Notes 21
Quadrature Coupler and Rat-Race Coupler
A quadrature coupler is one in which the input is split into two signals (usually with a goal of equal magnitudes) that are 90 degrees apart in phase. Types of quadrature couplers include branchline couplers (also known as quadrature hybrid* couplers), Lange couplers and overlay couplers.

Taken from “Microwaves 101”
http://www.microwaves101.com/encyclopedia/Quadrature_couplers.cfm

This coupler is very useful for obtaining circular polarization:
There is a 90° phase difference between ports 2 and 3.

Note:
The term “hybrid” denotes the fact that there is an equal (3 dB) power split to the output ports.
The quadrature hybrid is a lossless 4-port (the $S$ matrix is unitary).

All four ports are matched.

The device is reciprocal (the $S$ matrix is symmetric.)

Port 4 is isolated from port 1, and ports 2 and 3 are isolated from each other.
The quadrature coupler is usually used as a splitter:

- The signal from port 1 splits evenly between ports 2 and 3, with a $90^\circ$ phase difference.
  
  \[ S_{21} = jS_{31} \]  
  Can be used to produce right-handed circular polarization.

- The signal from port 4 splits evenly between ports 2 and 3, with a $-90^\circ$ phase difference.
  
  \[ S_{24} = -jS_{34} \]  
  Can be used to produce left-handed circular polarization.

**Note:** A matched load is usually placed on port 4.
A microstrip realization of a quadrature hybrid (branch-line coupler) is shown here.

Notes:
- We only need to study what happens when we excite port 1, since the structure is physically symmetric.
- We use even/odd mode analysis (exciting ports 1 and 4) to figure out what happens when we excite port 1.

An analysis of the branch-line coupler is given in the Appendix.
Quadrature Coupler (cont.)

Summary

The input power to port 1 divides evenly between ports 2 and 3, with ports 2 and 3 being 90° out of phase.

Note: A matched load is usually placed on port 4.
A **coupled-line** coupler is one that uses coupled lines (microstrip, stripline) with no direct connection between all of the ports.

Please see the Pozar book for more details.

This coupler has a 90° phase difference between the output ports (ports 2 and 3), and can be used to obtain an equal (-3 dB) power split or another split ratio.
Circularly-polarized microstrip antennas can be fed with a 90° coupler.

One feed port produces RHCP, the other feed port produced LHCP.

**Note:** This is a better way (higher bandwidth) to get CP than with a simple 90° delay line.
Rat-Race Ring Coupler (180° Coupler)

“Applications of rat-race couplers are numerous, and include mixers and phase shifters. The rat-race gets its name from its circular shape, shown below.”

Taken from “Microwaves 101”

http://www.microwaves101.com/encyclopedia/ratrace_couplers.cfm

Photograph of a microstrip ring coupler

Courtesy of M. D. Abouzahra, MIT Lincoln Laboratory
The rat race is a lossless 4-port (the $S$ matrix is unitary).

- All four ports are matched.
- The device is reciprocal (the $S$ matrix is symmetric).
- Port 4 is isolated from port 1, and ports 2 and 3 are isolated from each other.
Rat-Race Coupler (cont.)

The rat race can be used as a splitter:

1. The signal from the “sum port” $\Sigma$ (port 1) splits evenly between ports 2 and 3, in phase. This could be used as a power splitter (alternative to Wilkinson).
   \[ S_{21} = S_{31} \]

2. The signal from the “difference port” $\Delta$ (port 4) splits evenly between ports 1 and 2, $180^\circ$ out of phase. This could be used as a balun.
   \[ S_{24} = -S_{34} \]

Note: A matched load is usually placed on port 4.
The rat race can be used as a combiner:

- The signal from the sum port $\Sigma$ (port 1) is the sum of the input signals 1 and 2.
  \[ S_{12} = S_{13} \]

- The signal from the difference port $\Delta$ (port 4) is the difference of the input signals 1 and 2.
  \[ S_{42} = -S_{43} \]
A microstrip realization is shown here.

\[
[S] = \frac{-j}{\sqrt{2}} \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0
\end{bmatrix}
\]

An analysis of the rat-race coupler is given in the Appendix.
A waveguide realization of a 180° coupler is shown here, called a “Magic T.”

