Chapter 3
Uniform Plane Waves

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What is a “wave” ?

Mechanism by which a disturbance is propagated from one place to another

water, heat, sound, gravity, and **EM** (radio, light, microwaves, uv,IR)

Notice how the media itself is **NOT** propagated

One Dimensional Wave Equation

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] p(x,t) = 0$$

Given $p(x,0) = f(x)$

A solution $p(x,t) = \frac{1}{2}[f(x-vt) + f(x+vt)]$

Unique solution depends on physical problem

$$\frac{\partial^2}{\partial x^2} p(x,t) = f'' \qquad \frac{\partial^2}{\partial t^2} p(x,t) = v^2 f''$$

time harmonic case $\frac{\partial}{\partial t} \Rightarrow j\omega$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{v^2}\right] p(x) = 0$$

Maxwell's Equations

$$\nabla \times \underset{\sim}{\mathbf{E}} = -\, j\omega\mu_0 \underset{\sim}{\mathbf{H}}$$

$$\nabla \times \underset{\sim}{\mathbf{H}} = j\omega\varepsilon_0 \underset{\sim}{\mathbf{E}}$$

$$\nabla \bullet \underset{\sim}{\mathbf{H}} = 0$$

$$\nabla \bullet \underset{\sim}{\mathbf{E}} = 0$$

Source Free $\Rightarrow \rho_v = 0 \;\; ; \;\; \mathbf{J} = 0$

Time Harmonic case $\Rightarrow$ $e^{j\omega t}$ Time dependent

Linear medium $\Rightarrow \;\; \mathbf{B} = \mu_0 \mathbf{H} \;\; ; \quad \mathbf{D} = \varepsilon_0 \mathbf{E}$

Vector Identity $\nabla\times(\nabla\times\underset{\sim}{\mathbf{E}}) = \nabla(\nabla\bullet\underset{\sim}{\mathbf{E}}) - \nabla^2\underset{\sim}{\mathbf{E}}$

$$-j\omega\mu_0(\nabla\times\underset{\sim}{\mathbf{H}}) = \nabla(\nabla\bullet\underset{\sim}{\mathbf{E}}) - \nabla^2\underset{\sim}{\mathbf{E}}$$

$$-j\omega\mu_0(j\omega\varepsilon_0\underset{\sim}{\mathbf{E}}) = 0 \quad -\nabla^2\underset{\sim}{\mathbf{E}}$$

$$\nabla^2\underset{\sim}{\mathbf{E}} + \omega^2\mu_0\varepsilon_0\underset{\sim}{\mathbf{E}} = 0$$ Wave equation for $\underset{\sim}{\mathbf{E}}$

for $E = \mathrm{E}_x\hat{\boldsymbol{x}}$ and $\mathrm{E}_x(z)$

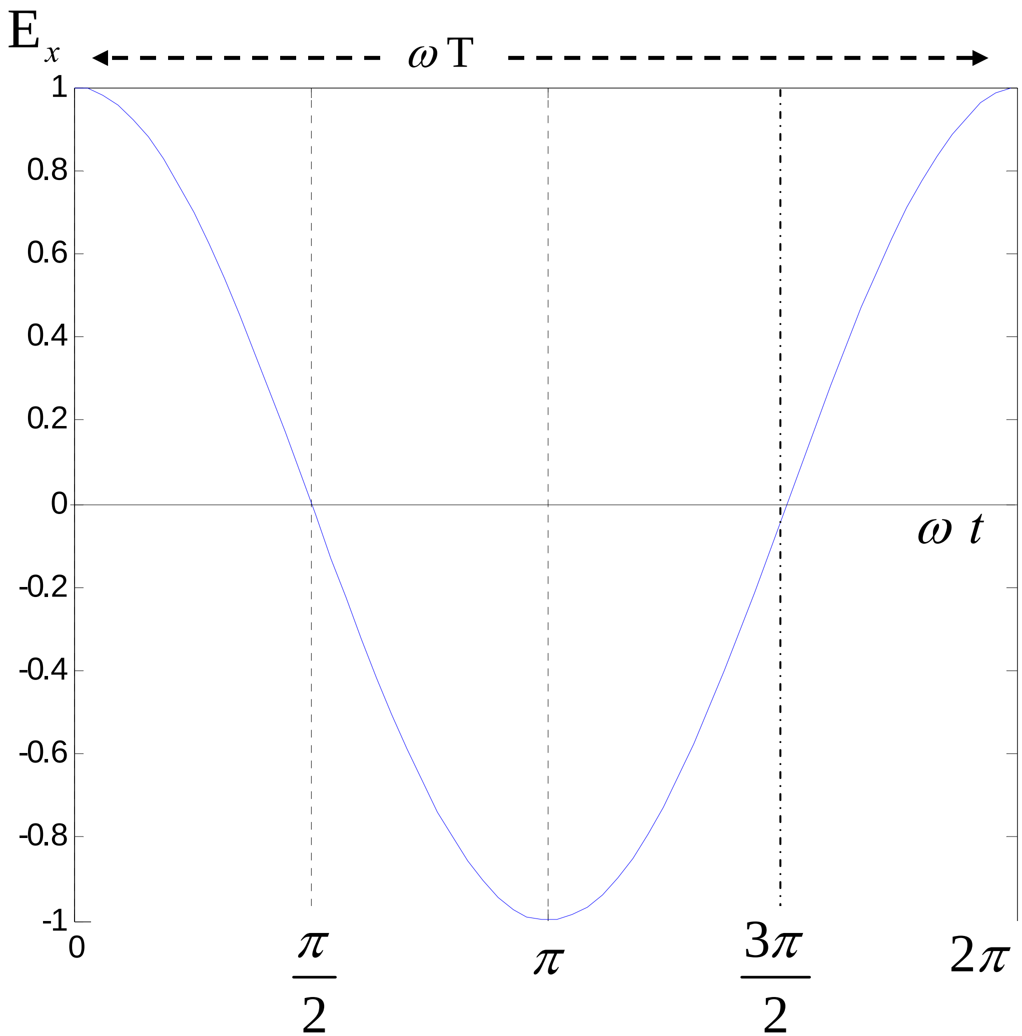
$$\frac{\partial^2\mathrm{E}_x}{\partial z^2} + \omega^2\mu_0\varepsilon_0\mathrm{E}_x = 0$$ (1-dim. case)

try soln of form $\underset{\sim}{\mathbf{E}} = \hat{\boldsymbol{x}}\, E_0 e^{-jkz}$

$$[-k^2 + \omega^2\mu_0\varepsilon_0]E_0 = 0$$

$$k^2 = \omega^2\mu_0\varepsilon_0$$ Dispersion Relation

$$E(z,t) = \mathrm{Re}\left\{\underset{\sim}{\mathbf{E}} e^{j\omega t}\right\} = \hat{\boldsymbol{x}} E_0\cos(\omega t - kz)$$



periodic in time
period T

$$E_x = E_0 \cos(\omega t)$$

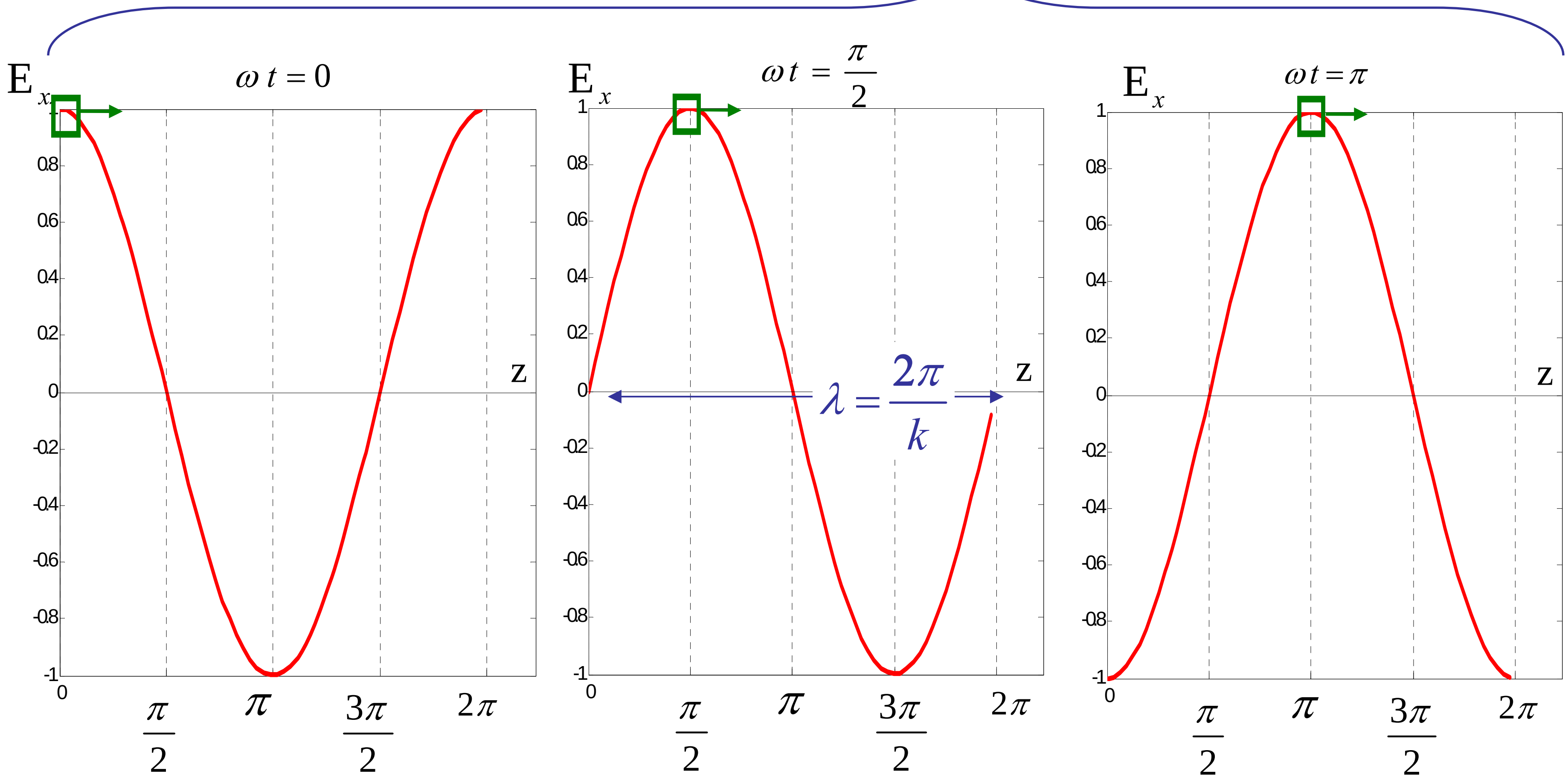
x component of the electric field at z=0 as a function of time

