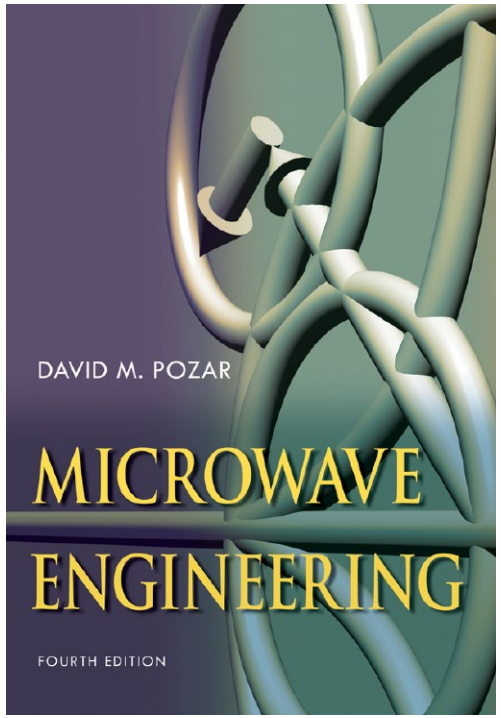
Adapted from notes by
Prof. Jeffery T. Williams

ECE 5317-6351
Microwave Engineering

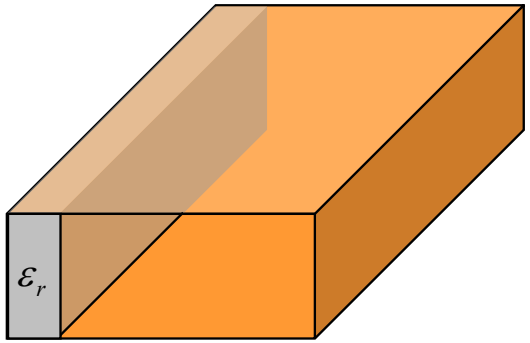
Fall 2026

Prof. David R. Jackson
Dept. of ECE



Notes 14

Transverse Resonance Method



Transverse Resonance Method

This is a general method that can be used to help us calculate various important quantities:

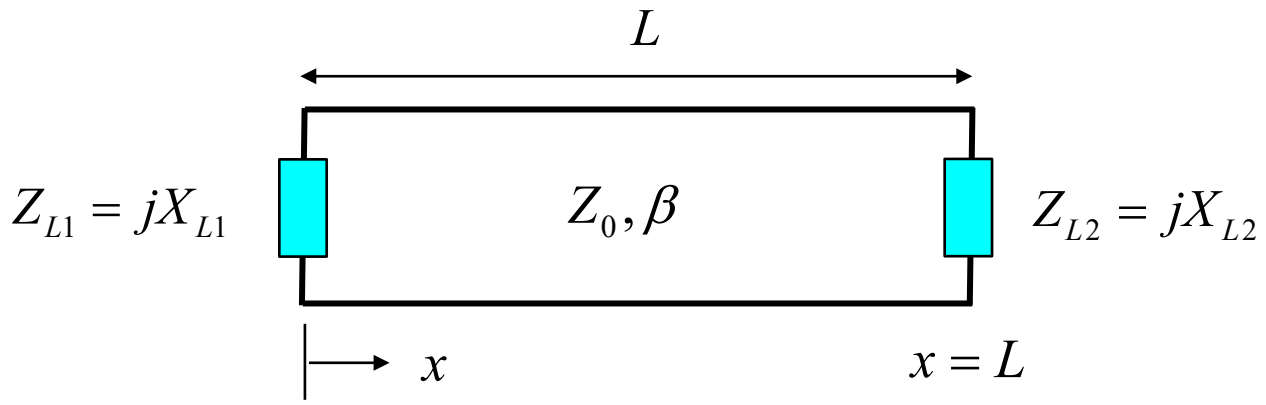
- Wavenumbers for complicated waveguiding structures (dielectric-loaded waveguides, surface waves, etc.)
- Resonance frequencies of resonant cavities (resonators)

The transverse resonance method involves establishing a reference plane and enforcing the KVL and KCL.

This leads to a "Transverse Resonance Equation (TRE)."

Transverse Resonance Method

To illustrate the method, consider a lossless resonator formed by a lossless transmission line with reactive loads at the ends.



Transmission line resonator

Note:
Having lossless (reactive) loads will give us real resonance frequencies.

We wish to find the resonance frequencies of this transmission-line resonator.

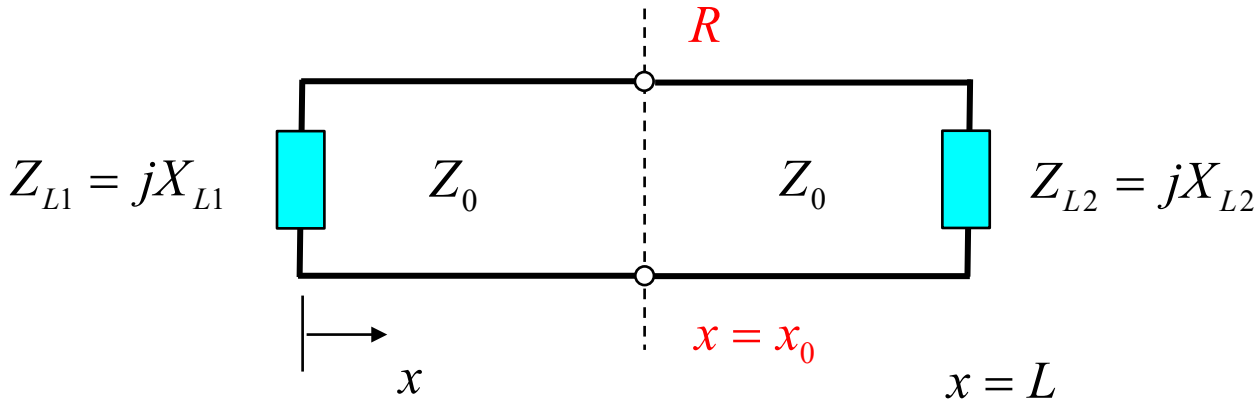
(Here we develop the method. We will do the actual algebra for this structure a little later as an example.)

A resonator can have nonzero fields at a resonance frequency, when there is no source.

Note: The transmission line in the model might be a TEN model for a waveguide type of problem.

Transverse Resonance Method (cont.)

We start by selecting an (arbitrary) reference plane R.

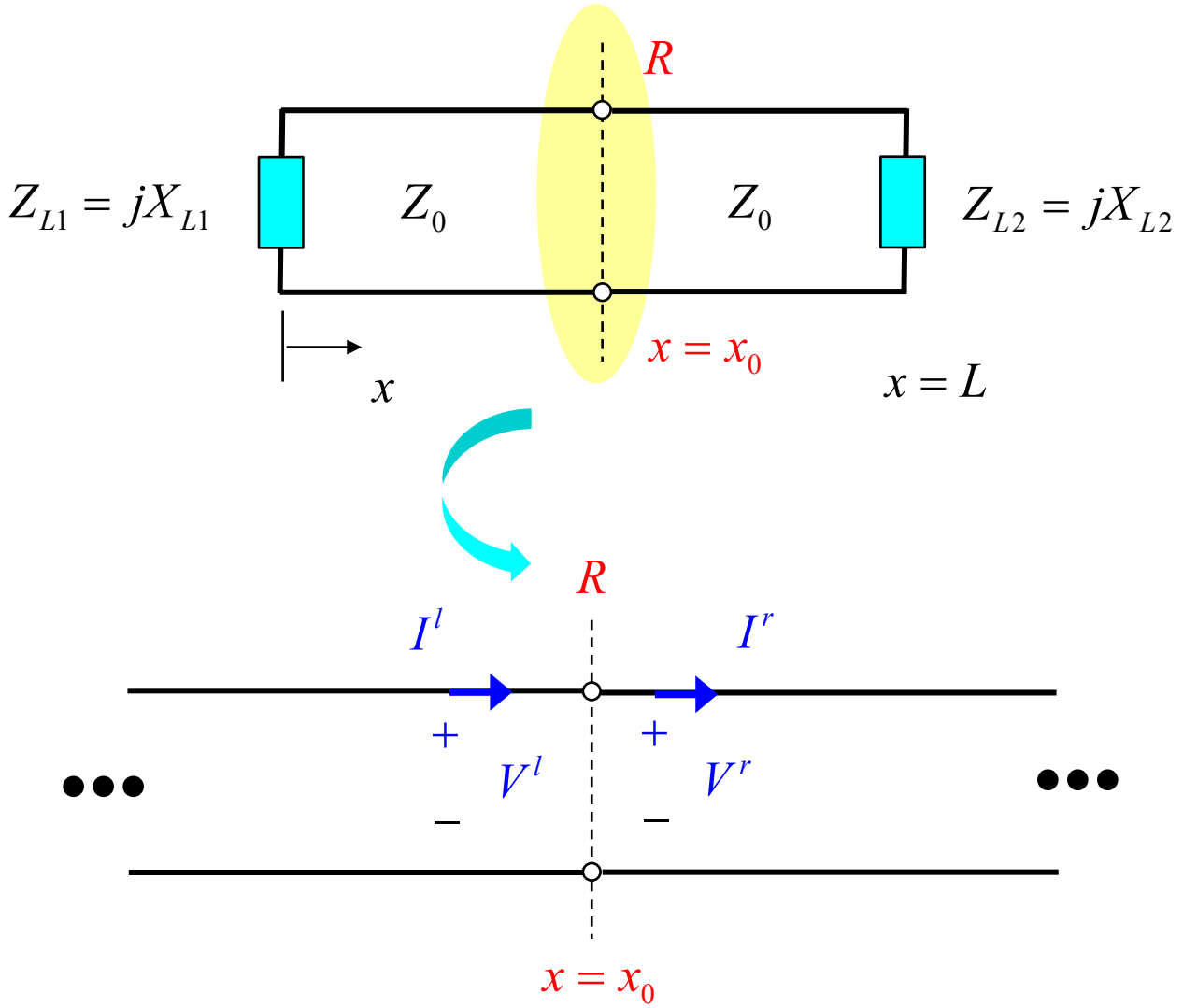


R = reference plane at arbitrary $x = x_0$

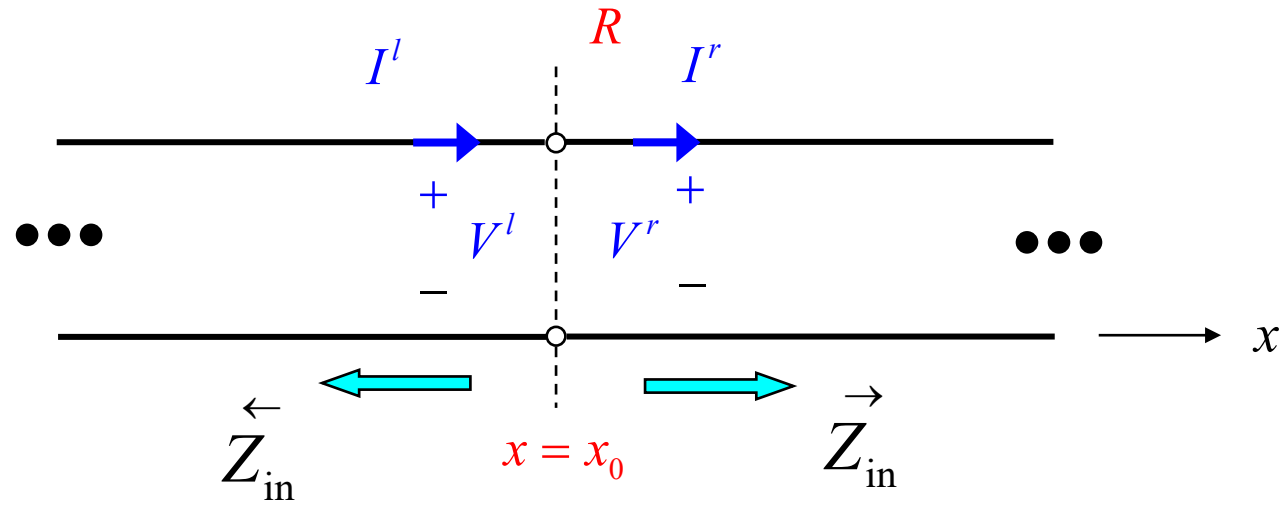
Note:
Although the location of the reference plane is arbitrary, a good choice will often simplify the derivation of the TRE and the complexity of the final TRE.

Transverse Resonance Method (cont.)

Examine the voltages and currents at the reference plane:



Transverse Resonance Method (cont.)



Define impedances:

$$\overrightarrow{Z}_{\text{in}} = \frac{V^r}{I^r}$$

$$\overleftarrow{Z}_{\text{in}} = \frac{V^l}{-I^l}$$

Boundary conditions:

$$V^r = V^l$$

$$I^r = I^l$$

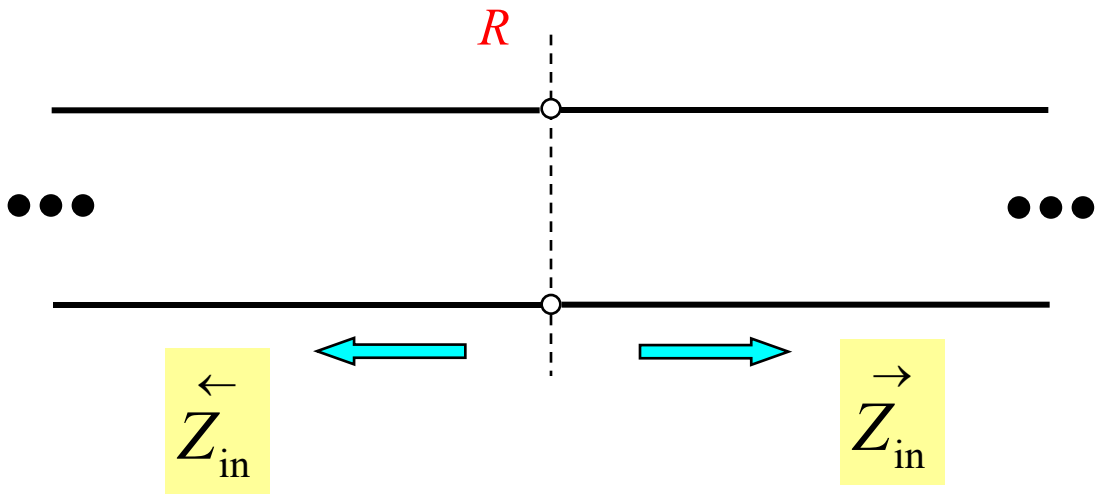
Hence:

$$\overleftarrow{Z}_{\text{in}} = -\overrightarrow{Z}_{\text{in}}$$

TRE

Transverse Resonance Method (cont.)