Note: Irises are usually used to obtain matching at the ports.

\[
[S] = \frac{-j}{\sqrt{2}} \begin{bmatrix}
  0 & 1 & 1 & 0 \\
  1 & 0 & 0 & -1 \\
  1 & 0 & 0 & 1 \\
  0 & -1 & 1 & 0 
\end{bmatrix}
\]
Monopulse Radar

Rat-Race couplers are often used in monopulse radar.

\[
\Sigma = A + B + C + D
\]

\[
\Delta_{AZ} = (B + C) - (A + D)
\]

\[
\Delta_{EL} = (C + D) - (B + A)
\]

The difference signals are used to determine the azimuth and elevation of the target.

\[
\Delta \phi = k_0 D \sin \theta
\]

\[
A - B = 1 - e^{-j \Delta \phi} = e^{-j \Delta \phi/2} \left( e^{+j \Delta \phi/2} - e^{-j \Delta \phi/2} \right) = e^{-j \Delta \phi/2} \left( 2j \sin \left( \Delta \phi / 2 \right) \right)
\]

The difference between the two antenna signals maps into the phase difference \( \Delta \phi \), which maps into the angle \( \theta \).
Here we analyze the quadrature coupler.

Port 1 Excitation
“even” analysis

Input admittance of open-circuited stub:

\[ Y_s = jY_0 \tan \left( \beta_s l_s \right) = jY_0 \tan \left( \pi / 4 \right) = jY_0 \]
Appendix A (cont.)

Port 1 Excitation
“odd” problem

\[ V_{1}^{\circ} = -V_{2}^{\circ} \]
\[ V_{4}^{\circ} = -V_{1}^{\circ} \]

Input admittance of short-circuited stub:
\[ Y_{s} = -jY_{0} \cot \left( \beta_{s} l_{s} \right) \]
\[ = -jY_{0} \cot \left( \pi / 4 \right) \]
\[ = -jY_{0} \]
Consider the general case:

\[ Y_s = \pm jY_0 \quad (+ \text{ for even}) \quad (- \text{ for odd}) \]

\[
[ABCD]_Y = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \\
[ABCD]_{\lambda/4} = \begin{bmatrix} 0 & \frac{jZ_0}{\sqrt{2}} \\ \frac{j\sqrt{2}}{Z_0} & 0 \end{bmatrix} \\
[ABCD_{\text{line}}] = \begin{bmatrix} \cos(\beta \ell) & jZ_0^{\text{line}} \sin(\beta \ell) \\ (j/Z_0^{\text{line}}) \sin(\beta \ell) & D = \cos(\beta \ell) \end{bmatrix}
\]

Shunt load on line  
Quarter-wave line

In general:

[ABCD] = [ABCD]_Y [ABCD]_{\lambda/4} [ABCD]_Y

Here:
\[ Z_0^{\text{line}} = \frac{Z_0}{\sqrt{2}} \]
\[ \beta \ell = \pi / 2 \]
Hence we have:

\[
[ABCD] = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \begin{bmatrix} 0 & jZ_0 \\ \frac{j\sqrt{2}}{Z_0} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \begin{bmatrix} jZ_0Y \sqrt{2} & jZ_0 \sqrt{2} \\ j\sqrt{2}Z_0 & j\sqrt{2} \end{bmatrix}
\]

\[
= \begin{bmatrix} jZ_0Y \sqrt{2} + j\sqrt{2}Z_0 & jZ_0Y \sqrt{2} \sqrt{2} \\ jZ_0Y \sqrt{2} + j\sqrt{2}Z_0 & jZ_0Y \sqrt{2} \end{bmatrix}
\]

\[
Y = \pm jY_0 = \pm \frac{j}{Z_0} \quad \text{(+ for even)} \quad \text{(- for odd)}
\]
Continuing with the algebra, we have:

\[
[ABCD] = \frac{1}{\sqrt{2}} \begin{bmatrix}
    jZ_0 \left( \pm \frac{j}{Z_0} \right) & jZ_0 \\
    jZ_0 \left( \pm \frac{j}{Z_0} \right)^2 + j2 & jZ_0 \left( \pm \frac{j}{Z_0} \right)
\end{bmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{bmatrix}
    j(\pm j) & jZ_0 \\
    -j \left( \frac{1}{Z_0} \right) + j2 & j(\pm j)
\end{bmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{bmatrix}
    \mp 1 & jZ_0 \\
    j \left( \frac{1}{Z_0} \right) & \mp 1
\end{bmatrix}
\]
Hence we have:

\[
[ABCD]_0 = \frac{1}{\sqrt{2}} \begin{bmatrix}
\mp 1 & jZ_0 \\
-j \left( \frac{1}{Z_0} \right) & \mp 1
\end{bmatrix}
\]

Convert this to $S$ parameters (use Table 4.2 in Pozar):

\[
[S]_0 = \begin{bmatrix}
0 & \mp 1 - j \\
\mp 1 - j & 0
\end{bmatrix}
\]

Note:
We are describing a two-port device here, in the even and odd mode cases.