spatially repeating every wavelength λ

angular frequency $\omega=2\pi f$

$$E_x = E_0 \cos(\omega t - kz)$$

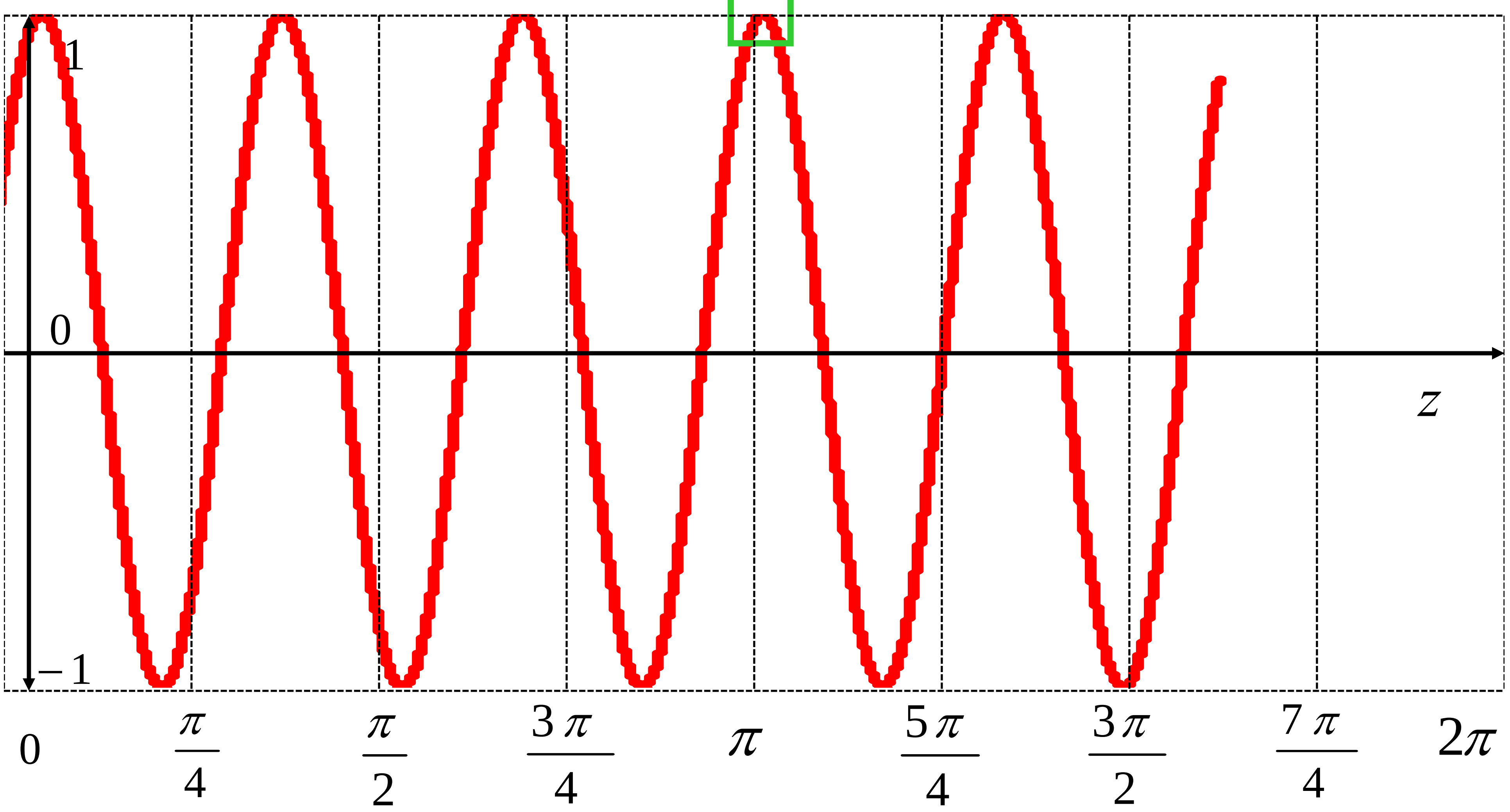
Electric field as a function of z at different times



$$v = \frac{\Delta z}{\Delta t} = \frac{\lambda/2}{\pi/\omega} = \frac{\omega}{k}$$

$$\mathrm{E}_x = E_0 \cos(\omega t - kz)$$

Electric field as a function of z at different times



$$v = \frac{\Delta z}{\Delta t} = \frac{\lambda/2}{\pi/\omega} = \frac{\omega}{k}$$

Quick Review

- The wave spacially repeats at point $z = \lambda$ where $k\lambda = 2\pi$.
- The quantity λ, where $\lambda = \dfrac{2\pi}{k}$ is called the wavelenght.
- The number of wavelengths contained in a spatial distribution of 2π is given by $k = \dfrac{2\pi}{\lambda}$ and it is called the wavenumber.
- The velocity of the peak of the wave (position of constant phase) requires that $\omega t - kz = \text{constant}$ so the velocity of propagation is given by

$$\frac{\partial z}{\partial t} = v = \frac{\omega}{k} \qquad \text{[m/sec]}$$

- The velocity in free space is given by

$$v = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 3 \times 10^8 \qquad \text{[m/sec]}$$

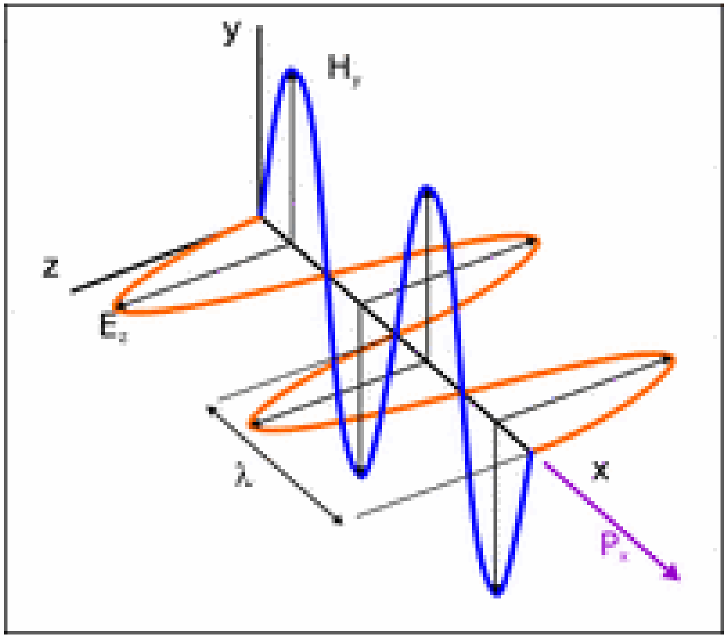
So far we have come across some useful expressions such as:

Period	$T = \frac{1}{f}$	[sec]	Phase Velocity	$v = \frac{\omega}{k}$	[m/sec]
Angular Frequency	$\omega = 2\pi f$	[rad]	Velocity in free space	$c \approx 3\times10^{8}$	[m/sec]
Frequency	$f = \frac{1}{T}$	[Hz]	Wavenumber	$k = \omega\sqrt{\mu_0\varepsilon_0}$	[1/m]
Wavelength	$\lambda = \frac{2\pi}{k}$	[m]	Note: $f[GHz]\bullet\lambda[cm] \approx 30$		

Also, remember that the orientation of the **E** field of a uniform plane electromagnetic wave is perpendicular to the **H** field of that wave and that both are perpendicular to the direction from which the wave propagates

Uniform Plane Waves

Waves with constant phase fronts (plane waves) and whose amplitude (E_0) is uniform



http://www.elec.york.ac.uk/cpd/img/em-wave.png

Recall $\nabla \times \underset{\sim}{\mathbf{E}} = -j\omega\mu_0 \underset{\sim}{\mathbf{H}}$