Summary



TRE:

$$\overleftarrow{Z}_{\text{in}} = -\overrightarrow{Z}_{\text{in}}$$

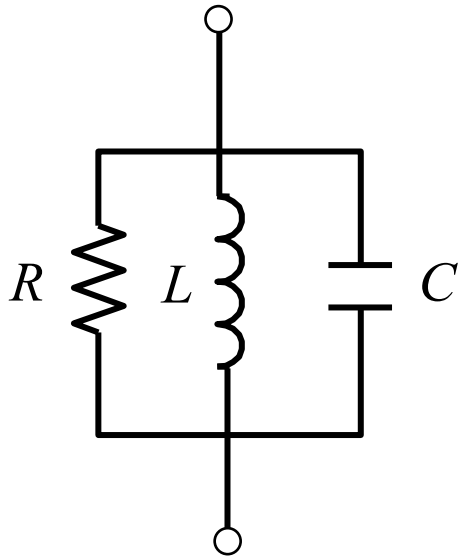
or

$$\overleftarrow{Y}_{\text{in}} = -\overrightarrow{Y}_{\text{in}}$$

RLC Resonator

Example:

Derive the resonance frequency of a parallel RLC resonator.



Note:
A complex resonance frequency corresponds to the fields being damped (decaying) with time.

$$\omega_0 = \omega_0' + j\omega_0''$$

$$\omega_0' = 2\pi f_0'$$

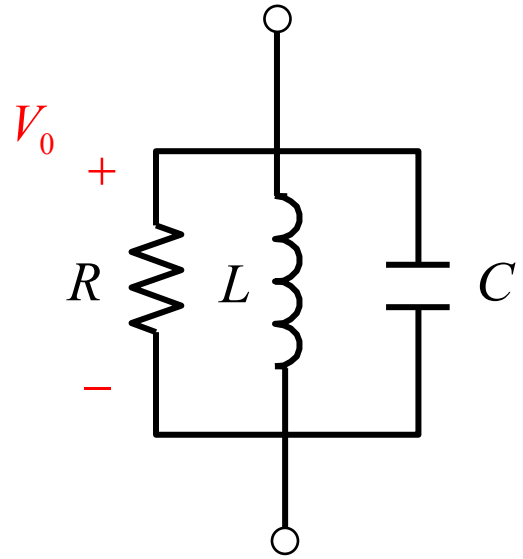
$$\omega = \omega_0 \ \text{(complex resonance frequency)}$$

At the resonance frequency, voltages and currents exist with no sources.

RLC Resonator (cont.)

In the time domain we have:

$$v(t) = \mathrm{Re}\left(V_0 e^{j\omega_0 t}\right)$$
$$= \mathrm{Re}\left(|V_0| e^{j\phi} e^{j\omega_0' t} e^{-\omega_0'' t}\right)$$
$$= |V_0| e^{-\omega_0'' t} \cos\left(\omega_0' t + \phi\right)$$



$$\omega_0 = \omega_0' + j\omega_0''$$

$$v(t) = |V_0| e^{-\omega_0'' t} \cos\left(\omega_0' t + \phi\right)$$

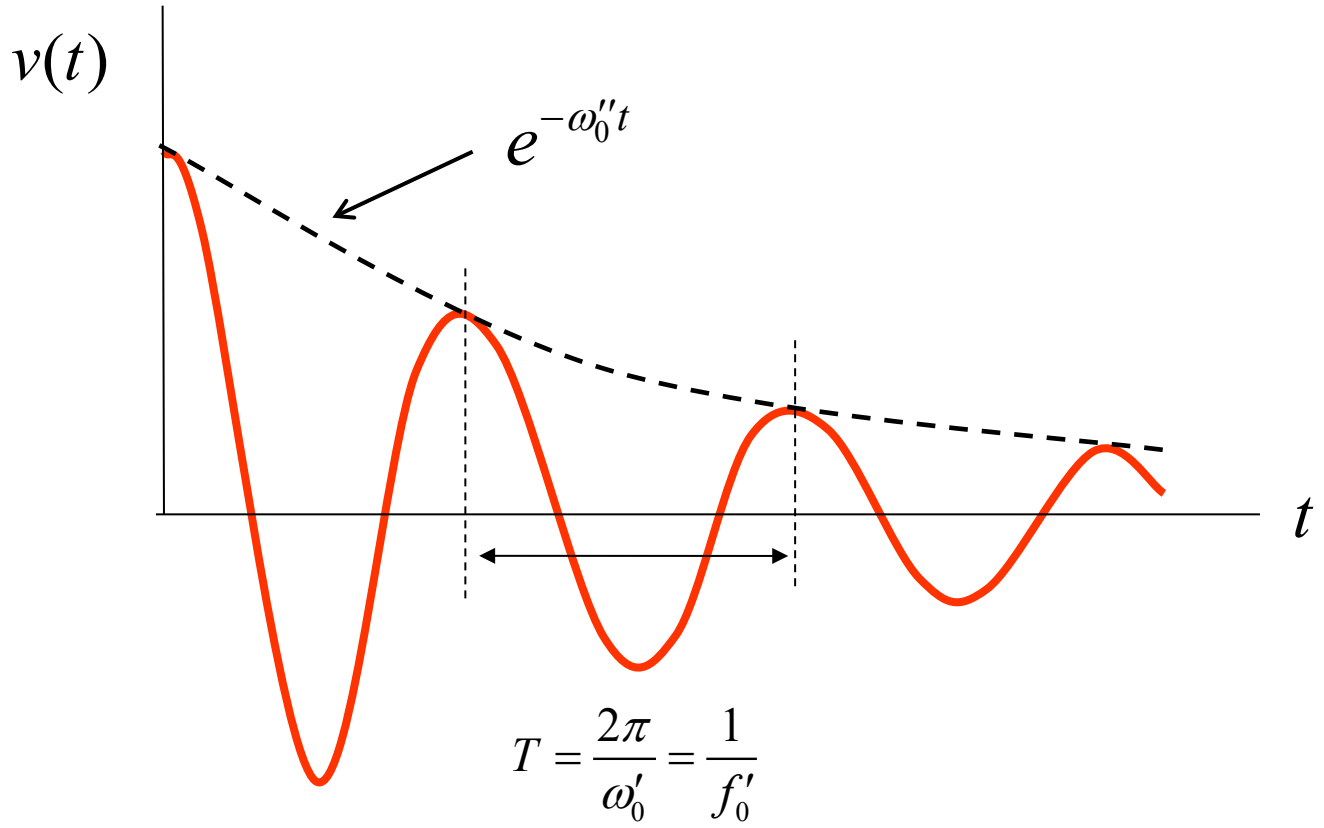
In the phasor domain:

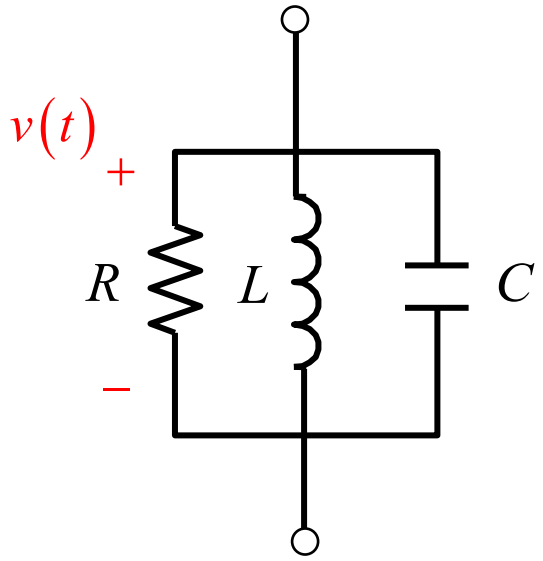
$$V_0 = |V_0| e^{j\phi}$$

Note: The phasor domain concept applies to complex frequencies as well as real frequencies.

RLC Resonator (cont.)

$$v(t) = |V_0| e^{-\omega_0'' t} \cos(\omega_0' t + \phi)$$

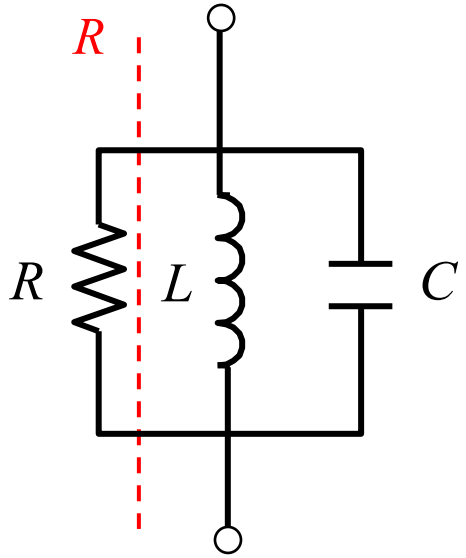




RLC Resonator (cont.)

Apply the Transverse Resonance Equation (TRE):

- A reference plane is first chosen (arbitrary).

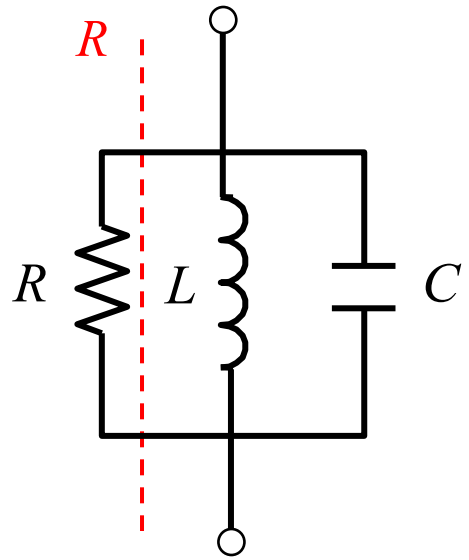


RLC Resonator (cont.)

The TRE is obtained.

$$\overleftarrow{Z}_{\text{in}} = -\overrightarrow{Z}_{\text{in}}$$

$$\overleftarrow{Y}_{\text{in}} = -\overrightarrow{Y}_{\text{in}}$$



$$G = \frac{1}{R}$$

$$\Rightarrow \quad G = -\left[\frac{1}{j\omega_0 L} + j\omega_0 C\right]$$

RLC Resonator (cont.)

Simplify the TRE:

$$G = -\frac{1}{j\omega_0 L} - j\omega_0 C$$



$$-j\omega_0 LG = 1 - \omega_0^2 LC$$



$$\omega_0^2 (LC) + \omega_0 (-jLG) - 1 = 0$$

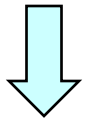


Solve the quadratic equation.

$$\omega_0 = \frac{jLG \pm \sqrt{-L^2G^2 + 4LC}}{2LC}$$

RLC Resonator (cont.)

$$\omega_0 = \frac{jLG \pm \sqrt{4LC - L^2G^2}}{2LC}$$



Factor out $4LC$ from the square root.

$$\omega_0 = j\left(\frac{G}{2C}\right) \pm \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{LG^2}{4C}}$$

For the lossless limit, $G \to 0$: $\omega_0 \to +\frac{1}{\sqrt{LC}}$ (must be a positive real number)

Hence, the plus sign is the correct choice.

RLC Resonator (cont.)

Hence, we have:

$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{LG^2}{4C}} + j\left(\frac{G}{2C}\right)$$

Complex resonance frequency

We can write this as $\omega_0 = \omega_0' + j\omega_0''$

where

$$\omega_0' = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{LG^2}{4C}} \qquad \omega_0'' = \left(\frac{G}{2C}\right)$$

RLC Resonator (cont.)

Ratio of imaginary and real parts of complex frequency (low-loss resonator):

$$\omega_0' = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{LG^2}{4C}} \approx \frac{1}{\sqrt{LC}} \qquad \omega_0'' = \left(\frac{G}{2C}\right)$$

so

$$\frac{\omega_0'}{\omega_0''} = \left(\frac{2C}{G}\right)\frac{1}{\sqrt{LC}} = 2R\sqrt{\frac{C}{L}} = \frac{2R}{L}\sqrt{LC} = 2\left(\frac{R}{\omega_0' L}\right)$$

or

$$\frac{\omega_0'}{\omega_0''} = 2Q$$

where

$$Q \equiv \frac{R}{\omega_0' L}$$

Quality factor of RLC resonator

Q of a General Resonator

The quality factor (Q) for a general resonator is defined as:

$$Q \equiv \omega_0' \frac{U}{P_d^{\text{ave}}}$$

$$U = U_E + U_H = \text{energy stored}$$

$$P_d^{\text{ave}} = \text{average power dissipated (averaged over one period } T\text{)}$$

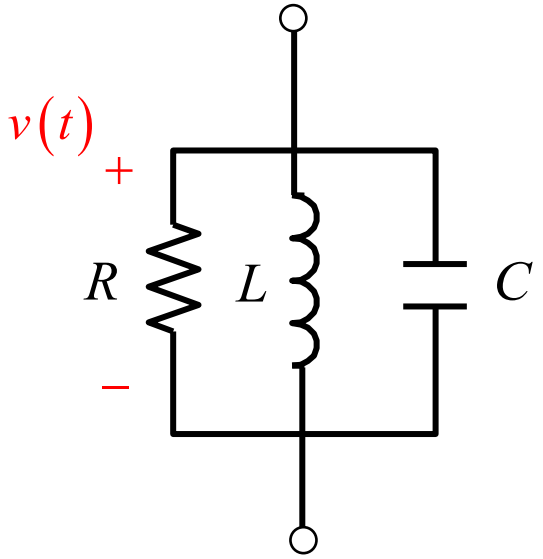
Note:
ω_0' is often denoted simply as ω_0 in this equation.

Q for RLC Resonator

For the RLC resonator we have:

$$Q \approx \frac{1}{\sqrt{LC}} \frac{U}{P_d^{\text{ave}}}$$

$$P_d^{\text{ave}} = \left\langle Gv^2(t) \right\rangle = G \left\langle v^2(t) \right\rangle$$



$$\left\langle v(t)v(t) \right\rangle = \frac{1}{T} \int_{t_0}^{t_0+T} v(t)^2 \, dt \qquad \left(T = \frac{2\pi}{\omega_0'} = \frac{1}{f_0'} \right)$$

$$= \frac{1}{T} \int_{t_0}^{t_0+T} |V_0|^2 e^{-2\omega_0'' t} \cos^2(\omega_0' t + \phi) \, dt$$

$$\approx \frac{1}{T} |V_0|^2 e^{-2\omega_0'' t_0} \int_{t_0}^{t_0+T} \cos^2(\omega_0' t + \phi) \, dt$$

$$= \frac{1}{T} |V_0|^2 e^{-2\omega_0'' t_0} \left(\frac{T}{2} \right)$$

$$= \frac{1}{2} |V_0|^2 e^{-2\omega_0'' t_0}$$

$$v(t) = |V_0| e^{-\omega_0'' t} \cos(\omega_0' t + \phi)$$

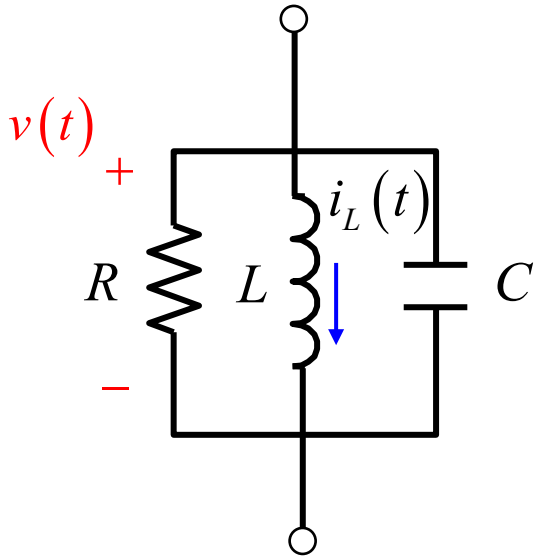
Hence

$$P_d^{\text{ave}} = \frac{1}{2} G |V_0|^2 e^{-2\omega_0'' t_0}$$

Q for RLC Resonator (cont.)

