array of identical elements is:

$$F_{\text{total}}(\theta, \phi) = F_{\text{element}}(\theta, \phi) \times AF(\theta, \phi). \tag{22.5}$$

The total pattern equals the element pattern multiplied by the array factor.

22.3.3 Beam Steering

By adjusting the progressive phase shift α, the main beam can be steered to angle θ_0:

$$\alpha = -k_0 d \cos\theta_0. \tag{22.6}$$

This is the principle behind **phased array** antennas used in radar, 5G base stations, and satellite communications.

22.4 Practical Antenna Types

Yagi-Uda antenna: A directional antenna consisting of a driven element (dipole), a reflector, and one or more directors. Commonly used for TV reception. Typical gain: 6–15 dBi.

Patch (microstrip) antenna: A flat, low-profile antenna fabricated on a PCB. Used extensively in mobile phones, GPS receivers, and WLAN devices. Typical gain: 5–8 dBi.

Horn antenna: A flared waveguide opening. Used as a standard-gain antenna for measurements and as a feed for dish reflectors. Typical gain: 10–25 dBi.

Parabolic reflector (dish): Uses a paraboloidal reflector with a feed antenna at the focus. Capable of very high gain (30–60 dBi). Used for satellite communications, radio astronomy, and radar.

Exercises

1. Plot the E-plane and H-plane patterns of a half-wave dipole (in dB) and measure the HPBW.

2. Compute the directivity of a half-wave dipole by numerically integrating $|F(\theta)|^2 \sin\theta$ over the sphere.

3. A 4-element uniform linear array with $d = \lambda/2$ and $\alpha = 0$ (broadside). Sketch the array factor pattern and find the HPBW.

4. Design a 4-element array with $d = \lambda/2$ to steer the main beam to $\theta_0 = 60°$. What progressive phase shift α is needed?

5. A Yagi-Uda antenna has gain $G = 12\,\text{dBi}$. What is its effective area at $f = 500\,\text{MHz}$?

6. Compare the radiation resistance of a half-wave dipole ($73\,\Omega$) with that of a Hertzian dipole of the same total length. Why is the half-wave dipole so much more efficient?