Where the **E** field of a uniform plane wave is given by

$$\underset{\sim}{\mathbf{E}} = \hat{\boldsymbol{x}} E_0 e^{-jkz}$$

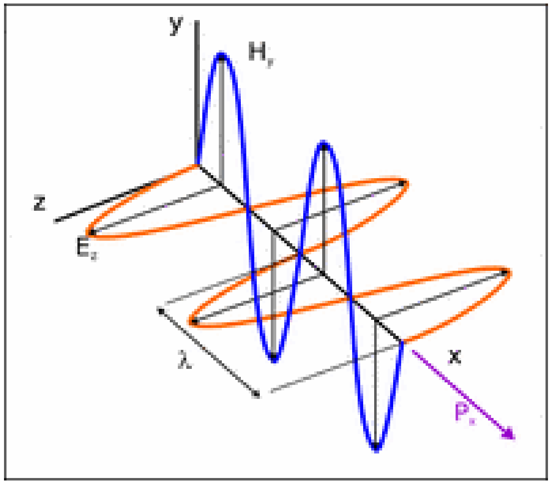
The magnetic field is then

$$\underset{\sim}{\mathbf{H}} = \hat{\boldsymbol{y}} \frac{E_0 e^{-jkz}}{\eta_0}$$

E field is in $\hat{\boldsymbol{x}}$ direction

H field is in $\hat{\boldsymbol{y}}$ direction

Wave propagating in $+\hat{\boldsymbol{z}}$ direction



http://www.elec.york.ac.uk/cpd/img/em-wave.png

Or in the time domain

$$\boldsymbol{E}(z,t) = \mathrm{Re}\left\{\mathbf{E}e^{j\omega t}\right\}\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}E_0\cos(\omega t - kz)$$

Similarly

$$\boldsymbol{H}(z,t) = \mathrm{Re}\left\{\frac{\mathbf{E}e^{j\omega t}}{\eta_0}\right\}\hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}\frac{E_0}{\eta_0}\cos(\omega t - kz)$$

Where the η_0 is the intrinsic impedance of free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377 \ \text{[Ohms]}$$

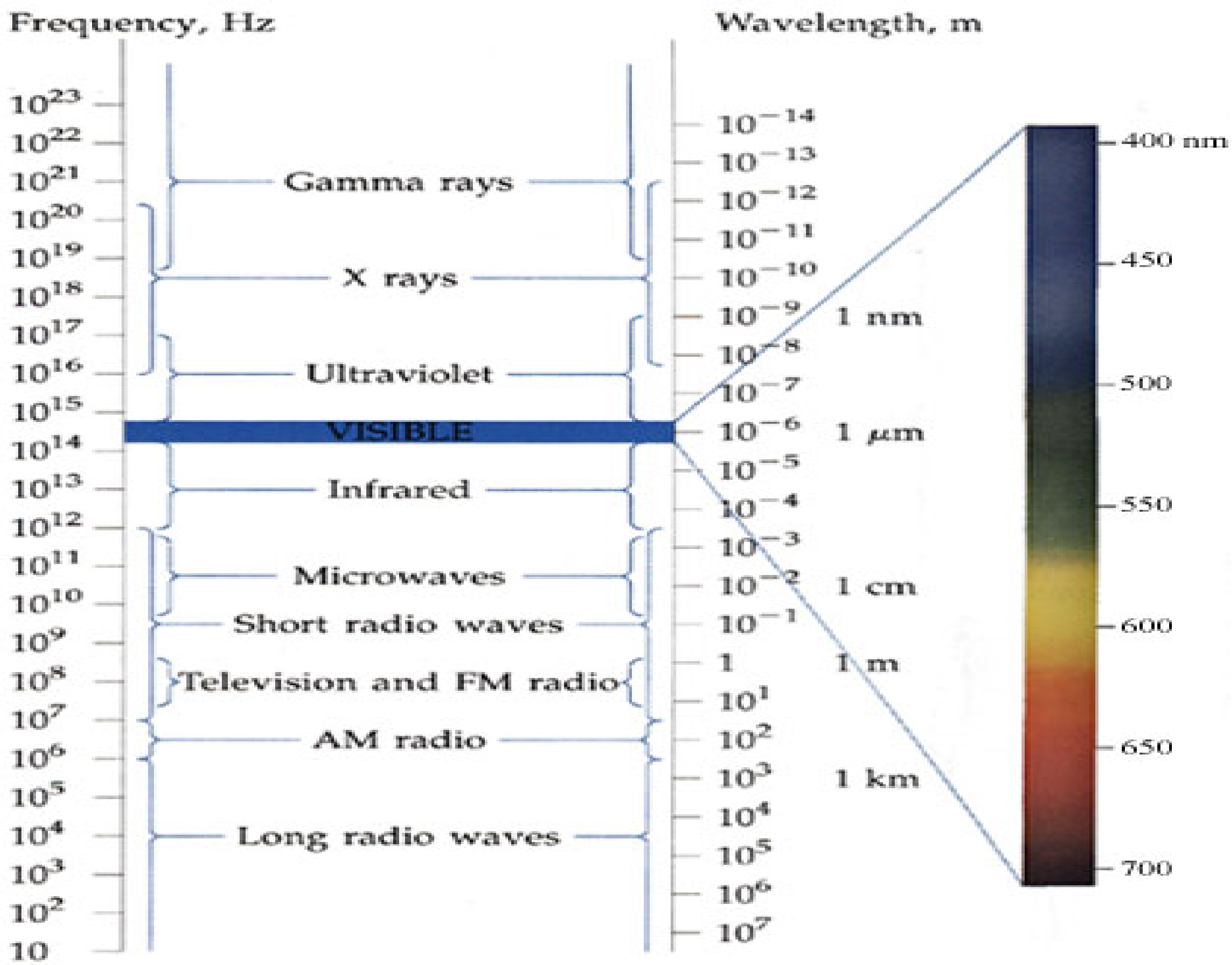
Permittivity $\boldsymbol{\varepsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}}$ $\left[\mathrm{F}/\mathrm{m}\right]$

Permeability $\boldsymbol{\mu_0 = 4\pi \times 10^{-7}}$ $\left[\mathrm{H}/\mathrm{m}\right]$

The Electromagnetic Spectrum

Source	Freq.[Hz]	Freq. (common units)	Wavelength [m]	Wavelength (common units)
U.S A-C Power	60	60 Hz	$5x10^{6}$	5000 Km
ELF Subm. Comm.	500	500 Hz	$6x10^{5}$	600 Km
AM radio	10^{6}	1000 Hz	300	300 m
CB radio	$2.7x10^{7}$	27 MHz	11	11 m
Cordless phone	$4.9x10^{7}$	49 MHz	6.1	6.1 m
TV ch. 2	$6x10^{7}$	60 MHz	5	5 m
FM radio	10^{8}	100 MHz	3	3 m
TV ch. 8	$1.8x10^{8}$	180 MHz	1.7	1.7 m
UHF Aircraft Comm.	$5x10^{8}$	500 MHz	.6	60 cm
TV ch. 39	$6.2x10^{8}$	620 MHz	.48	48 cm
Cellular phone	$8.7x10^{8}$	870 MHz	.34	34 cm
μ-wave oven	$2.45x10^{9}$	2.45 GHz	.12	12 cm
"C" band	$6x10^{9}$	6 GHz	.05	5 cm
Police radar	$1.05x10^{10}$	10.5 GHz	.0285	2.85 cm
mm wave	10^{11}	100 GHz	.003	3 mm
He-Ne Laser	$4.7x10^{14}$		$6.3x 10^{-7}$	6300 Å
Light	10^{15}		$3x10^{-7}$	3000 Å
X-ray	10^{18}		$3x10^{-7}$	3 Å

The Electromagnetic Spectrum



http://www.impression5.org/solarenergy/misc/emspectrum.html

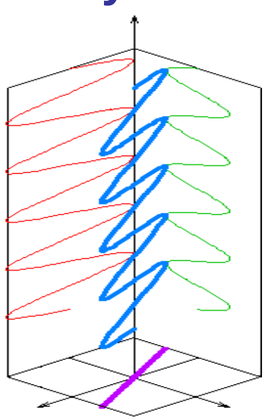
The Electromagnetic Spectrum

Name	Wavelengths	Used for	Source
Radio Wave	1000 – 1 m	Radio, TV, Communications, Cellular phones	Radio transmitters, electric cables
Microwaves	1000 – 1 mm	Radar, telecom, microwave ovens, speed measuring	Klystrons, magnetrons, masers
Infrared (IR)	1000 – 0.8 u	Radiation heaters, infrared heating, remote control	Hot objects, IR lamps, fires, LEDs Laser
Visible Lights	800 – 400 nm	Illumination, photography, imaging, holography	Light-bulbs, flash lamps, candles, LEDs, Laser
Ultraviolet (UV)	400 – 1 nm	Solarium, curing of plastics, sterilization	UV lamps, Lasers, accelerators
X-rays	1000 – 1 pm	X-raying, radiation treatment of tumors	X-rays tubes, accelerators
Gamma Rays	1000 – 1 fm	Radiation treatment of cancer, sterilization of food	Radioactive isotopes, particles accelerators

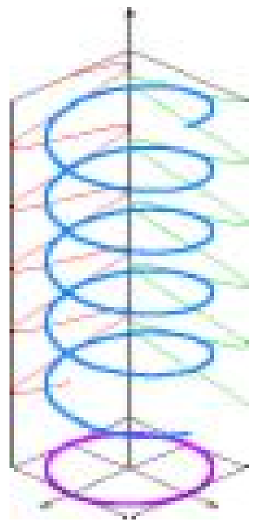
Polarization

The **polarization** of a wave is described by the locus of the tip of the **E** vector as time progresses at a fixed point in space.