Similarly:

$$Q \approx \frac{1}{\sqrt{LC}} \frac{U}{P_d^{\text{ave}}}$$



$$v(t) = |V_0| e^{-\omega_0'' t} \cos(\omega_0' t + \phi)$$

$$\begin{aligned}
U &= \left(\frac{1}{2} C |v(t)|^2 + \frac{1}{2} L |i_L(t)|^2 \right) \\
&\approx \left(\frac{1}{2} C |V_0|^2 \cos^2(\omega_0' t) + \frac{1}{2} L \left| \frac{V_0}{\omega_0 L} \right|^2 \sin^2(\omega_0' t) \right) e^{-2\omega_0'' t_0} \\
&= \frac{|V_0|^2}{2} \left(C \cos^2(\omega_0' t) + L \left| \frac{1}{\omega_0 L} \right|^2 \sin^2(\omega_0' t) \right) e^{-2\omega_0'' t_0} \\
&= \frac{|V_0|^2}{2} \left(C \cos^2(\omega_0' t) + C \sin^2(\omega_0' t) \right) e^{-2\omega_0'' t_0} \\
&= \frac{|V_0|^2}{2} (C) e^{-2\omega_0'' t_0} \\
&= \frac{C}{2} |V_0|^2 \, e^{-2\omega_0'' t_0}
\end{aligned}$$

$$V_0 = |V_0| e^{j\phi}$$

$$I_L = \frac{V_0}{j\omega L} = \frac{|V_0| e^{j\phi}}{j\omega L} = \frac{|V_0| e^{j(\phi - \pi/2)}}{\omega L}$$

$$\Rightarrow \quad i_L(t) = \left| \frac{V_0}{\omega L} \right| e^{-\omega_0'' t} \cos(\omega_0' t + \phi - \pi/2) = \left| \frac{V_0}{\omega L} \right| e^{-\omega_0'' t} \sin(\omega_0' t + \phi)$$

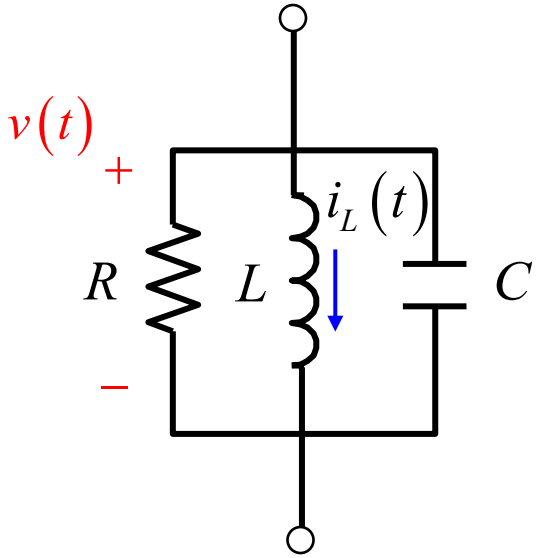
Q for RLC Resonator (cont.)

$$Q \approx \frac{1}{\sqrt{LC}} \frac{U}{P_d^{\text{ave}}}$$

We have:

$$P_d^{\text{ave}} = \frac{1}{2} G \left| V_0 \right|^2 e^{-2\omega_0'' t_0}$$

$$U = \frac{1}{2} C \left| V_0 \right|^2 e^{-2\omega_0'' t_0}$$



Hence, we have:

$$Q \approx \frac{1}{\sqrt{LC}} \frac{C}{G} = R\sqrt{\frac{C}{L}} = R\sqrt{LC}\,\frac{1}{L} = \frac{R}{\omega_0' L}$$

$$v(t) = \left| V_0 \right| e^{-\omega_0'' t} \cos\left(\omega_0' t + \phi\right)$$

so

$$Q \approx \frac{R}{\omega_0' L}$$

Note: This can also be written as $Q \approx \omega_0' RC$

General Q Formulas

These formulas hold for any resonator:

$$Q \equiv \omega_0' \frac{U}{P_d^{\text{ave}}}$$

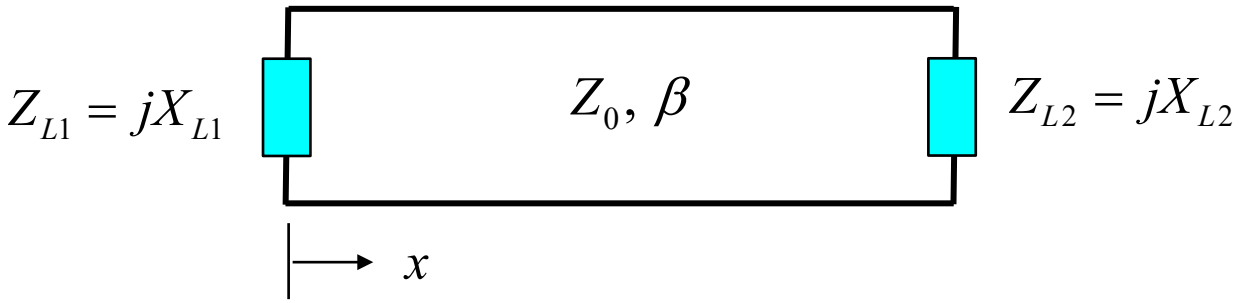
$$Q = \frac{1}{2}\left(\frac{\omega_0'}{\omega_0''}\right)$$

$$\omega_0 = \omega_0' + j\omega_0''$$

Transmission Line Resonator

Example:

Derive a transcendental equation for the resonance frequency of this lossless transmission-line resonator.



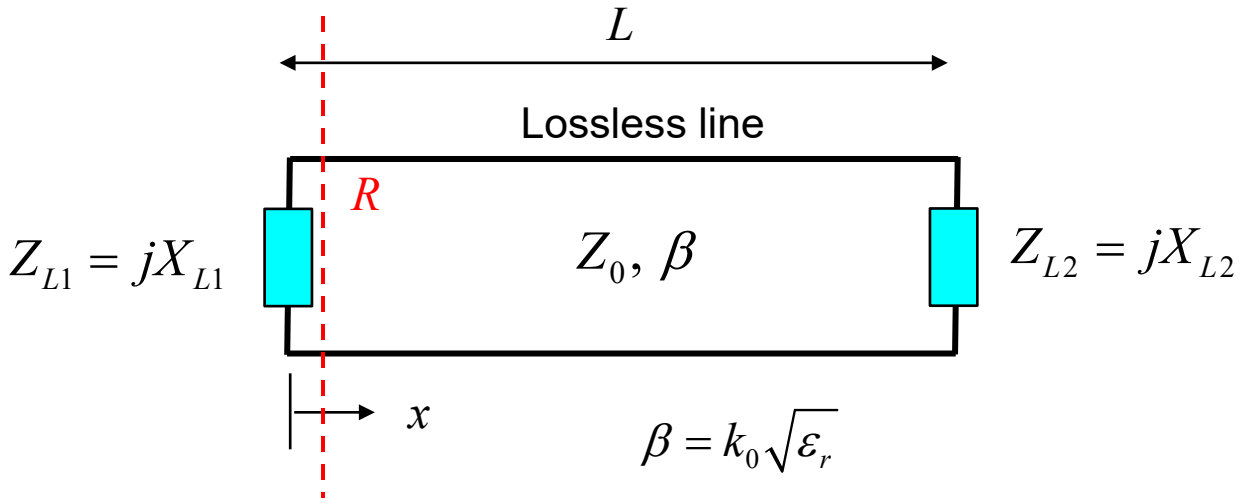
$$\beta = k = k_0\sqrt{\varepsilon_r}$$

We choose a reference plane at $x = 0^+$.

The resonance frequencies will be real if the loads are lossless.

Note: The load reactances may be functions of frequency.

Transmission Line Resonator (cont.)



Apply TRE: $\overleftarrow{Z}_{\text{in}} = -\overrightarrow{Z}_{\text{in}}$

$$\Rightarrow \quad Z_{L1} = -\left[Z_0 \left(\frac{Z_{L2} + jZ_0 \tan(\beta L)}{Z_0 + jZ_{L2} \tan(\beta L)} \right) \right]$$

Lossless: $k_z = \beta$

Transmission Line Resonator (cont.)

$$Z_{L1} = -\left[Z_0 \left(\frac{Z_{L2} + jZ_0 \tan(\beta L)}{Z_0 + jZ_{L2} \tan(\beta L)} \right) \right]$$

$$jX_{L1} = -\left[Z_0 \left(\frac{jX_{L2} + jZ_0 \tan(\beta L)}{Z_0 + j(jX_{L2}) \tan(\beta L)} \right) \right]$$

TEM: $\beta = k_1 = k_0\sqrt{\varepsilon_r}$

$$jX_{L1} = -\left[Z_0 \left(\frac{jX_{L2} + jZ_0 \tan\left(k_0 L\sqrt{\varepsilon_r}\right)}{Z_0 + j(jX_{L2}) \tan\left(k_0 L\sqrt{\varepsilon_r}\right)} \right) \right]$$

Note:

$$k_0 = 2\pi f_n^{\text{res}} \sqrt{\mu_0 \varepsilon_0}$$

$$jX_{L1}\left(Z_0 + j(jX_{L2}) \tan\left(k_0 L\sqrt{\varepsilon_r}\right)\right) = -Z_0\left(jX_{L2} + jZ_0 \tan\left(k_0 L\sqrt{\varepsilon_r}\right)\right)$$

Transmission Line Resonator (cont.)

After solving for the tangent term, we have:

$$\tan\left(k_0 L\sqrt{\varepsilon_r}\right) = \frac{Z_0\left(X_{L1}+X_{L2}\right)}{X_{L1}X_{L2}-Z_0^2}$$

Special cases:

$$X_{L1}=X_{L2}=0 \quad \Rightarrow \quad k_0 L\sqrt{\varepsilon_r}=n\pi, \quad n=1,2,\ldots$$

$$X_{L1}=X_{L2}=\infty \quad \Rightarrow \quad k_0 L\sqrt{\varepsilon_r}=n\pi, \quad n=1,2,\ldots$$

$$X_{L1}=0,\ X_{L2}\to\infty \quad \Rightarrow \quad k_0 L\sqrt{\varepsilon_r}=(2n-1)\pi/2, \quad n=1,2,\ldots$$

Transmission Line Resonator (cont.)

For the resonance frequencies, we have

$$k_0 = 2\pi f_n^{\text{res}} \sqrt{\mu_0 \varepsilon_0} = 2\pi \frac{f_n^{\text{res}}}{c}$$

$$c = 2.99792458 \times 10^8 \ [\text{m/s}]$$

f_n^{res} = resonance frequnecy of nth mode

We then have

$$\tan\left(2\pi L \frac{f_n^{\text{res}}}{c}\sqrt{\varepsilon_r}\right) = \frac{Z_0\left(X_{L2} + X_{L1}\right)}{X_{L1}X_{L2} - Z_0^2}$$

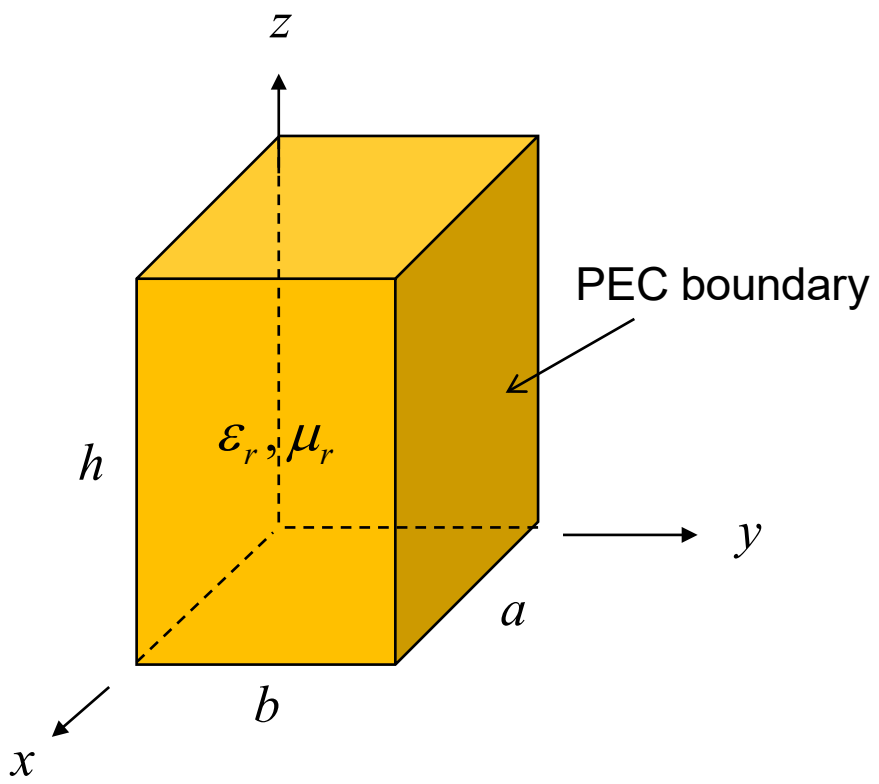
The RHS may also be a function of frequency.

Rectangular Resonator

Example:

Derive a transcendental equation for the resonance frequencies of a lossless rectangular resonator.

Orient the structure so that $b < a < h$.



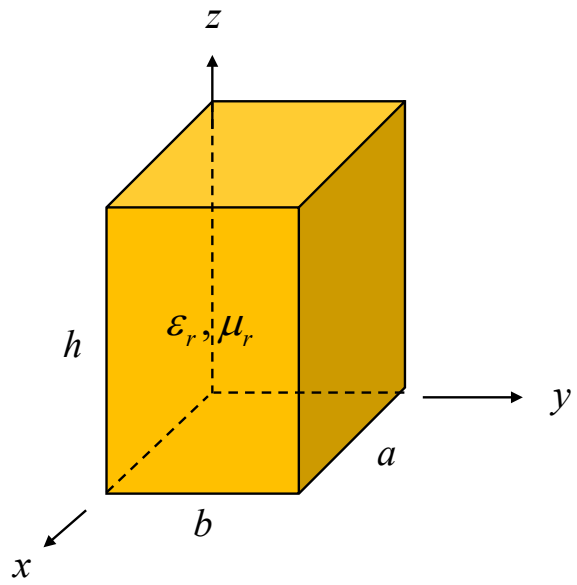
The structure is thought of as supporting rectangular waveguide modes bouncing back and forth in the z direction.