This is a 2×2 matrix, not a 4×4 matrix.
Adding even and odd mode cases together:

\[ V_1^+ = V^+ + V^+ \]

\[ S_{11} = \frac{V^-}{V^+} \]

Hence \( S_{11} = 0 \)  By symmetry: \( S_{11} = S_{22} = S_{33} = S_{44} = 0 \)
\[ V_1^+ = V^+ + V^+ \]

By symmetry and reciprocity:

\[ S_{21} = S_{12} = S_{43} = S_{34} = \frac{-j}{\sqrt{2}} \]
$V_1^+ = V^+ + V^+$

By symmetry and reciprocity:

$$S_{31} = S_{13} = S_{24} = S_{42} = \frac{-1}{\sqrt{2}}$$
By symmetry and reciprocity: \[ S_{41} = S_{14} = S_{23} = S_{32} = 0 \]
Appendix B

Here we analyze the Rat-Race Ring coupler.
Port 1 Excitation
“even” problem

\[ Y_0 = 1 / Z_0 \]
\[ Y_{0s} = Y_0 / \sqrt{2} \]

\[ Y_{s1} = jY_{0s1} \tan(\beta_s l_{s1}) \]
\[ = j\left(Y_0 / \sqrt{2}\right) \tan(\pi / 4) \]
\[ = jY_0 / \sqrt{2} \]

\[ Y_{s2} = jY_{0s2} \tan(\beta_s l_{s2}) \]
\[ = j\left(Y_0 / \sqrt{2}\right) \tan(3\pi / 4) \]
\[ = -jY_0 / \sqrt{2} \]
Appendix B (cont.)

Port 1 Excitation
“odd” problem

$Y_0 = 1 / Z_0$

$Y_{0s} = Y_0 / \sqrt{2}$

$Y_{s1} = -jY_{0s1} \cot (\beta_s l_s)$

$= -j\left(\frac{Y_0}{\sqrt{2}}\right) \cot \left(\frac{\pi}{4}\right)$

$= -jY_0 / \sqrt{2}$

$Y_{s1} = -jY_{0s2} \cot (\beta_s l_s)$

$= -j\left(\frac{Y_0}{\sqrt{2}}\right) \cot \left(\frac{3\pi}{4}\right)$

$= jY_0 / \sqrt{2}$
Proceeding as for the 90° coupler, we have:

\[
[ABCD]_0 = \begin{bmatrix}
1 & 0 \\
\pm jY_0 / \sqrt{2} & 1
\end{bmatrix}
\begin{bmatrix}
0 & j\sqrt{2}Z_0 \\
j/\sqrt{2}Z_0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\mp jY_0 / \sqrt{2} & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 \\
\pm jY_0 / \sqrt{2} & 1
\end{bmatrix}
\begin{bmatrix}
\pm 1 & j\sqrt{2}Z_0 \\
j/\sqrt{2}Z_0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\pm 1 & j\sqrt{2}Z_0 \\
j\sqrt{2} \div Z_0 & \mp 1
\end{bmatrix}
\]
Converting from the \( ABCD \) matrix to the \( S \) matrix, we have:

\[
[ABCD]_e = \begin{bmatrix}
\pm 1 & j\sqrt{2}Z_0 \\
j\frac{\sqrt{2}}{Z_0} & \mp 1
\end{bmatrix}
\]

\[
[S]_e = \frac{-j}{\sqrt{2}} \begin{bmatrix}
\pm 1 & 1 \\
1 & \mp 1
\end{bmatrix}
\]

**Note:**
We are describing a two-port device here, in the even and odd mode cases.
This is a 2\( \times \)2 matrix, not a 4\( \times \)4 matrix.

Use Table 4.2 in Pozar
For the $S$ parameters coming from port 1 excitation, we then have:

\[ S_{11} = \frac{V_1^-}{V_1^+} \bigg|_{a_2=a_3=a_4=0} \]

\[ S_{11} = S_{33} = 0 \]
(symmetry)

\[ S_{21} = \frac{V_2^-}{V_1^+} \bigg|_{a_2=a_3=a_4=0} \]

\[ S_{21} = S_{12} = S_{34} = S_{43} = \frac{-j}{\sqrt{2}} \]
(symmetry and reciprocity)

\[ S_{11} = \frac{V_1^- + V_1^-}{2V^+} = \frac{1}{2} (S_{11}^e + S_{11}^o) \]

\[ = \frac{1}{2} \left( \frac{-j}{\sqrt{2}} + \frac{j}{\sqrt{2}} \right) \]

\[ = 0 \]

\[ S_{21} = \frac{V_2^- + V_2^-}{2V^+} = \frac{1}{2} (S_{21}^e + S_{21}^o) \]

\[ = \frac{1}{2} \left( \frac{-j}{\sqrt{2}} + \frac{-j}{\sqrt{2}} \right) \]

\[ = \frac{-j}{\sqrt{2}} \]
Similarly, exciting port 2, and using symmetry and reciprocity, we have the following results (derivation omitted):

\[
S_{22} = S_{44} = 0
\]

\[
S_{23} = S_{32} = S_{14} = S_{41} = 0
\]

\[
S_{24} = S_{42} = \frac{j}{\sqrt{2}}
\]