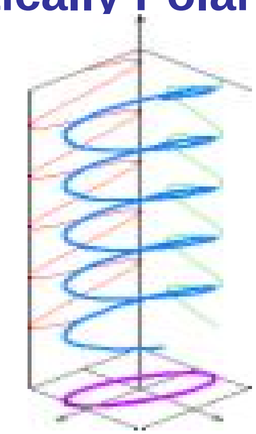
If locus is a straight line the wave is said to be **Linearly Polarized**



If locus is a circle the wave is said to be **Circularly Polarized**



If locus is an ellipse the wave is said to be **Elliptically Polarized**



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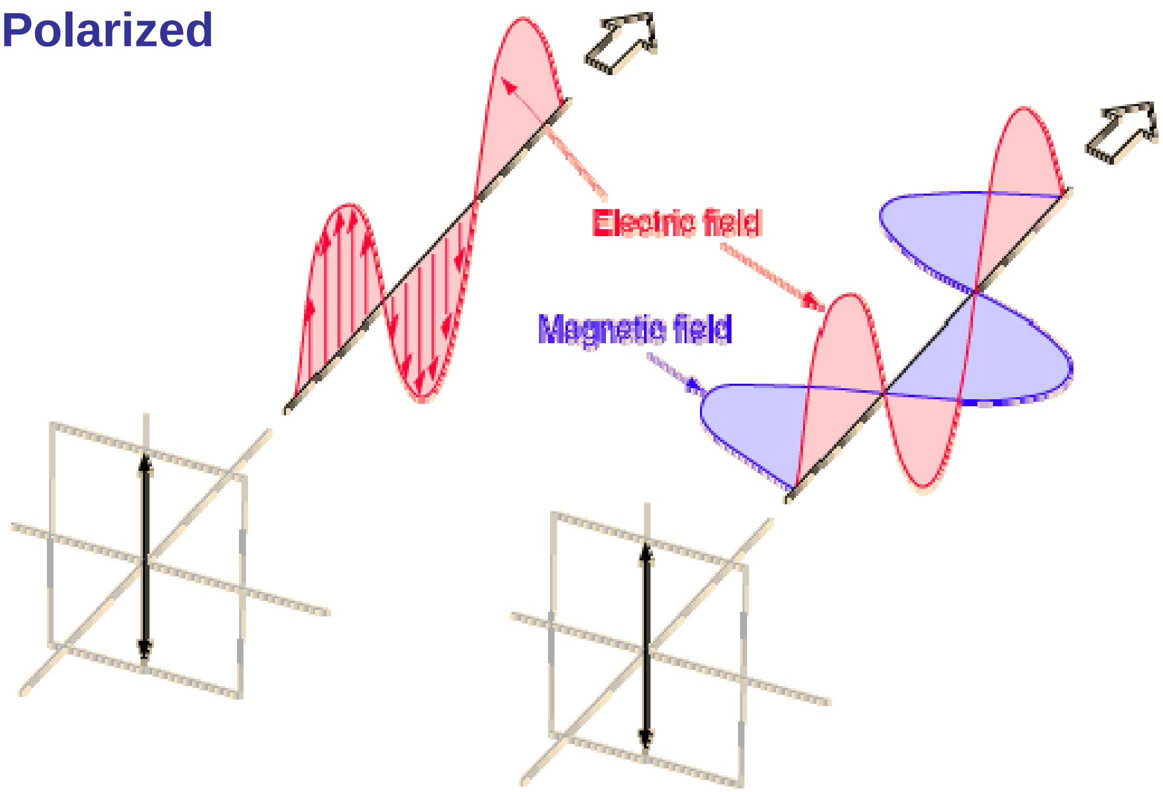

http://hyperphysics.phy-astr.gsu.edu/hbase/phyopt/imgpho/pollin.gif

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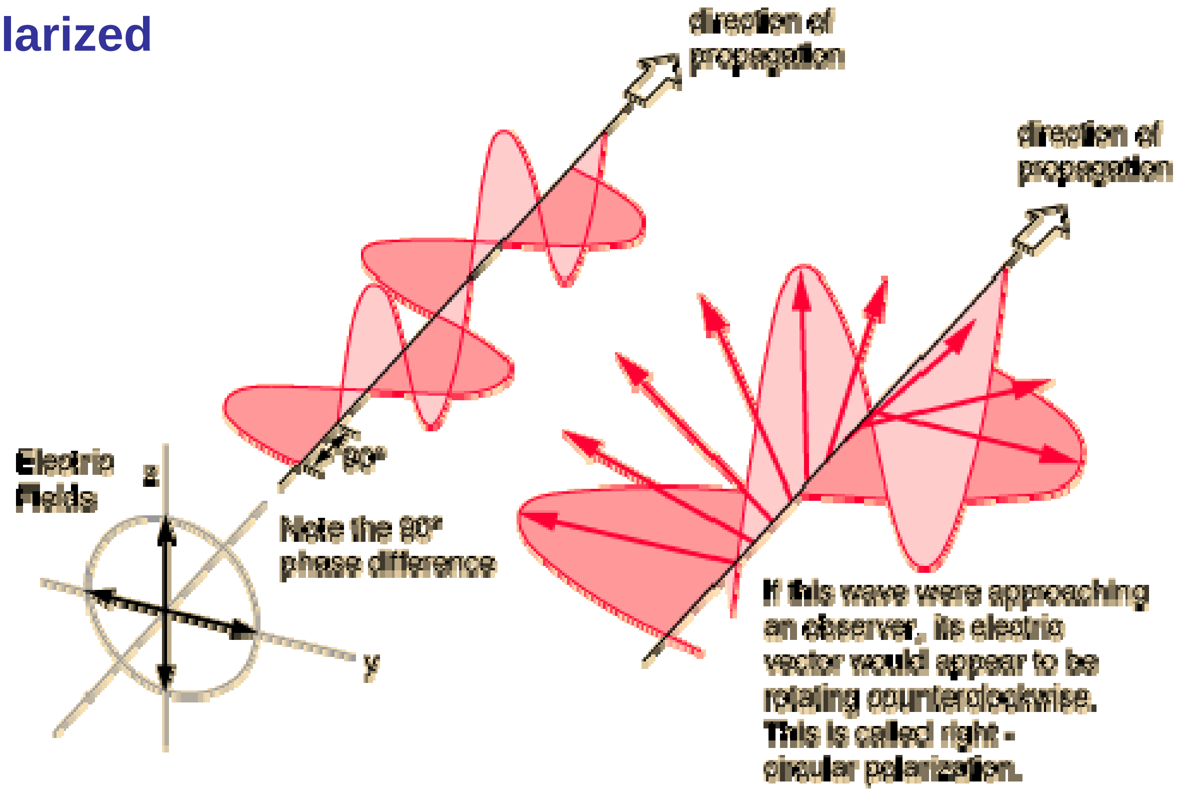

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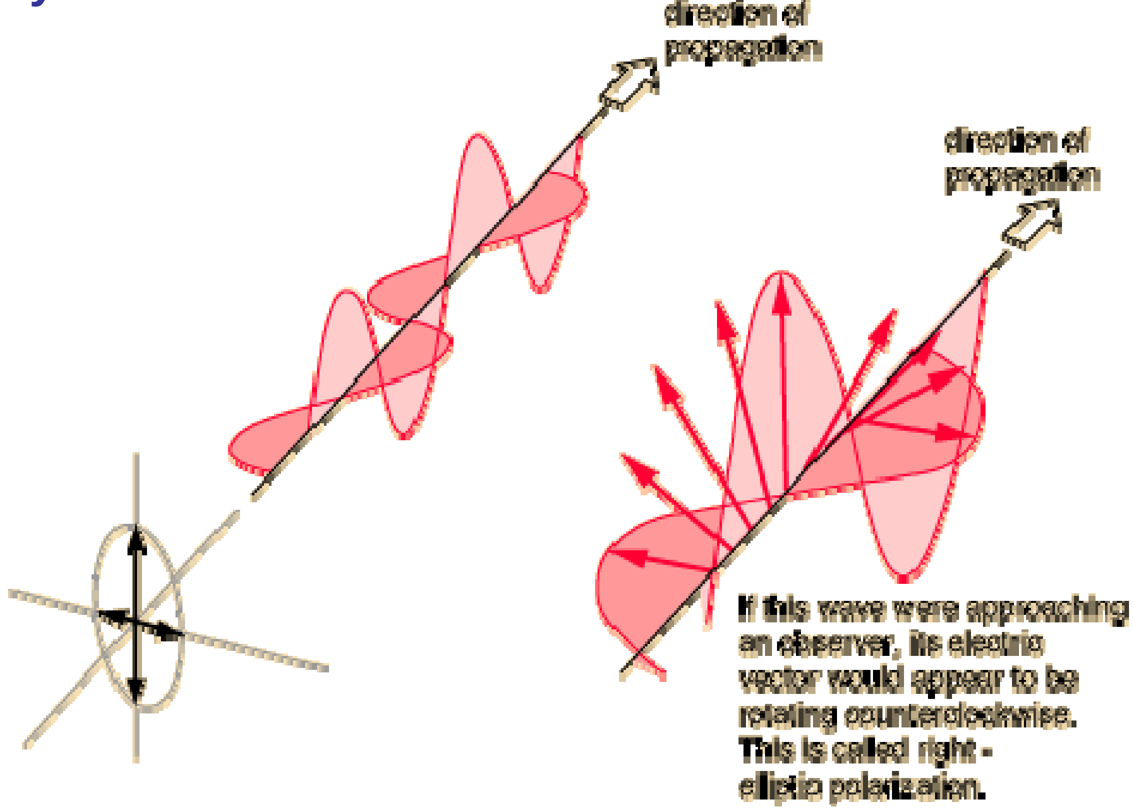