We have TM_{mnp} and TE_{mnp} modes.

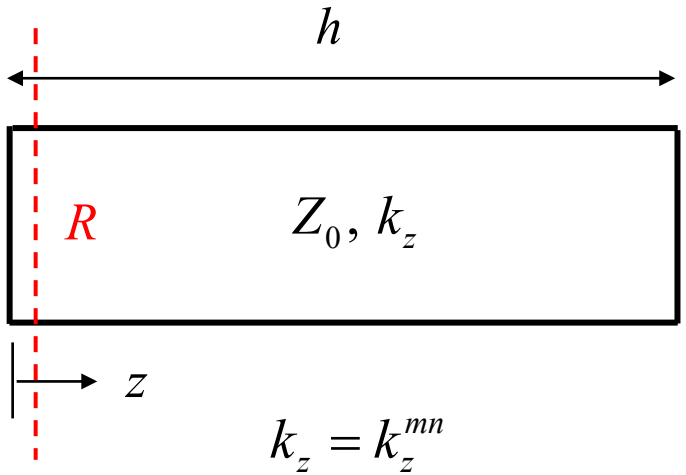
The index p describes the variation in the z direction.

Rectangular Resonator (cont.)

We use a Transverse Equivalent Network (TEN) to model any one of the waveguide modes:



We choose a reference plane at $z = 0^+$:



$$k_z = k_z^{mn}$$

$$Z_0 = Z_{\text{TE,TM}}^{(m,n)}$$

(Choose Z_0 to be the wave impedance.)

TRE:

$$\overleftarrow{Z}_{\text{in}} = -\overrightarrow{Z}_{\text{in}}$$

$$\overleftarrow{Z}_{\text{in}} = 0 \quad \text{(PEC bottom)}$$

Hence $\overrightarrow{Z}_{\text{in}} = 0$

Rectangular Resonator (cont.)

Hence, we have:

$$\vec{Z}_{\text{in}} = jZ_0 \tan(\beta h) = 0$$

$$\Downarrow$$

$$jZ_{\text{TE,TM}}^{(m,n)} \tan\left(k_z^{(m,n)} h\right) = 0$$

$$\Downarrow$$

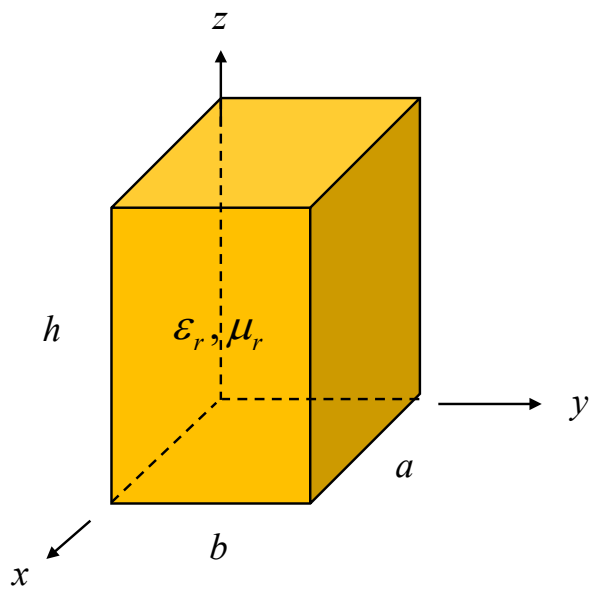
$$\tan\left(k_z^{(m,n)} h\right) = 0$$

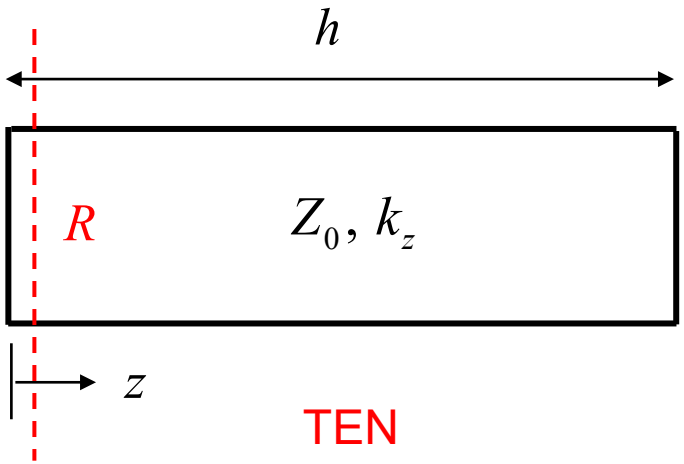
$$\Downarrow$$

$$k_z^{(m,n)} h = p\pi, \quad p = 0,1,2\ldots$$

$$\Downarrow$$

$$k_z^{(m,n)} = \frac{p\pi}{h}, \quad p = 0,1,2\ldots$$





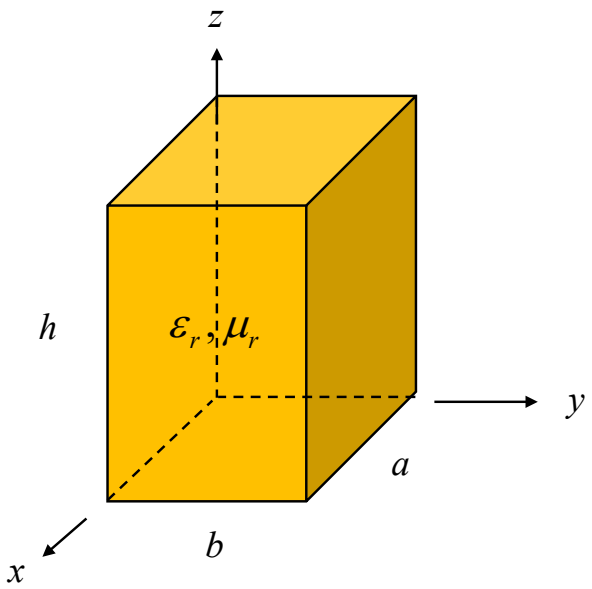
Note: $p \neq 0$ for TE modes (H_z must be zero on the top and bottom walls).

Rectangular Resonator (cont.)

$$k_z^{(m,n)} = \frac{p\pi}{h}, \quad p = 0,1,2\ldots$$

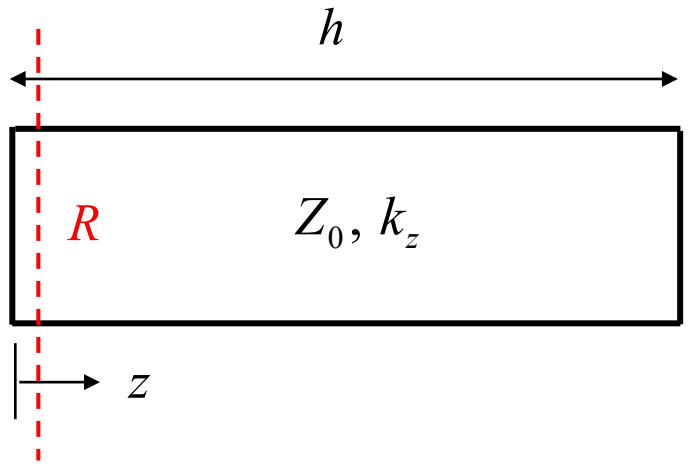
Also, we have

$$k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$



Hence, we have:

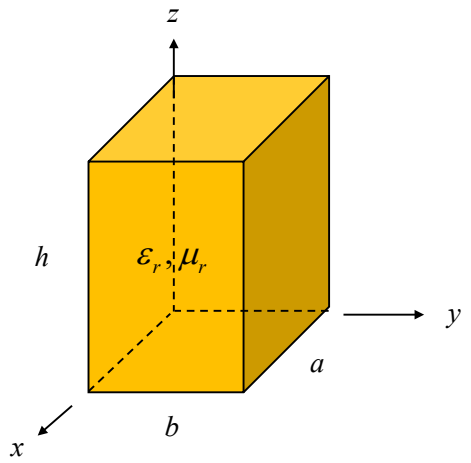
$$\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \frac{p\pi}{h}$$



Rectangular Resonator (cont.)

Solving for the wavenumber k we have:

$$k = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$



Hence, we have:

$$2\pi f_{mnp}\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

Note:
The TM_z and TE_z modes have the same resonance frequency.

or

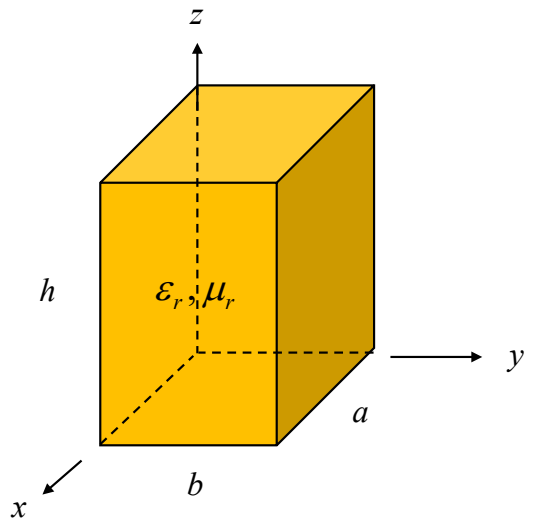
$$f_{mnp} = \frac{c}{2\pi}\frac{1}{\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

$$c = 2.99792458\times10^8 \text{ [m/s]}$$

Rectangular Resonator (cont.)

Summary

$$f_{mnp} = \frac{c}{2\pi}\frac{1}{\sqrt{\mu_r \varepsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$



TE$_{mnp}$ mode:

$m = 0,1,2,\ldots$

$n = 0,1,2,\ldots$

$p = 1,2,\ldots$

$(m,n) \neq (0,0)$

TM$_{mnp}$ mode:

$m = 1,2,\ldots$

$n = 1,2,\ldots$

$p = 0,1,2,\ldots$

Note: p cannot be zero for a TE$_z$ mode, since H_z must vanish at the top and bottom walls.

The lowest mode is the TE$_{101}$ mode.

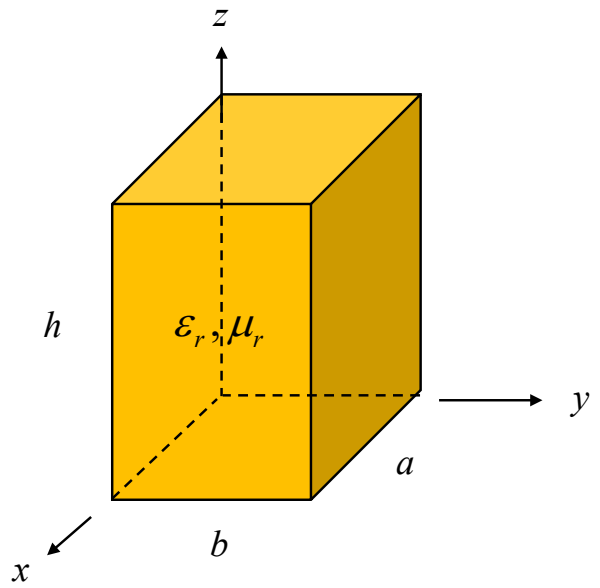
$$(b < a < h)$$

Rectangular Resonator (cont.)

TE$_{101}$ mode:

$$f_{101} = \frac{c}{2}\frac{1}{\sqrt{\mu_r \varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{h}\right)^2}$$

$$H_z(x,y,z) = H_0 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{h}\right)$$



Note: The sin choice ensures the boundary condition on the PEC top and bottom plates:

$$\underline{H} \cdot \underline{\hat{n}} = \pm H_z = 0$$

The other field components, E_y and H_x, can be found from H_z.

Rectangular Resonator (cont.)

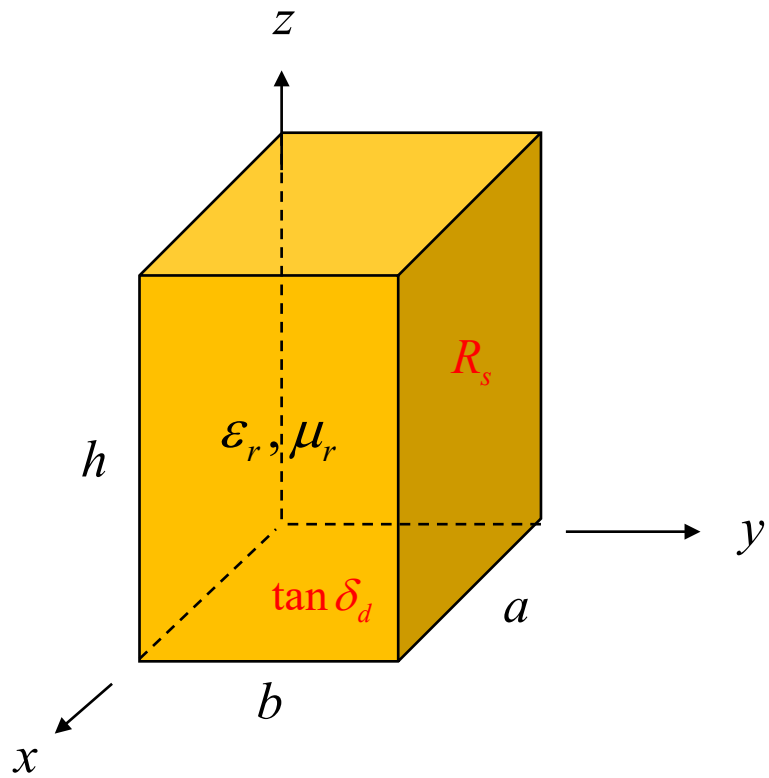
Here we examine the Q of the resonator:

$$Q \equiv \omega_0' \frac{U}{P_{\text{diss}}^{\text{ave}}}$$

$$P_{\text{diss}}^{\text{ave}} = P_d + P_c \qquad \text{(dielectric and conductor loss)}$$

$$\Rightarrow \quad \frac{1}{Q} = \frac{1}{\omega_0'} \frac{P_{\text{diss}}^{\text{ave}}}{U} = \frac{1}{\omega_0'} \frac{P_d}{U} + \frac{1}{\omega_0'} \frac{P_c}{U}$$

$$\Rightarrow \quad \frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c}$$



We now allow for dielectric and conductor loss in the resonator.

$$Q_d \equiv \omega_0' \frac{U}{P_d} \quad , \quad Q_c \equiv \omega_0' \frac{U}{P_c}$$

Q_d = dielectric Q
Q_c = conductor Q

Rectangular Resonator (cont.)