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Polarization

Consider a plane wave propagating in the **positive** $\mathbf{z}$ direction.

$$\boldsymbol{E} = E_0 \cos(\omega t - kz)$$

The associated electric field can be expressed in the form of

$$\boldsymbol{E} = \hat{\boldsymbol{x}} E_x + \hat{\boldsymbol{y}} E_y$$

where the two components are, in general terms, $\left\{ \begin{aligned} E_x &= a\cos(\omega t - kz + \phi_a) \\ E_y &= b\cos(\omega t - kz + \phi_b) \end{aligned} \right\}$

The **polarization** of this plane wave is determined by the quantity

$$\frac{E_y}{E_x} = A \angle \phi$$

Where

$$A = \frac{|E_y|}{|E_x|} = \frac{b}{a}$$ and $$\phi = \phi_b - \phi_a$$

Polarization Classification

If $\mathbf{E}$ field is traveling in the **positive** $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{x}}$ or $\hat{\boldsymbol{z}}$ direction $A\angle\phi$ can be found respectively by

$$\frac{E_x\angle\phi_x}{E_z\angle\phi_z} \text{ or } \frac{E_z\angle\phi_z}{E_y\angle\phi_y} \text{ or } \frac{E_y\angle\phi_y}{E_x\angle\phi_x}$$

$\mathrm{A}=0 \;;\; \phi = 0 \text{ or } \pm\pi$	Linear Polarization (LP)
$\mathrm{A}\to\infty$	Linear Polarization (LP)
$\mathrm{A}=1 \;;\; \phi = \pi/2$	Left-Hand Circular Polarization (LHCP)
$\mathrm{A}=1 \;;\; \phi = -\pi/2$	Right-Hand Circular Polarization (RHCP)
$0<\phi<\pi$	Left-Hand Elliptical Polarization (LHEP)
$-\pi<\phi<0$	Right-Hand Elliptical Polarization (RHEP)

Polarization Classification

If $\mathbf{E}$ field is traveling in the **negative** $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{x}}$ or $\hat{\boldsymbol{z}}$ direction $A\angle\phi$ can be found respectively by

$$\frac{E_z\angle\phi_x}{E_x\angle\phi_z} \text{ or } \frac{E_y\angle\phi_z}{E_z\angle\phi_y} \text{ or } \frac{E_x\angle\phi_y}{E_y\angle\phi_x}$$

$\mathrm{A}=0 \quad ; \quad \phi=0 \text{ or } \pm\pi$	Linear Polarization (LP)
$\mathrm{A}\rightarrow\infty$	Linear Polarization (LP)
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where the two components are, in general terms, $\begin{Bmatrix} E_x = a\cos(\omega t - kz + \phi_a) \\ E_y = b\cos(\omega t - kz + \phi_b) \end{Bmatrix}$

The complex representation is given can be expressed by

$$\underset{\sim}{\mathbf{E}} = \hat{\boldsymbol{x}}\, a e^{-j(kz-\phi_a)} + \hat{\boldsymbol{y}} b e^{-j(kz-\phi_b)}$$

Polarization

Look at $z = 0$ and $\phi_b = \phi$; $\phi_a = 0$

$$E_x = a\cos\omega t$$

$$E_y = b\cos(\omega t + \phi_b)$$

$$\left(\frac{E_x}{a}\right)^2 - 2\left(\frac{E_x E_y}{ab}\right)\cos\phi + \left(\frac{E_y}{b}\right)^2 = \sin^2\phi$$

Recall that the general quadratic equation is given by

$$Ax^2 + Bxy + cy^2 + Dx + Ey + F = 0$$

where

$$A = \frac{1}{a^2} \;;\; B = -\frac{2\cos\phi}{ab} \;;\; C = \frac{1}{b^2} \;;\; D = 0 \;;\; E = 0 \;;\; F = -\sin^2\phi$$

Polarization

$$Ax^2 + Bxy + cy^2 + Dx + Ey + F = 0$$

where

$$A = \frac{1}{a^2} \;\; ; \;\; B = -\frac{2\cos\phi}{ab} \;\; ; \;\; C = \frac{1}{b^2} \;\; ; \;\; D = 0 \;\; ; \;\; E = 0 \;\; ; \;\; F = -\sin^2\phi$$

If $B^2 - 4AC < 0$ this becomes equ of an ellipse

$$\cos^2\phi\left(-\frac{2}{ab}\right)^2 - 4\left(\frac{1}{a^2}\right)\left(\frac{1}{b^2}\right) = \frac{4}{a^2b^2}\left(\cos^2\phi - 1\right) \le 0$$

rotated by an angle $\theta \Rightarrow \cot 2\theta = \dfrac{A - C}{B}$

$$\cot 2\theta = \left[\frac{1}{a^2} - \frac{1}{b^2}\right]\left[\frac{ab}{-2\cos\phi}\right]$$

Example

Let $\phi = 0$ $(or \;\; \phi = \pi)$

$$\left(\frac{E_x}{a}\right)^2 - 2\left(\frac{E_x E_y}{ab}\right)\cos\phi + \left(\frac{E_y}{b}\right)^2 = \sin^2\phi$$

$$\left(\frac{E_x}{a}\right)^2 - 2\left(\frac{E_x E_y}{ab}\right) + \left(\frac{E_y}{b}\right) = 0 \Rightarrow \left(\frac{E_x}{a} - \frac{E_y}{b}\right)^2 = 0$$

$$\frac{E_x}{a} = \frac{E_y}{b} \Rightarrow E_y = \frac{b}{a} E_x$$

Linear polarization (line of slope b/a)

Example

Let $a = b$; $\phi = \dfrac{\pi}{2}$

$$\left(\frac{E_x}{a}\right)^2 - 2\left(\frac{E_x E_y}{ab}\right)\cos\phi + \left(\frac{E_y}{b}\right)^2 = \sin^2\phi$$

$$\left(\frac{E_x}{a}\right)^2 + \left(\frac{E_y}{a}\right)^2 = 1$$

Circular polarization (circle of radius "a")

Example

Let $b = 2a \quad ; \quad \phi = \dfrac{\pi}{2}$

$$\left(\frac{E_x}{a}\right)^2 - 2\left(\frac{E_x E_y}{ab}\right)\cos\phi + \left(\frac{E_y}{b}\right)^2 = \sin^2\phi$$

$$\left(\frac{E_x}{a}\right)^2 + \left(\frac{E_y}{2a}\right)^2 = 1$$

Elliptical polarization (equ of an ellipse with major radius = $2a$ and minor radius $= a$)

Polarization Example

Find the polarization of the following field:

(a) $\underset{\sim}{\mathbf{E}} = (j\hat{x} + \hat{y})e^{-jkz}$

$$E_x = (1\angle -kz + 90^\circ)$$

$$E_y = (1\angle -kz)$$

$$A\angle\phi = \frac{E_y}{E_x} = \frac{(1\angle -kz)}{(1\angle -kz + 90^\circ)} = \frac{|1|}{|1|}\angle -kz - (-kz + 90^\circ)$$

$$A\angle\phi = 1\angle -90^\circ \Rightarrow RHCP$$

Polarization Example

Find the polarization of the following field:

(b) $\underset{\sim}{\mathbf{E}} = ((2+j)\hat{x} + (3-j)\hat{z})e^{-jky}$

$$\mathrm{E_x} = (\sqrt{5}\angle -ky + 26.5651^\circ)$$

$$\mathrm{E_z} = (\sqrt{10}\angle -ky - 18.4349^\circ)$$

$$A\angle\phi = \frac{\mathrm{E_x}}{\mathrm{E_z}} = \frac{(\sqrt{5}\angle -ky + 26.5651^\circ)}{(\sqrt{10}\angle -ky - 18.4349^\circ)} = \frac{|\sqrt{5}|}{|\sqrt{10}|}\angle -ky + 26.5651^\circ - (-ky - 18.4349^\circ)$$

$$A\angle\phi = \frac{1}{\sqrt{2}}\angle 45^\circ \Rightarrow LHEP$$

Polarization Example

Find the polarization of the following field:

(c) $\underset{\sim}{\mathbf{E}} = ((1+j)\hat{y} + (1-j)\hat{z})e^{-jkx}$

$$\mathrm{E_y} = (\sqrt{2}\angle -kx + 45)$$

$$\mathrm{E_z} = (\sqrt{2}\angle -kx - 45)$$

$$A\angle\phi = \frac{\mathrm{E_z}}{\mathrm{E_y}} = \frac{(\sqrt{2}\angle -kx - 45^\circ)}{(\sqrt{2}\angle -kx + 45^\circ)} = \frac{|\sqrt{2}|}{|\sqrt{2}|}\angle -kx - 45^\circ - (-kx - 45^\circ)$$

$$A\angle\phi = 1\angle -90^\circ \Rightarrow RHCP$$

Plane Waves in Dissipative Media

For isotropic conductors Ohm's Law states that

$$\underset{\sim}{\mathbf{J}}_c = \sigma \underset{\sim}{\mathbf{E}}$$

where $\underset{\sim}{\mathbf{J}}_c$ conduction current ; σ conductivity $\left[℧/\mathrm{m} \right]$

$\underset{\sim}{\mathbf{J}}_0$ source current

Consequently Ampere's Law becomes

$$\nabla \times \underset{\sim}{\mathbf{H}} = j\omega \mathbf{D} + \underset{\sim}{\mathbf{J}}_c + \underset{\sim}{\mathbf{J}}_0$$

$$\nabla \times \underset{\sim}{\mathbf{H}} = j\omega \left[\varepsilon - j\frac{\sigma}{\omega} \right] \underset{\sim}{\mathbf{E}} + \underset{\sim}{\mathbf{J}}_0$$

Where

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \varepsilon - j\frac{\sigma}{\omega} \quad \text{Complex Permittivity}$$

Plane Waves in Dissipative Media

In a source free conducting medium $(\mathbf{J}=0)$ Ampere's Law states

$$\nabla \times \underset{\sim}{\mathbf{H}} = j\omega\underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\mathbf{E}}$$

As derived earlier, the wave equation is given by

$$\left(\nabla^2 + \omega^2\mu\underset{\sim}{\boldsymbol{\varepsilon}}\right)\underset{\sim}{\mathbf{E}} = 0$$

As we have seen, $\underset{\sim}{\boldsymbol{\varepsilon}}$ is complex for a conducting medium.

Note: The wave number and the intrinsic impedance are now **complex numbers.**

$$\boxed{\underset{\sim}{\mathbf{k}}^2 = \omega^2\mu\,\underset{\sim}{\boldsymbol{\varepsilon}}} \qquad ; \qquad \boxed{\underset{\sim}{\eta} = \sqrt{\mu/\underset{\sim}{\boldsymbol{\varepsilon}}}}$$

Plane Waves in Dissipative Media

The wave number and the intrinsic impedance can also be written as

$$\underset{\sim}{\mathbf{k}} = k_R - jk_I$$

$$\underset{\sim}{\eta} = |\eta| \mathrm{e}^{j\phi}$$

The electromagnetic fields of a uniform plane wave in a dissipative medium are given by

$$\underset{\sim}{\mathbf{E}} = \hat{x} E_0 e^{-j\underset{\sim}{\mathbf{k}} z}$$

$$\underset{\sim}{\mathbf{H}} = \hat{y} \frac{E_0 e^{-j\underset{\sim}{\mathbf{k}} z}}{\underset{\sim}{\eta}}$$

Plane Waves in Dissipative Media

The electromagnetic fields can also be written as

$$\underset{\sim}{\mathbf{E}} = \hat{x} E_0 e^{-k_I z} e^{-jk_R z}$$

$$\underset{\sim}{\mathbf{H}} = \hat{y} \frac{E_0 e^{-k_I z} e^{-jk_R z} \mathrm{e}^{j\phi}}{|\eta|}$$

Or in the time domain

$$E_x(z,t) = E_0 e^{-k_I z} \cos(\omega t - k_R z)$$

$$H_y(z,t) = \frac{E_0 e^{-k_I z} \cos(\omega t - k_R z - \phi)}{|\eta|}$$

Plane Waves in Dissipative Media

From the electromagnetic fields we can observe that

1) The wave travels in the $+\hat{\boldsymbol{z}}$ direction with a velocity

$$\mathrm{v} = \frac{\omega}{k_R}$$

where k_R is called the wavenumber.

2) The amplitude is attenuated exponentially at the rate k_I nepers per meter, where k_I is the attenuation constant.

3) The magnetic field H_y is out of phase by ϕ.

Attenuation

One neper attenuation if

$$\ln\left[\frac{\text{Amplitude}_{\text{start}}}{\text{Amplitude}_{\text{end}}}\right]=1$$

The attenuation in nepers after lenght d is given by

$$\text{Attenuation[nepers]}=\ln\left[\frac{E_0 e^{-k_I z}}{E_0 e^{-k_I(z+d)}}\right]=k_I d$$

The relationship between nepers and dB is given by

$$1\text{[neper]}=8.686\text{ [dB]}$$

Example

The electric field is decreased by a factor of 0.707.

Find the attenuation in nepers and dB

$$\ln\left[\frac{E_F}{E_I}\right] = \ln[0.707] = \boxed{-0.3467\ [\text{nepers}]}$$

$$-0.3467[\text{nepers}]\bullet 8.686\left[\frac{\text{dB}}{\text{nepers}}\right] = \boxed{-3.01\ [\text{dB}]}$$

or

$$20\log\left(\frac{E_F}{E_I}\right) = 20\log(0.707) = \boxed{-3.01\ [\text{dB}]}$$

Note on dB Scale

If dealing with electric field use

$$20\log\left[\frac{E_F}{E_I}\right]$$

If dealing with power use

$$10\log\left[\frac{P_F}{P_I}\right]$$

This is because $P \sim E^2$

when $E_F = 0.707 E_I$

then $P_F = 0.707^2 P_I \Rightarrow P_F = 0.5 P_I$

$$20\log\left[0.707\right] = 10\log\left[0.5\right] = -3.01\ [\text{dB}]$$

General Medium

The penetration depth (d_p) such that $\left|E_{(z=d_p)}\right| = \left(\frac{1}{e}\right) \left|E_{(z=0)}\right|$ is given by

$$k_I d_p = 1$$

Where for a conducting media

$$\underset{\sim}{\mathbf{k}} = \omega\sqrt{\mu\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]} = \omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} = k_R - jk_I$$

Keep in mind that

If a>>1 then

$$\sqrt{1 + ja} \approx \sqrt{\frac{a}{2}}(1 + j)$$

or

If a<<1 then

$$\sqrt{1 + ja} \approx 1 + j\frac{a}{2}$$

Slightly Conducting Media

(Good Dielectric) $\frac{\sigma}{\omega\varepsilon} << 1$

$$\underset{\sim}{\mathbf{k}} \approx \omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{2\omega\varepsilon}\right]$$

$$\underset{\sim}{\mathbf{k}} = k_R - jk_I \quad ; \quad k_I = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$d_p = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}} \quad ; \quad k_R = \omega\sqrt{\mu\varepsilon}$$

Highly Conducting Media

(Good Conductor) $\frac{\sigma}{\omega\varepsilon} >> 1$

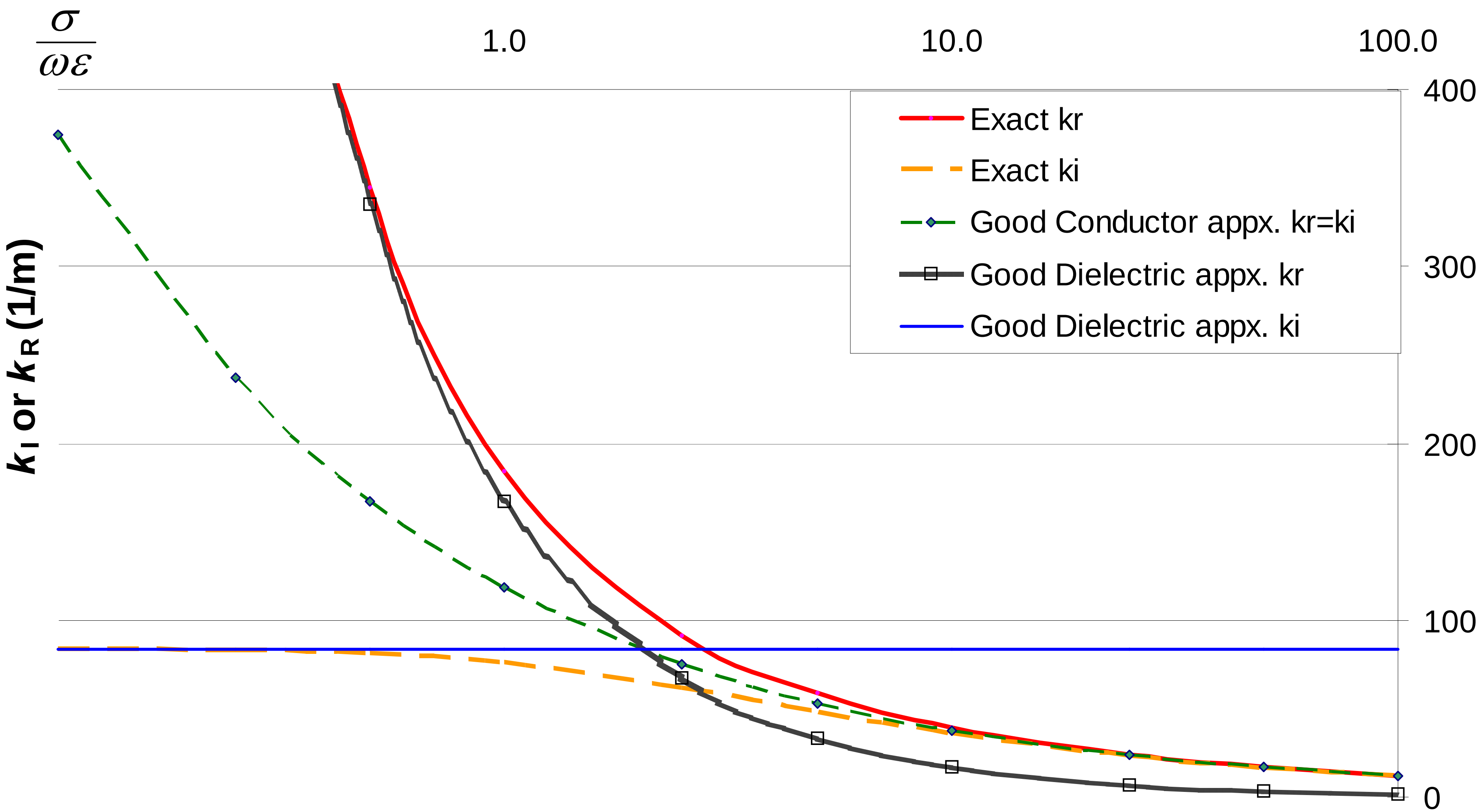
$$\underset{\sim}{\mathbf{k}} \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1-j)$$

$$\underset{\sim}{\mathbf{k}} = k_R - jk_I \quad ; \quad k_I = \sqrt{\frac{\omega\,\mu\sigma}{2}}$$

$$d_p = \sqrt{\frac{2}{\omega\,\mu\sigma}} \equiv \delta \quad ; \quad k_R = \sqrt{\frac{\omega\,\mu\sigma}{2}}$$