Q of TE_{101} mode

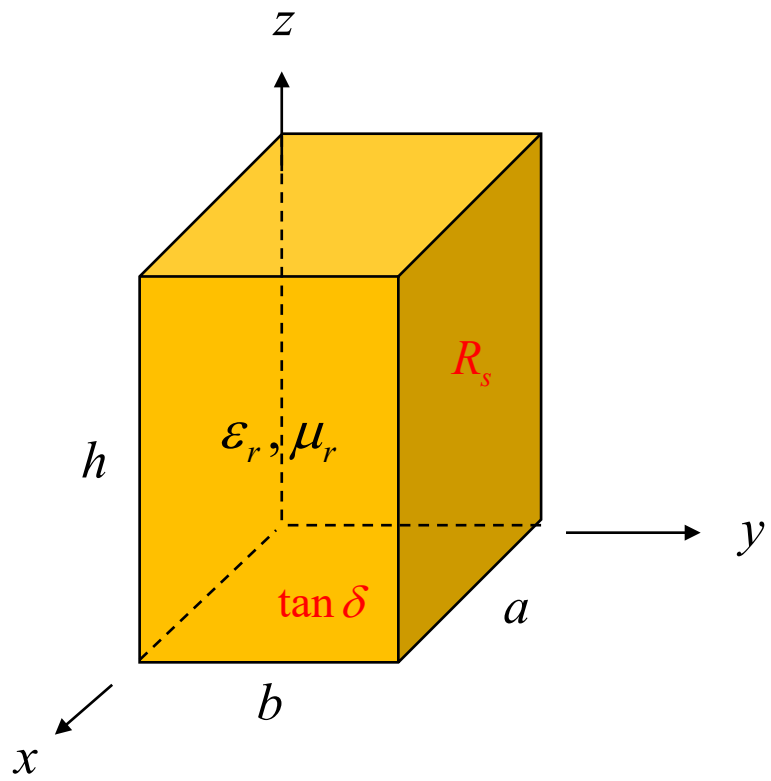
$$U = U_E + U_H = 2U_E$$

$$\varepsilon_c = \varepsilon_c' - j\varepsilon_c'' = \varepsilon_0\varepsilon_r\left(1 - j\tan\delta_d\right)$$

$$U_E = \frac{1}{4}\varepsilon_c'\int_V \left|\underline{E}\right|^2 dV = \frac{1}{4}\varepsilon_c'\int_V \left|E_y^2\right| dV \qquad \left(\frac{\varepsilon_c''}{\varepsilon_c'} = \tan\delta_d\right)$$

$$P_d = \frac{1}{2}\left(\omega\varepsilon_c''\right)\int_V \left|\underline{E}\right|^2 dV = \frac{1}{2}\left(\omega\varepsilon_c''\right)\int_V \left|E_y^2\right| dV \qquad \left(\omega = \omega_0'\right)$$

$$P_c = \frac{1}{2}R_s\int_S \left|J_s^2\right| dS$$



Note:
For conductivity loss
$\sigma = \omega\varepsilon_c' \tan\delta = \omega\varepsilon_c''$

$$H_z\left(x,y,z\right) = H_0\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{h}\right)$$

$$\Rightarrow \begin{cases} E_y\left(x,y,z\right) = E_0\sin\left(\dfrac{\pi x}{a}\right)\sin\left(\dfrac{\pi z}{h}\right) \\ H_x\left(x,y,z\right) = -j\dfrac{E_0}{Z_{\mathrm{TE}}^{(1,0)}}\sin\left(\dfrac{\pi x}{a}\right)\cos\left(\dfrac{\pi z}{h}\right) \end{cases}$$

$$H_0 = \left(\frac{j\pi}{\omega\mu a}\right)E_0$$

Rectangular Resonator (cont.)

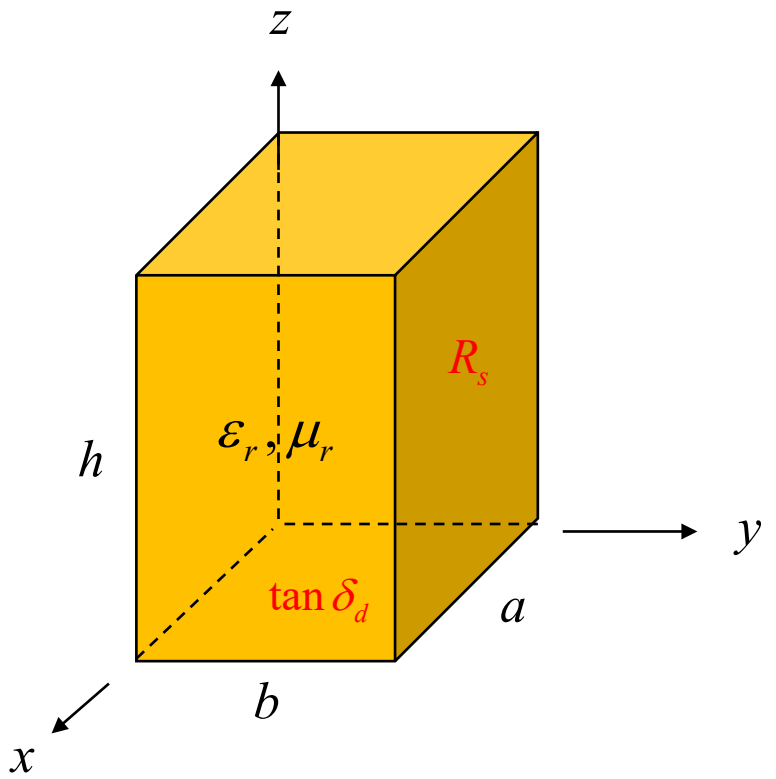
Results (from Pozar book):

Q of TE_{101} mode

$$U = \frac{1}{8}\varepsilon_c'(abh)|E_0|^2$$

$$P_d = \frac{1}{8}(\omega\varepsilon_c'')(abh)|E_0|^2 \qquad (\omega = \omega_0')$$

$$P_c = R_s|E_0|^2\frac{\lambda^2}{8\eta^2}\left(\frac{ab}{h^2}+\frac{bh}{a^2}+\frac{a}{2h}+\frac{h}{2a}\right) \qquad (\omega = \omega_0')$$



This gives:

$$Q_d = \frac{1}{\tan\delta_d}$$

(This holds for any mode.)

$$Q_c = \frac{1}{R_s}\frac{(kah)^3 b\eta}{2\pi^2}\frac{1}{2a^3b+2bh^3+a^3h+ah^3}$$

(This is for the TE_{101} mode.)

$$f_{101} = \frac{c}{2}\frac{1}{\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{h}\right)^2}$$

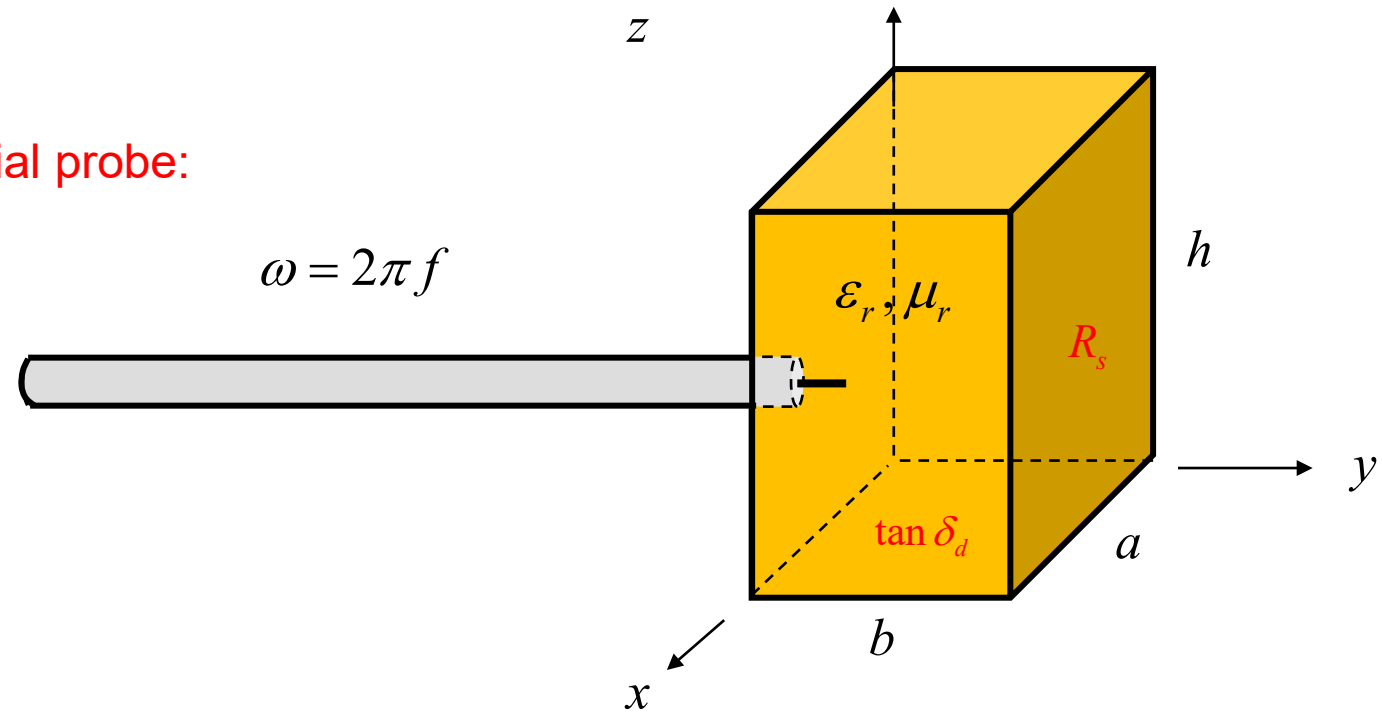
$$k = k_0\sqrt{\varepsilon_r\mu_r} \qquad \eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$$k_0 = \omega_0'\sqrt{\mu_0\varepsilon_0}$$

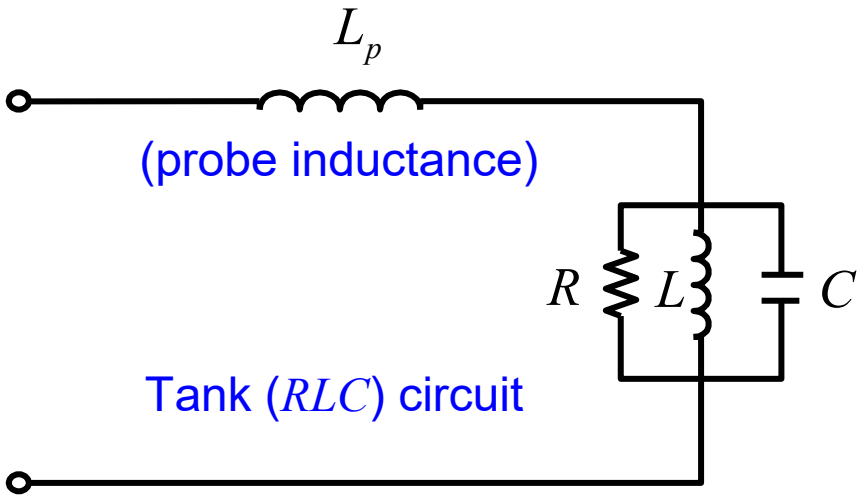
$$\omega_0' = 2\pi f_{101}$$

Excitation of Resonator

Practical excitation by a coaxial probe:



Circuit model:



Note:
The value of the circuit elements will depend on where the resonator is fed, and also the size of the feed probe. The values of Q and ω_0 do not depend on the feed.

$$Z_{\text{in}} = Z_{\text{probe}} + Z_{RLC}$$

Excitation of Resonator (cont.)

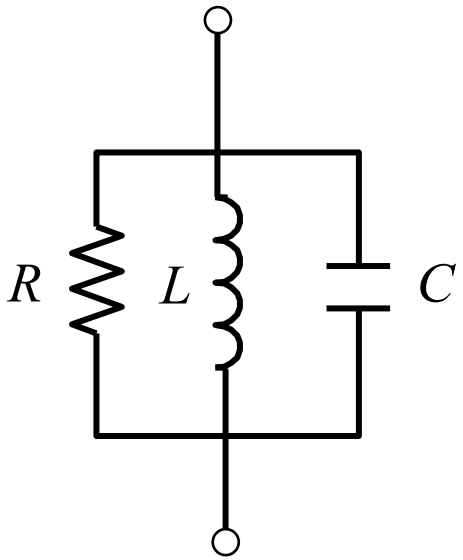
$$Z_{RLC} = \frac{R}{1 + jQ\left(\frac{\omega}{\omega_0'} - \frac{\omega_0'}{\omega}\right)}$$

(from circuit theory – see next slide)

$$\omega = 2\pi f$$

where

$$\omega_0' = \frac{1}{\sqrt{LC}} \qquad Q = \frac{R}{\omega_0' L} = \omega_0' RC$$



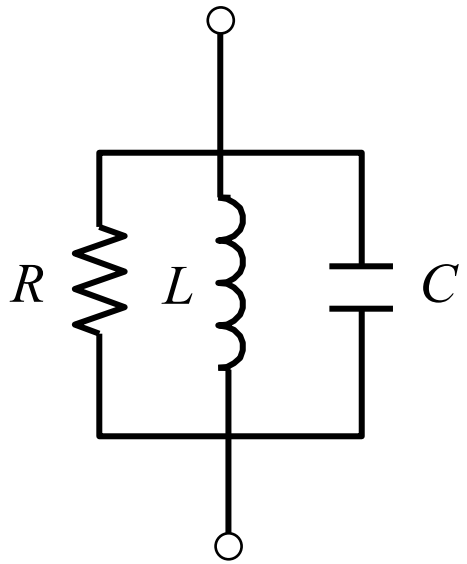
Approximate form (see slide 40):

$$Z_{RLC} \approx \frac{R}{1 + j2Q\left(\frac{\omega}{\omega_0'} - 1\right)}$$

Excitation of Resonator (cont.)

Derivation of Z_{RLC} formula:

Note:
Red color highlights where equivalent substitutions have been made. Blue color highlights where terms combine to unity.



$$
\begin{aligned}
Y_{RLC} &= G + j\omega C - j\frac{1}{\omega L} \\
&= G\left(1 + j\left(\frac{R}{\omega_0' L}\right)\frac{\omega_0' L}{R}\left(\omega\frac{C}{G} - \frac{1}{\omega}\frac{1}{LG}\right)\right) \\
&= G\left(1 + j\left(\frac{R}{\omega_0' L}\right)\left(\frac{\omega_0' L}{R}\omega\frac{C}{G} - \frac{\omega_0' L}{R}\frac{1}{\omega LG}\right)\right) \\
&= G\left(1 + jQ\left(\frac{\omega_0' L}{R}\omega\frac{C}{G} - \frac{\omega_0'}{\omega}\frac{L}{R}\frac{1}{LG}\right)\right) \\
&= G\left(1 + jQ\left(\frac{\omega_0' L}{R}\omega\frac{C}{G} - \frac{\omega_0'}{\omega}\right)\right) \\
&= G\left(1 + jQ\left(\frac{L\omega_0'^2}{R}\frac{\omega}{\omega_0'}\frac{C}{G} - \frac{\omega_0'}{\omega}\right)\right) \\
&= G\left(1 + jQ\left(\frac{\omega}{\omega_0'} - \frac{\omega_0'}{\omega}\right)\right)
\end{aligned}
$$

Excitation of Resonator (cont.)

Derivation of approximate form:

$$\frac{\omega}{\omega_0'}-\frac{\omega_0'}{\omega}=\frac{\omega_0'}{\omega}\left(\left(\frac{\omega}{\omega_0'}\right)^2-1\right)$$

$$\approx\left(\frac{\omega}{\omega_0'}\right)^2-1$$

$$=\left(\frac{\omega}{\omega_0'}-1\right)\left(\frac{\omega}{\omega_0'}+1\right)$$

$$\approx\left(\frac{\omega}{\omega_0'}-1\right)(2)$$

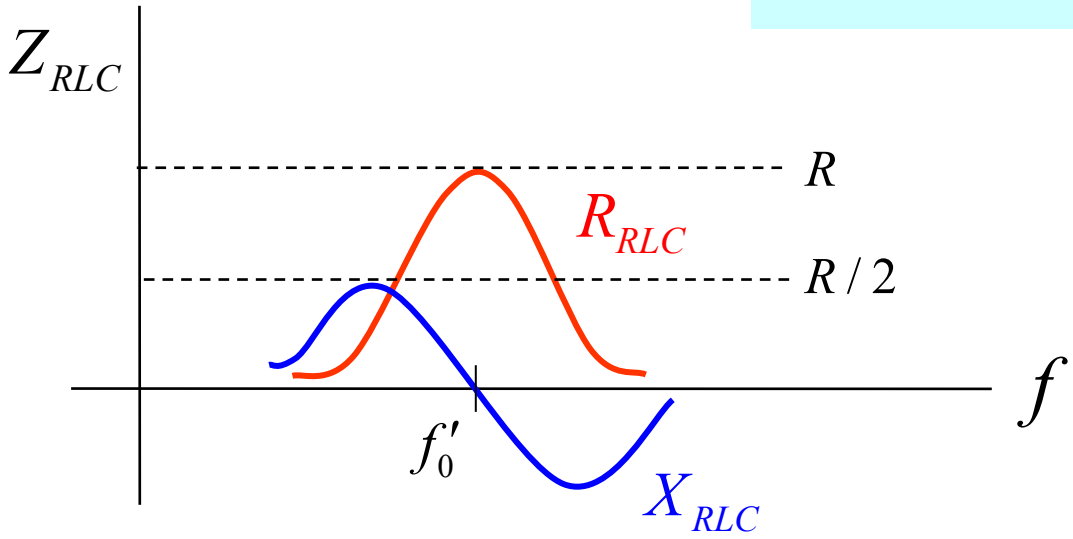
$\omega \approx \omega_0'$

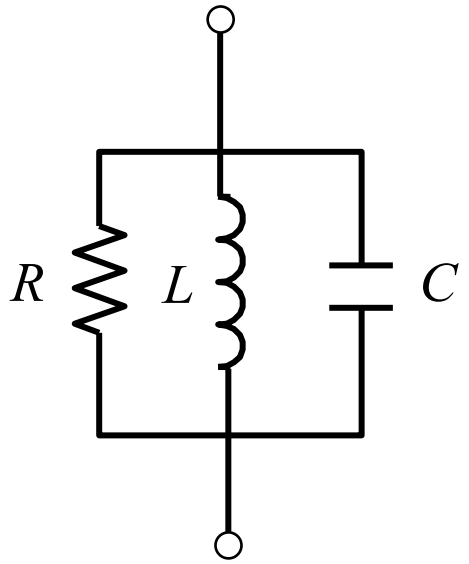
$$Z_{RLC}=\frac{R}{1+jQ\left(\frac{\omega}{\omega_0'}-\frac{\omega_0'}{\omega}\right)}$$

$$Z_{RLC}\approx\frac{R}{1+j2Q\left(\frac{\omega}{\omega_0'}-1\right)}$$

Excitation of Resonator (cont.)

$$Z_{RLC} \approx \frac{R}{1 + j2Q\left(\frac{\omega}{\omega_0'} - 1\right)}$$





Lossless resonator:

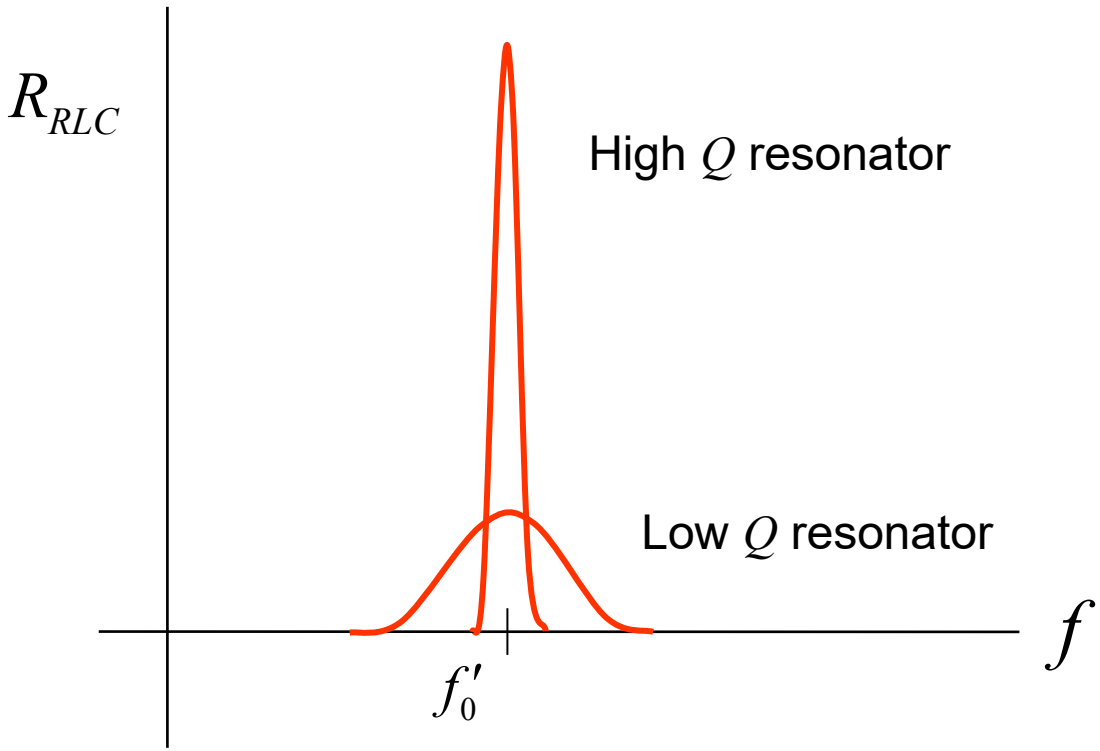
$$R \to \infty$$

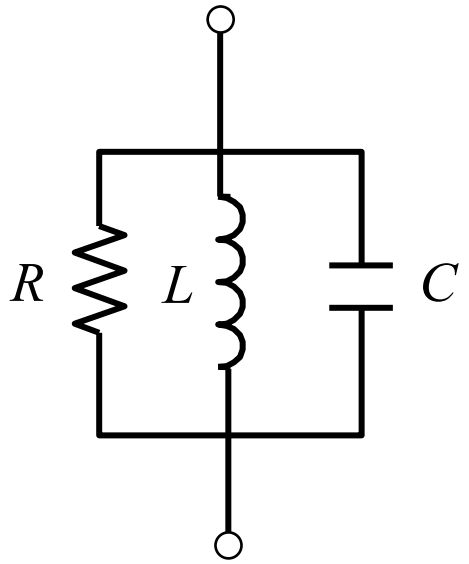
$$Q \to \infty$$

A larger Q of the resonator (less loss), means a more sharply peaked response, and a larger R value:

$$R = Q\omega_0' L$$

Excitation of Resonator (cont.)

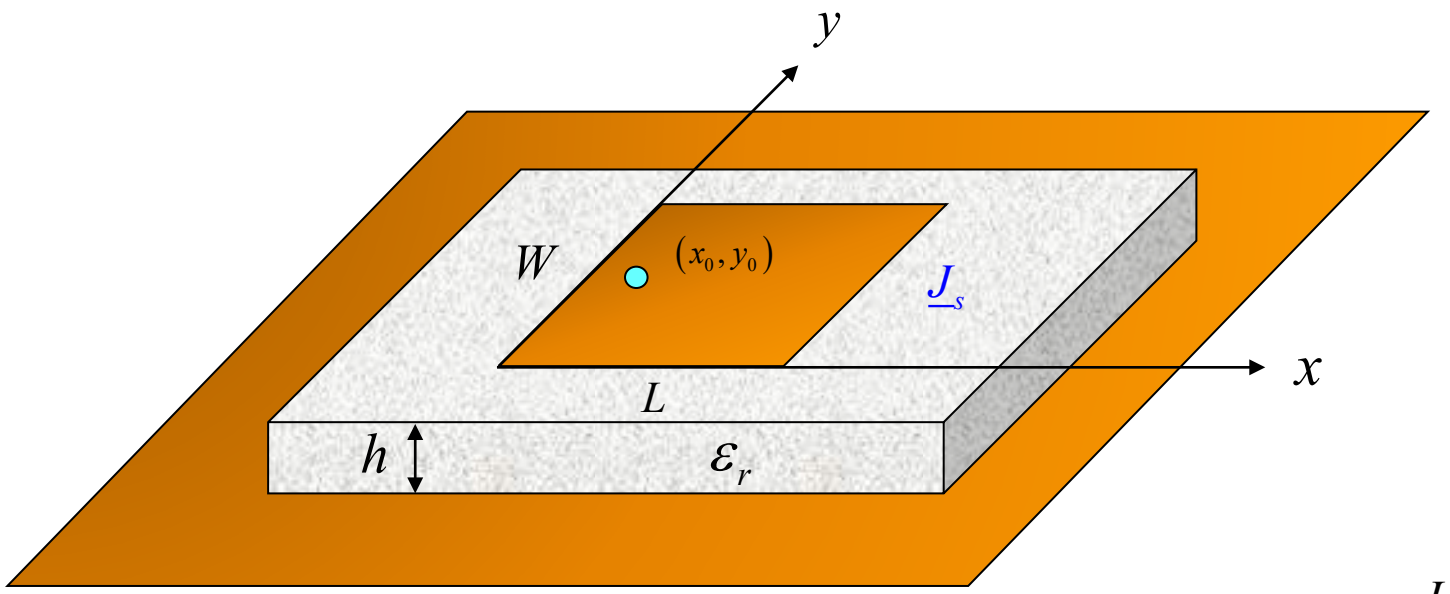


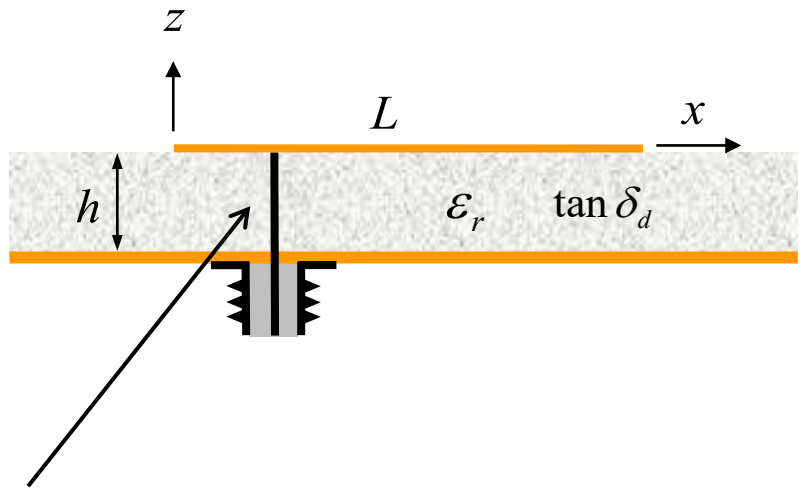


Note:
A larger Q of the resonator (less loss), means a more sharply peaked response, and a larger R value.

Microstrip Patch Antenna

A microstrip patch antenna acts as a lossy cavity resonator
(The "loss" is due to radiation, as well as actual dielectric and conductor loss.)

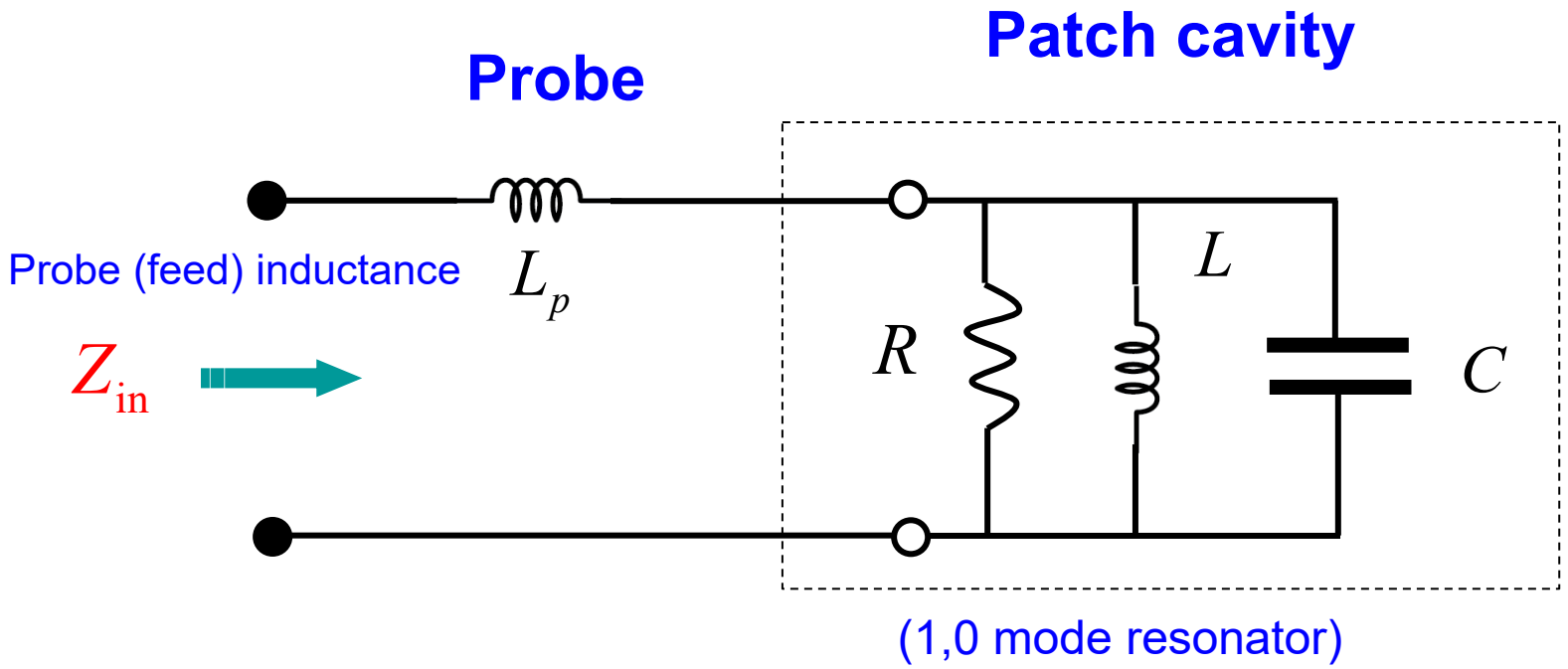




I_0 current feed at (x_0, y_0)