δ Also called the skin depth

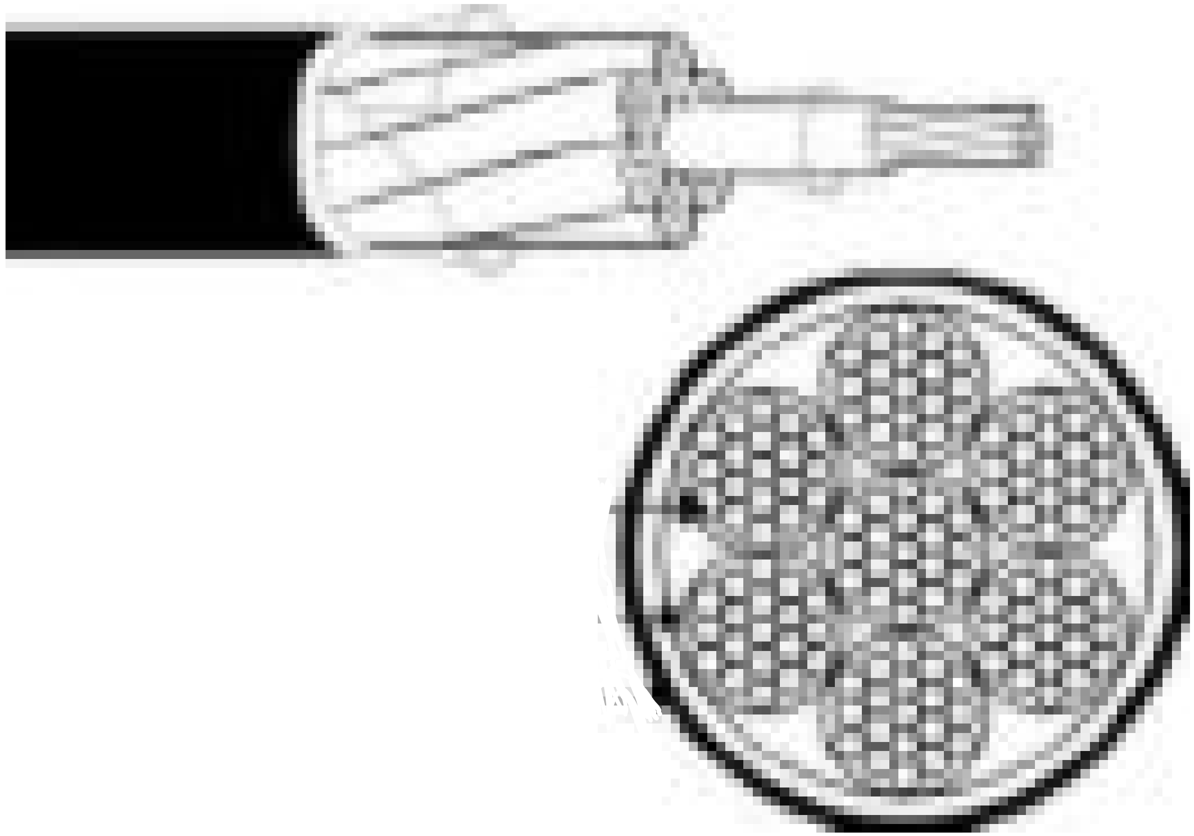
Behavior of k_I and k_R as a Function of Loss Tangent



Seawater

$$\mu = \mu_0 \quad ; \quad \varepsilon = 81\varepsilon_0 \quad ; \quad \sigma = 4\left[\frac{\text{mho}}{\text{m}}\right]$$

Conductors



"Good" Conductor

OHM'S LAW

$$\underset{\sim}{\mathbf{J}} = \sigma \underset{\sim}{\mathbf{E}} \Rightarrow \sigma \to \infty$$

Ordinary metal with very high values of σ approximate "perfect" conductors

Superconductive lead	$\sigma = 2.7 \times 10^{20}$	[mho/m]
Silver	$\sigma = 6.2 \times 10^{7}$	[mho/m]
Copper	$\sigma = 5.8 \times 10^{7}$	[mho/m]
Gold	$\sigma = 4.1 \times 10^{7}$	[mho/m]
Aluminum	$\sigma = 3.8 \times 10^{7}$	[mho/m]
Brass	$\sigma = 1.5 \times 10^{7}$	[mho/m]
Solder	$\sigma = 0.7 \times 10^{7}$	[mho/m]
Stainless steel	$\sigma = 0.1 \times 10^{7}$	[mho/m]
Graphite	$\sigma = 7 \times 10^{4}$	[mho/m]
Silicon	$\sigma = 1.2 \times 10^{3}$	[mho/m]
Sea water	$\sigma = 4$	[mho/m]
Distilled water	$\sigma = 2 \times 10^{-4}$	[mho/m]
Sandy soil	$\sigma = 10^{-5}$	[mho/m]
Granite	$\sigma = 10^{-6}$	[mho/m]
Bakelite	$\sigma = 10^{-9}$	[mho/m]
Diamond	$\sigma = 2 \times 10^{-13}$	[mho/m]
Polystyrene	$\sigma = 10^{-16}$	[mho/m]
Quartz	$\sigma = 10^{-17}$	[mho/m]

A perfect conductor is an idealized material in which no electric field can exits

Lossy Dielectrics

$\mathbf{D} = \underset{\sim}{\boldsymbol{\varepsilon}}\, \mathbf{E}$

$\underset{\sim}{\boldsymbol{\varepsilon}} = \varepsilon' - j\varepsilon''$

Can dissipate energy in oscillations of bound charge in a dielectric.

Can define an effective conductivity

$$\sigma_e = \omega\varepsilon''$$

Same effect as σ but from a different source

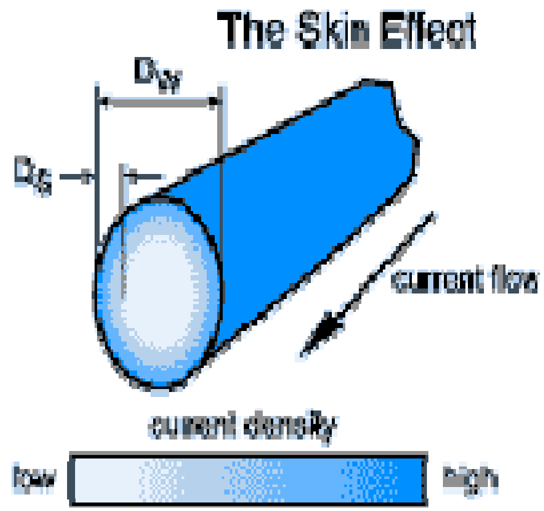
Table gives $[\tan\delta] = \dfrac{\varepsilon''}{\varepsilon'} = \dfrac{\sigma_e}{\omega\varepsilon'}$

	$\dfrac{\varepsilon'}{\varepsilon_0} = \varepsilon_r$	$\tan\delta$
Ice	4.2	0.1
Dry soil	2.8	0.07
Distilled water	80	0.04
Nylon	4	0.01
Teflon	2	0.0003
Glass	$4 \to 7$	0.0002
Dry wood	$1.5 \to 4$	0.01
Styrofoam	1.03	0.00003
Steak	40	0.3

Phase Lag caused by bound charge not "keeping up" with $\mathbf{E}$ Field

Skin Effect

The skin effect is the tendency of an **alternating electric current** to distribute itself within a **conductor** so that the current density near the surface of the conductor is greater than that at its core. That is, the electric current tends to flow at the "**skin**" of the conductor.



http://www.ee.surrey.ac.uk/Workshop/advice/coils/power_loss.html

For EM waves

$$\underset{\sim}{\mathbf{E}} = \hat{\mathbf{x}} E_0 e^{-k_I z} e^{-jk_R z}$$ Since $\underset{\sim}{\mathbf{J}} = \sigma \underset{\sim}{\mathbf{E}}$

$$\underset{\sim}{\mathbf{J}} = \hat{\mathbf{x}} \sigma E_0 e^{-k_I z} e^{-jk_R z}$$

Current is exponentially damped into material

Plane Waves in a Plasma

Plasma is a collection of (+) and (-) charged particles for which $<\rho_v>=0$

For low density plasma (few collisions)

$$\mu = \mu_0 \quad ; \quad \varepsilon = \varepsilon_0 \left[1 - \frac{\omega_p{}^2}{\omega^2} \right] \qquad \omega_p \rightarrow \text{Plasma freq.}$$

"Cold Plasma"

Note: ε is a function of $\omega \Rightarrow$ Dispersive medium

For $\omega > \omega_p$

$$k = \omega\sqrt{\mu_0 \varepsilon_0} \left[1 - \frac{\omega_p{}^2}{\omega^2} \right]^{\frac{1}{2}}$$

$$v = \frac{\omega}{k}$$

Plane Waves in a Plasma

For $\omega < \omega_p$ the wavenumber becomes imaginary

$$\underset{\sim}{\mathbf{k}} = -j\alpha = -j\omega\sqrt{\mu_0\varepsilon_0}\left[\frac{\omega_p{}^2}{\omega^2} - 1\right]^{\frac{1}{2}}$$