Microstrip Patch Antenna Model

CAD Circuit model of patch antenna (ECE 6345):



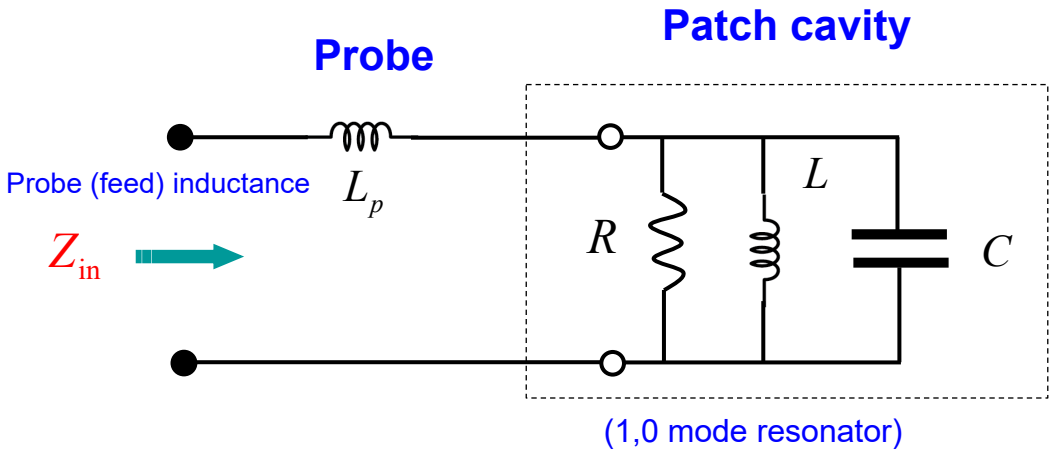
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{R}{\omega_0 L} = \omega_0 RC$$

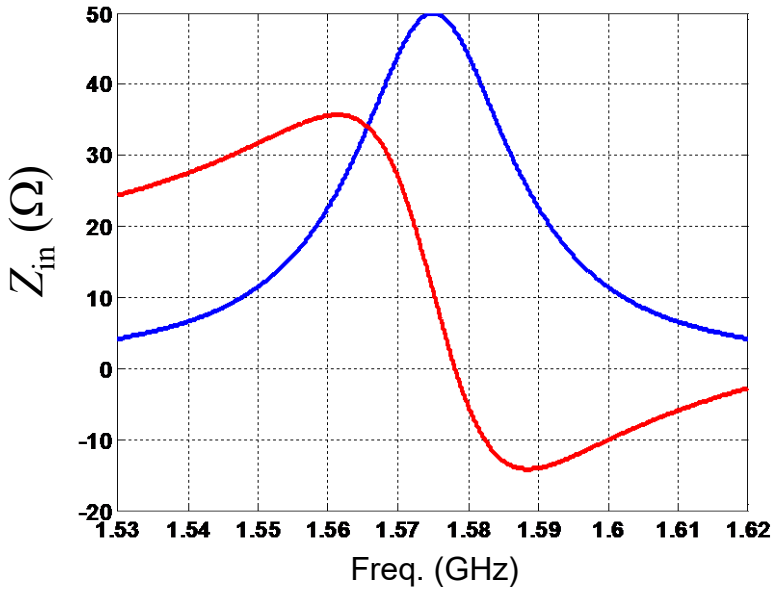
$$Z_{in} \approx j\omega L_p + \frac{R}{1 + jQ\left(\frac{f}{f_0} - \frac{f_0}{f}\right)}$$

Resonance frequency: $k_0 L_e \sqrt{\varepsilon_r'} = \pi$

Microstrip Patch Antenna Model (cont.)



CAD Circuit model of patch

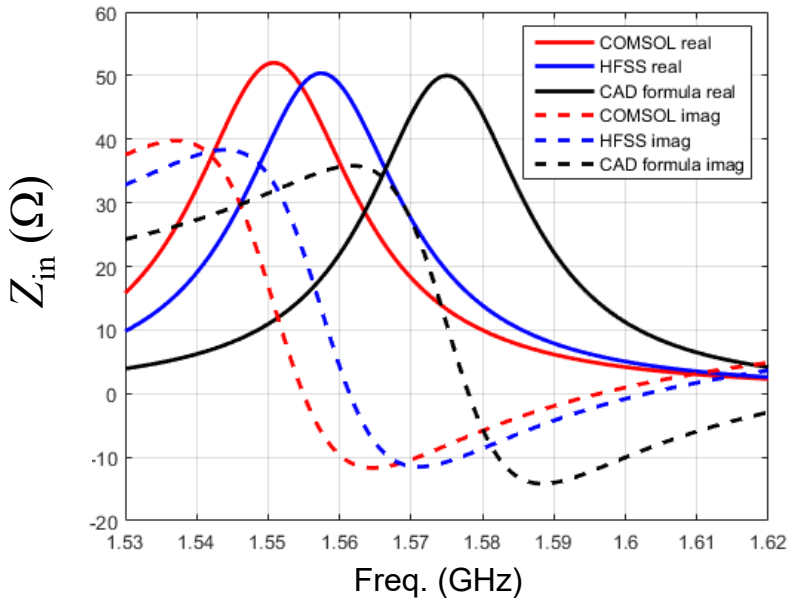


Patch	Feed
$\varepsilon_r = 2.2$	$x_0 = 1.85$ cm
$\tan\delta = 0.001$	$y_0 = W/2$
$h = 1.524$ mm	$a = 0.635$ mm

$L = 6.255$ cm

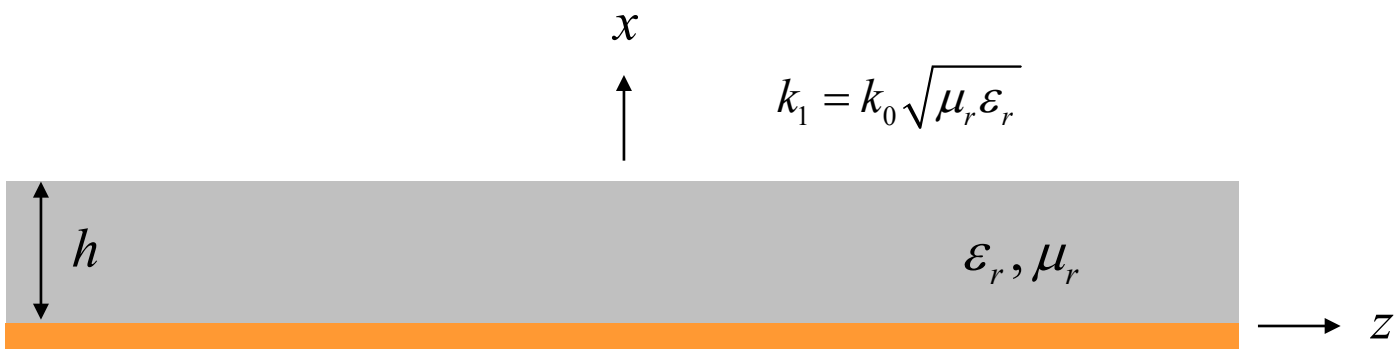
$W / L = 1.5$

$\sigma = 3.0 \times 10^7$ S/m



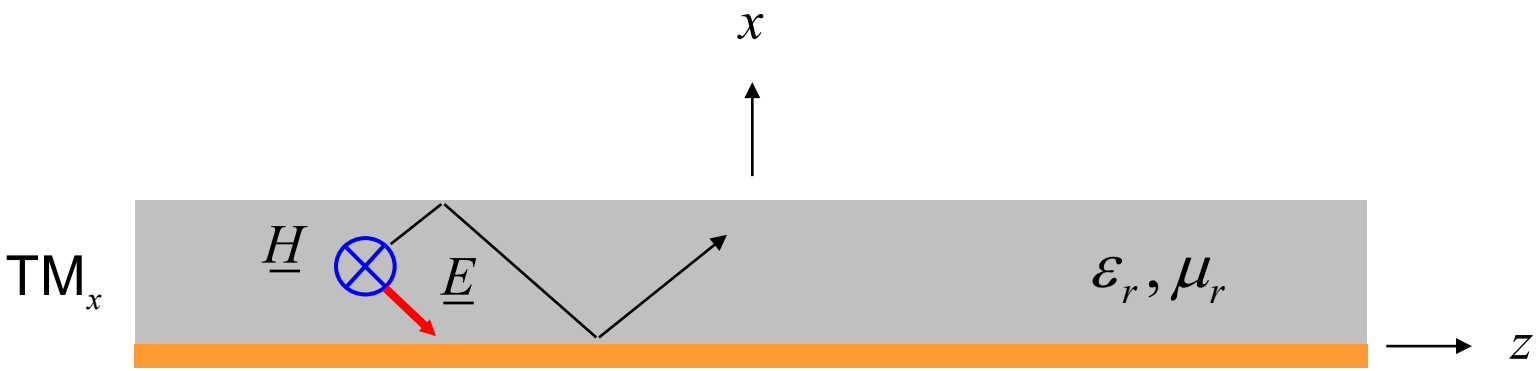
Grounded Dielectric Slab

Derive a transcendental equation for wavenumber of the TM_x surface waves by using the TRE.



Assumption: There is no variation of the fields in the y direction, and propagation is along the z direction.

Grounded Dielectric Slab (cont.)



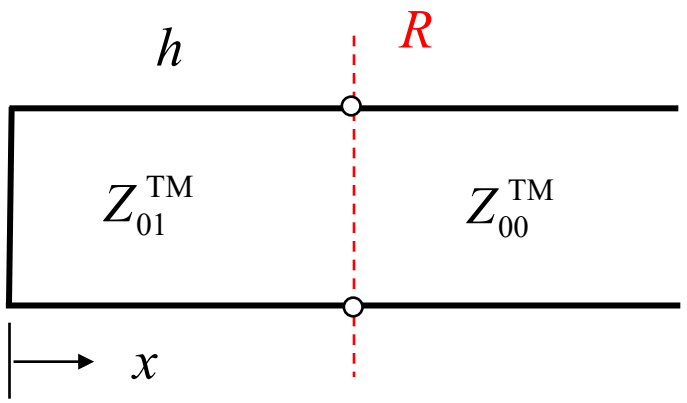
$$Z^{\mathrm{TM}_x} \equiv \frac{E_z}{-H_y} \quad \text{(defined for a wave going in the } +x \text{ direction)}$$

$$Z_{01}^{\mathrm{TM}} = \frac{k_{x1}}{\omega\varepsilon_1} \qquad Z_{00}^{\mathrm{TM}} = \frac{k_{x0}}{\omega\varepsilon_0}$$

We think of the transmission lines in the TEN as running in the x direction.

TM_x Surface-Wave Solution

TEN:



The reference plane R is chosen at the interface.

$$Z_{01}^{\mathrm{TM}} = \frac{k_{x1}}{\omega\varepsilon_1} \qquad Z_{00}^{\mathrm{TM}} = \frac{k_{x0}}{\omega\varepsilon_0}$$

$$k_{x1} = \sqrt{k_1^2 - k_z^2}$$

$$k_{x0} = (k_0^2 - k_z^2)^{1/2} = -j\sqrt{k_z^2 - k_0^2} = -j\alpha_{x0}$$

$$\overleftarrow{Z}_{\mathrm{in}} = jZ_{01}^{\mathrm{TM}} \tan(k_{x1}h), \qquad \overrightarrow{Z}_{\mathrm{in}} = Z_{00}^{\mathrm{TM}}$$

TM_x Surface-Wave Solution (cont.)

TRE:

$$\overleftarrow{Z}_{\text{in}} = -\overrightarrow{Z}_{\text{in}}$$

$$\Downarrow$$

$$jZ_{01}^{\text{TM}} \tan(k_{x1}h) = -Z_{00}^{\text{TM}}$$

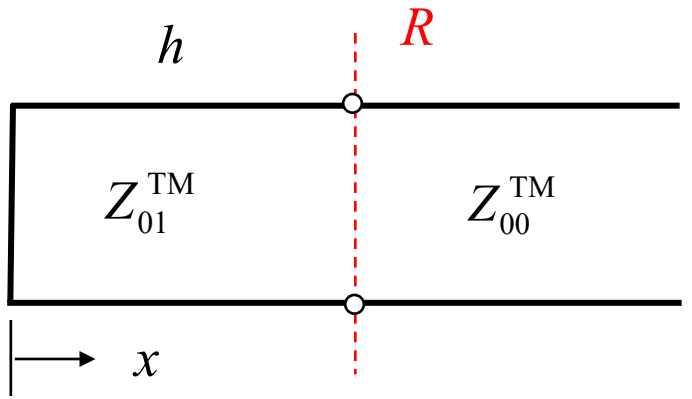
$$\Downarrow$$

$$j\frac{k_{x1}}{\omega\varepsilon_1}\tan(k_{x1}h) = -\frac{k_{x0}}{\omega\varepsilon_0}$$

$$\Downarrow$$

$$-j\left(\frac{k_{x1}}{k_{x0}}\right)\tan(k_{x1}h) = \varepsilon_r \quad \Rightarrow \quad \boxed{\varepsilon_r = -j\left(\frac{k_{x1}}{k_{x0}}\right)\tan(k_{x1}h)}$$

TEN:



TM_x Surface-Wave Solution (cont.)

Letting

$$k_{x0} = -j\alpha_{x0} \qquad \left(\alpha_{x0} = \sqrt{k_z^{\,2} - k_0^{\,2}}\right)$$

$$\varepsilon_r = -j\left(\frac{k_{x1}}{k_{x0}}\right)\tan(k_{x1}h)$$

we have

$$\varepsilon_r = \left(\frac{k_{x1}}{\alpha_{x0}}\right)\tan(k_{x1}h)$$

$$k_{x1} = \sqrt{k_1^2 - k_z^2}$$

$$k_{x0} = -j\sqrt{k_z^2 - k_0^2}$$

or

$$\varepsilon_r\sqrt{k_z^2 - k_0^2} = \sqrt{k_1^2 - k_z^2}\,\tan\left[\left(h\sqrt{k_1^2 - k_z^2}\right)\right]$$

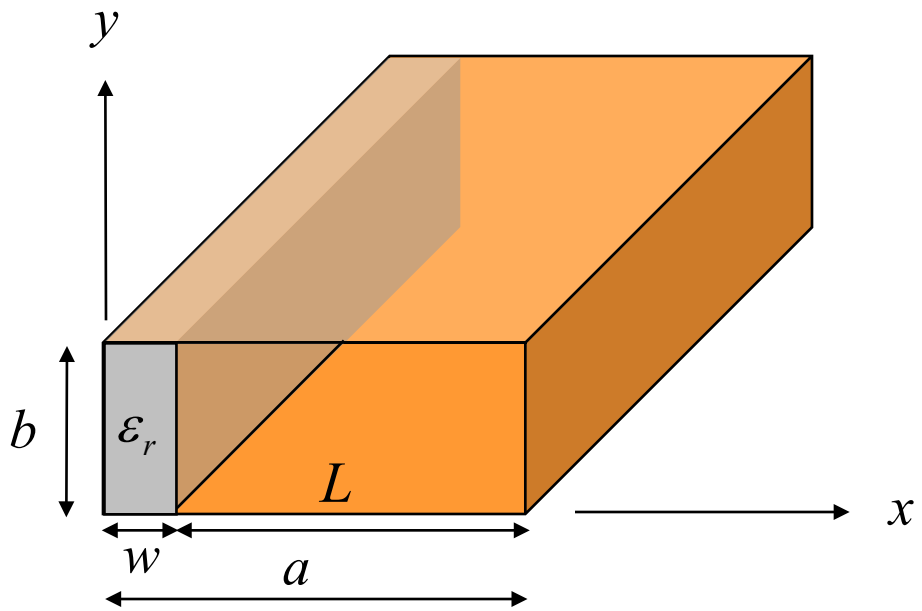
Note:
This method (TRE) is a lot simpler than doing EM analysis and applying the boundary conditions (Notes 13)!

Waveguide With Slab

Choose this representation:

TE_x^{mn} modes

TM_x^{mn} modes



$$k_y = \frac{n\pi}{b}$$

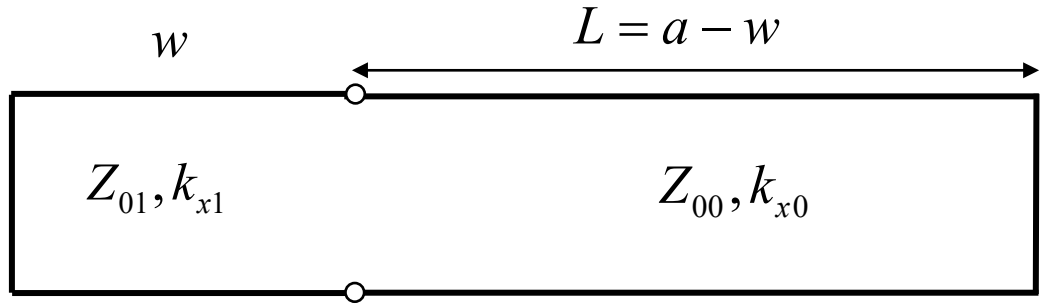
$$n = 0, 1, 2\ldots$$

$$Z_0^{\mathrm{TM}_x} = \frac{k_x}{\omega\varepsilon_c}$$

$$Z_0^{\mathrm{TE}_x} = \frac{\omega\mu}{k_x}$$

Note:
The modes are hybrid in the z direction (not TE_z or TM_z).

TEN:

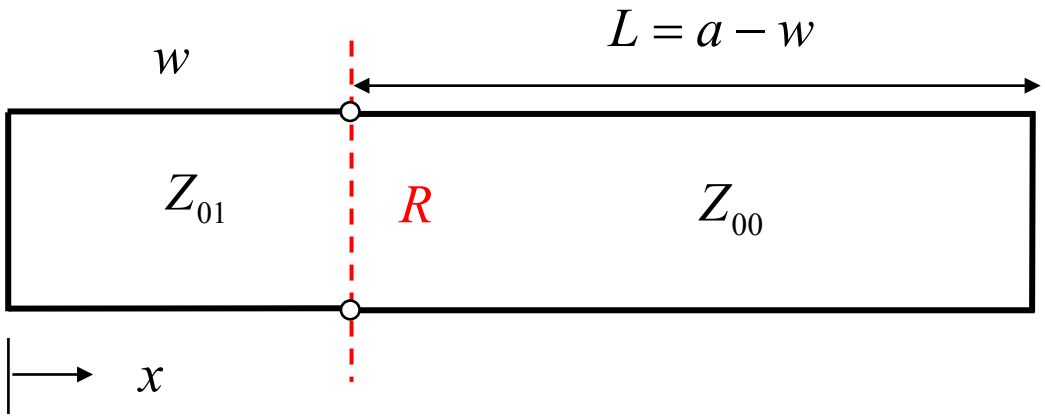


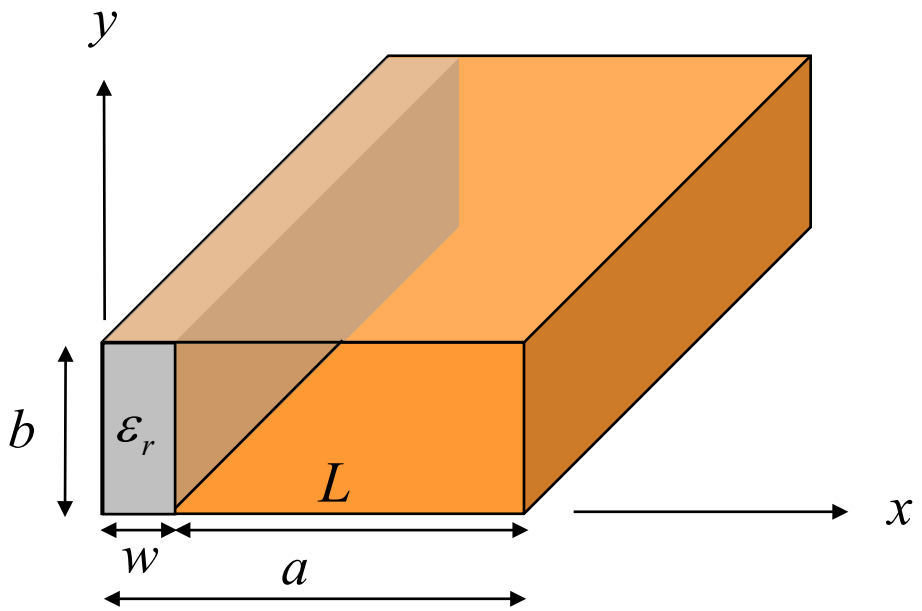
$k_x = k_{x0}$ or k_{x1}

$Z_0 = Z_0^{\mathrm{TM}_x}$ or $Z_0^{\mathrm{TE}_x}$

Waveguide With Slab (cont.)

TEN:





TRE: $\overleftarrow{Z}_{\text{in}} = -\overrightarrow{Z}_{\text{in}}$

$$\Rightarrow \quad jZ_{01}\tan\left(k_{x1}w\right) = -jZ_{00}\tan\left(k_{x0}L\right)$$

$$\Rightarrow \quad Z_{01}\tan\left(k_{x1}w\right) + Z_{00}\tan\left(k_{x0}L\right) = 0$$

Waveguide With Slab (cont.)

$$Z_{01}\tan\left(k_{x1}w\right)+Z_{00}\tan\left(k_{x0}L\right)=0$$

Choose TE$_x$:

$$\left(\frac{\omega\mu_0}{k_{x1}}\right)\tan\left(k_{x1}w\right)+\left(\frac{\omega\mu_0}{k_{x0}}\right)\tan\left(k_{x0}L\right)=0$$

$$Z_0^{\mathrm{TE}_x}=\frac{\omega\mu}{k_x}$$

Separation equations:

$$k_{x0}^2+k_y^2+k_z^2=k_0^2$$

$$k_{x1}^2+k_y^2+k_z^2=k_1^2=k_0^2\varepsilon_r$$

so

$$k_{x0}^2=k_0^2-\left(\frac{n\pi}{b}\right)^2-k_z^2$$

$$k_{x1}^2=k_0^2\varepsilon_r-\left(\frac{n\pi}{b}\right)^2-k_z^2$$

Waveguide With Slab (cont.)

Final transcendental equation for the unknown wavenumber k_z:

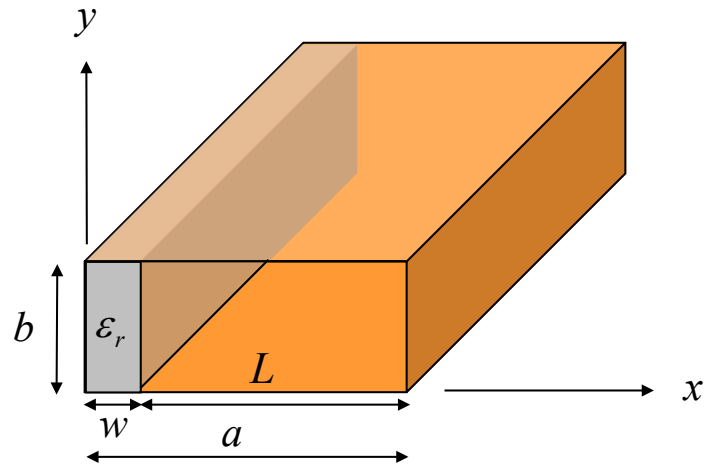
TE$_x$:

$$\left(\frac{1}{k_{x1}}\right)\tan\left(k_{x1}w\right)+\left(\frac{1}{k_{x0}}\right)\tan\left(k_{x0}L\right)=0$$

with

$$k_{x0}^2 = k_0^2 - \left(\frac{n\pi}{b}\right)^2 - k_z^2$$

$$k_{x1}^2 = k_0^2\varepsilon_r - \left(\frac{n\pi}{b}\right)^2 - k_z^2$$



Note:
The integer n is arbitrary but fixed.

The equation has an infinite number of solutions for k_z for a given n: $m = 1, 2, 3, \ldots$

$$\mathrm{TE}_{mn}^{(x)}: \underbrace{\mathrm{TE}_{10}^{(x)}, \mathrm{TE}_{20}^{(x)}, \mathrm{TE}_{30}^{(x)}, \ldots}_{n=0} \underbrace{\mathrm{TE}_{01}^{(x)}, \mathrm{TE}_{11}^{(x)}, \mathrm{TE}_{21}^{(x)} \ldots}_{n=1}$$

Waveguide With Slab (cont.)

Limiting case: $w \to 0$:

$$\left(\frac{1}{k_{x1}}\right)\tan\left(k_{x1}w\right)+\left(\frac{1}{k_{x0}}\right)\tan\left(k_{x0}L\right)=0$$

$$\Downarrow$$

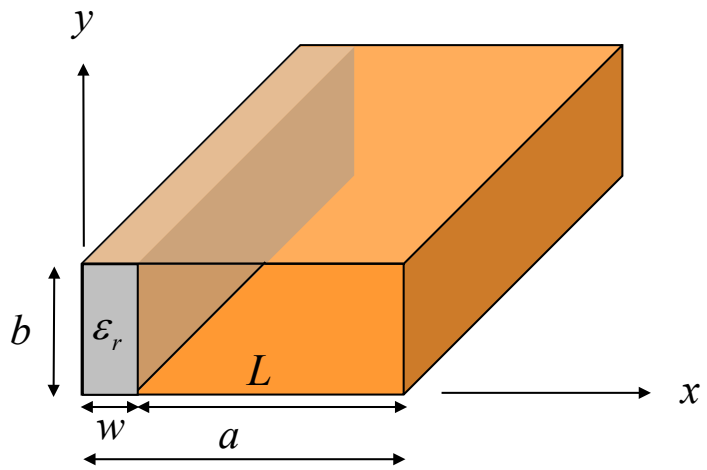
$$\tan\left(k_{x0}L\right)=0$$

$$\Downarrow \quad L \to a$$

$$\tan\left(k_{x0}a\right)=0$$

$$\Downarrow$$

$$k_{x0}=\frac{m\pi}{a}$$



$$\mathrm{TE}_{10}^{(x)} \to \mathrm{TE}_{10}^{(z)}$$

This mode becomes the usual TE_{10} mode of the hollow waveguide.

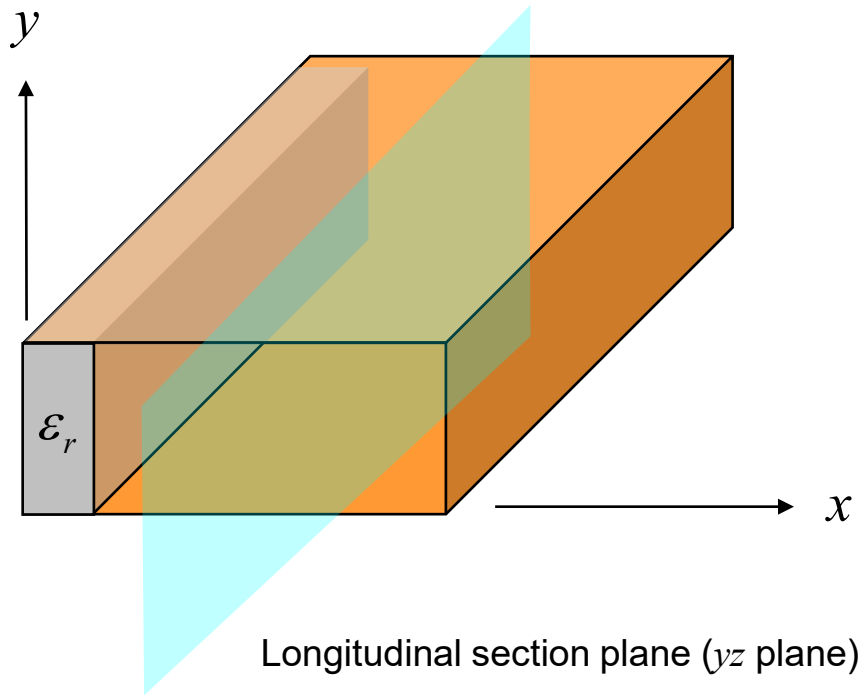
Hence, we have:

$$k_z=\sqrt{k_0^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}$$

Waveguide With Slab (cont.)

Alternative notation:

Another notation that is common in the literature for slab-loaded waveguides is the designation of “LSE” and “LSM” modes.



Longitudinal section plane (yz plane)

LSE mode: “longitudinal section electric” mode
LSM mode: “longitudinal section magnetic” mode

LSE: TE_x^{mn} (The electric field vector stays in the “longitudinal section”.)
LSM: TM_x^{mn} (The magnetic field vector stays in the “longitudinal section”.)