Then $\underset{\sim}{\mathbf{E}}(z) = \hat{x}E_0 e^{-j\underset{\sim}{\mathbf{k}}z} = \hat{x}E_0 e^{-\alpha z}$

and $\underset{\sim}{\mathbf{H}}(z) = \hat{y}\dfrac{\alpha}{j\omega\mu_0}E_0 e^{-\alpha z}$

Since **E** and **H** are both imaginary

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{E} \times \mathbf{H}^*\right] = 0$$

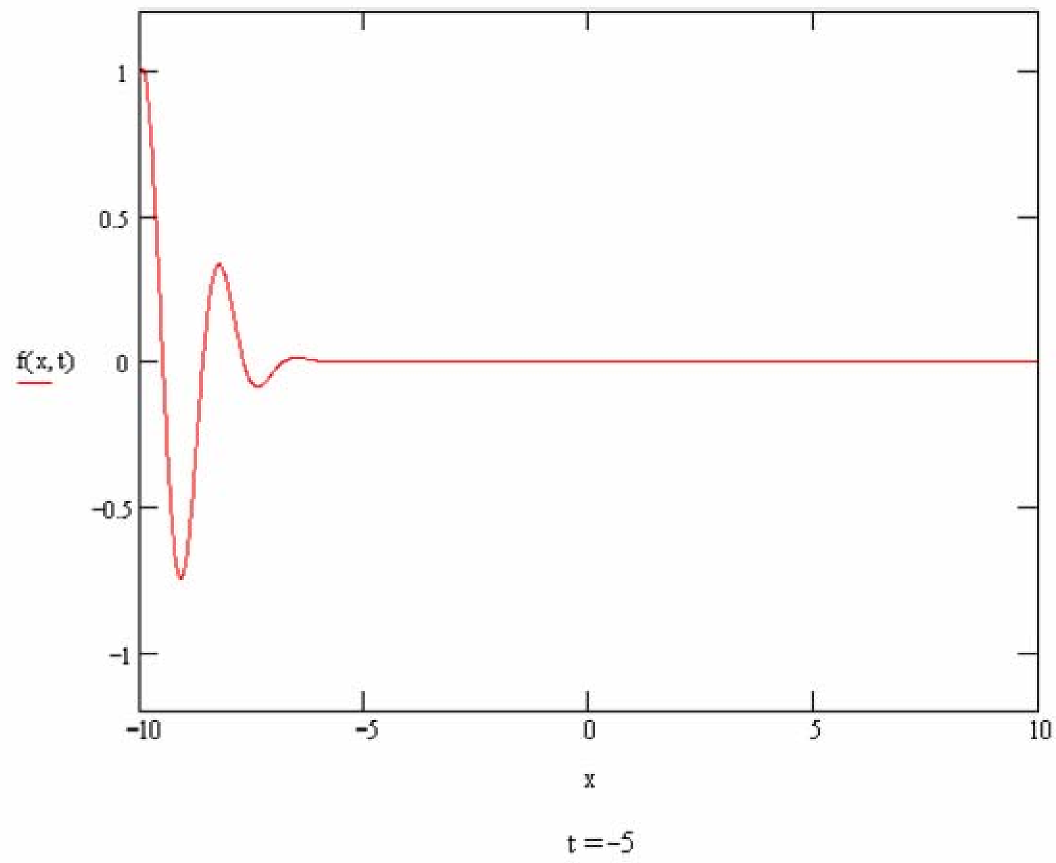
Evanescent Waves

Attenuation occurs but no real power is dissipated

Phase vs. Group Velocity

The **phase velocity** is the speed of the individual wave crests, whereas the **group velocity** is the speed of the wave packet as a whole (the envelope).

In this case, the phase velocity is greater than the group velocity.



http://www.geneseo.edu/~freeman/animations/phaseani_comp.avi

Phase vs. Group Velocity

Consider a plane wave propagating in the $+\hat{\boldsymbol{x}}$ direction

$$\boldsymbol{E}(x,t) = E_0 \cos\left(\omega t - kx\right)$$

with two frequencies $\omega_1 = \omega_0 - \Delta\omega$ and $\omega_2 = \omega_0 + \Delta\omega$

and with wavenumbers $k_1 = k_0 - \Delta k$ and $k_2 = k_0 + \Delta k$

For $\omega_1 \Rightarrow E_0 \cos\left((\omega_0 - \Delta\omega)t - (k_0 - \Delta k)x\right)$

For $\omega_2 \Rightarrow E_0 \cos\left((\omega_0 + \Delta\omega)t - (k_0 + \Delta k)x\right)$

Sum to get total field

$$\boldsymbol{E}(x,t)_{total} = E_0 \left\{\cos\left((\omega_0 - \Delta\omega)t - (k_0 - \Delta k)x\right) + \cos\left((\omega_0 + \Delta\omega)t - (k_0 + \Delta k)x\right)\right\}$$

Phase vs. Group Velocity

Using trig identities

$$\boldsymbol{E}(x,t)_{total} = 2E_0 \cos\left(\omega_0 t - k_0 x\right)\cos\left(\Delta\omega t - \Delta k x\right)$$

The 2 cosine factors give a slow variation superimposed over a more rapid one

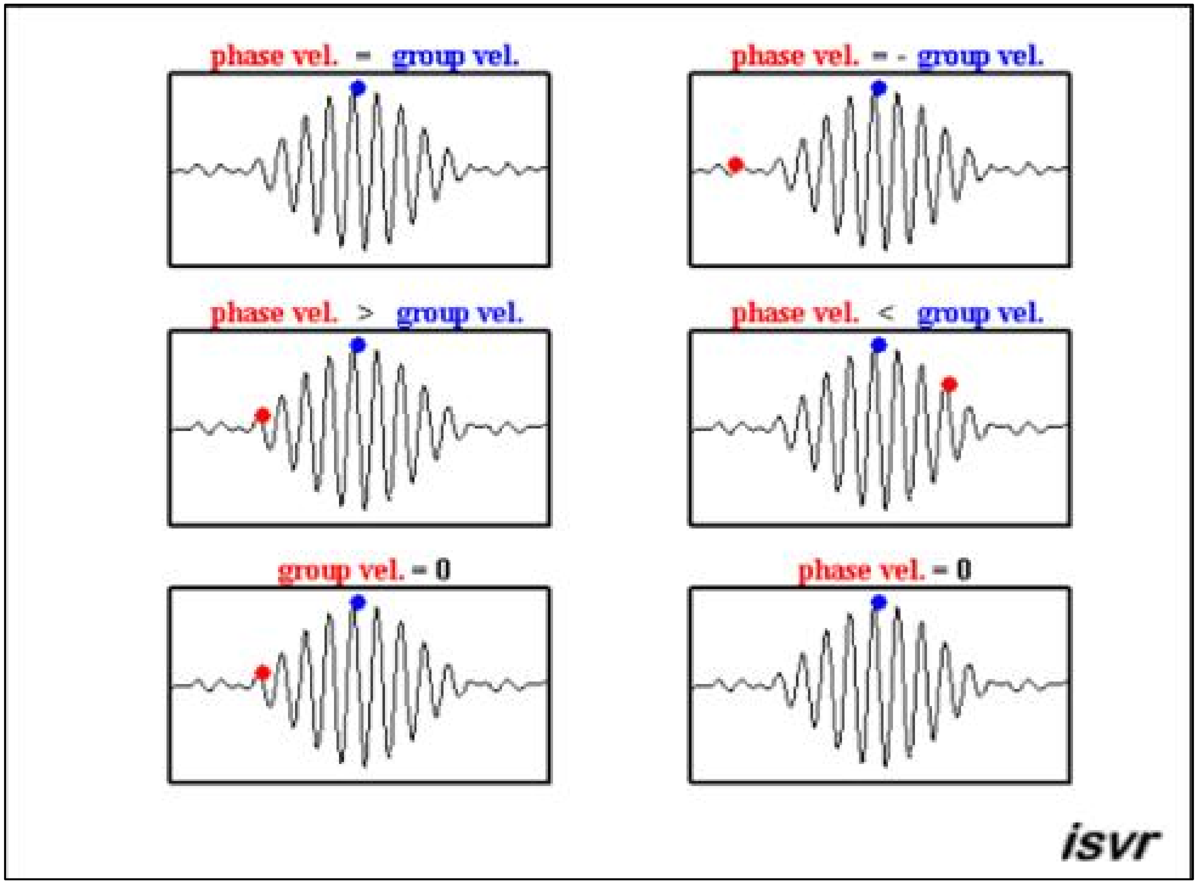
Constant phase on rapid (1st cos) term

$$\omega_0 t - k_0 x = \text{constant} \Rightarrow \frac{\delta x}{\delta t} = \frac{\omega_0}{k_0} = v_p$$ Phase Velocity

Constant argument on 2nd slower variation

$$\Delta\omega t - \Delta k x = \text{constant} \Rightarrow \frac{\delta x}{\delta t} = \frac{\Delta\omega}{\Delta k} = \frac{\delta\omega}{\delta k} = v_g$$ Group Velocity

Phase vs. Group Velocity



http://www.isvr.soton.ac.uk/SPCG/Tutorial/Tutorial/Tutorial_files/littlewavepackets.gif