ECE 6340
Intermediate EM Waves

Fall 2016

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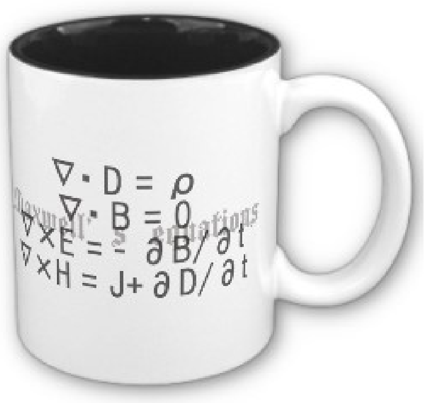


Notes 6

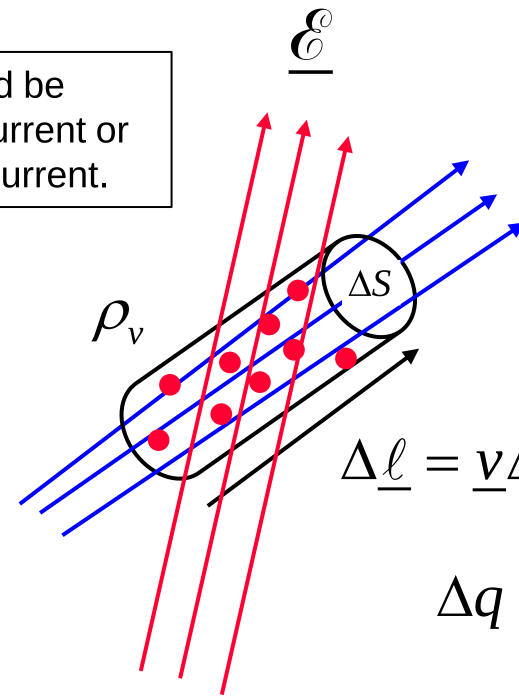
Power Dissipated by Current

Work given to a collection of electric charges moving in an electric field:

This could be conduction current or impressed current.

$$\Delta W = \left(\Delta q\, \underline{\mathscr{E}}\right)\cdot \Delta \underline{\ell}$$

$$= \left(\rho_v \Delta S\, \Delta \ell\, \underline{\mathscr{E}}\right)\cdot \Delta \underline{\ell}$$



$$\underline{\mathscr{J}} = \rho_v \underline{v}$$

$$\Delta \underline{\ell} = \underline{v}\Delta t$$

$$\Delta q = \rho_v\, \Delta \ell\, \Delta S$$

Note: The magnetic field never does any work:

$$\Delta q\left(\underline{v}\times \underline{\mathscr{B}}\right)\cdot \Delta \underline{\ell} = 0$$

Power dissipated per unit volume:

$$\mathscr{P}_d = \frac{\Delta W}{\Delta S\, \Delta \ell\, \Delta t} = \left(\rho_v \frac{\Delta \underline{\ell}}{\Delta t}\right)\cdot \underline{\mathscr{E}} = \left(\rho_v\, \underline{v}\right)\cdot \underline{\mathscr{E}} = \underline{\mathscr{J}}\cdot \underline{\mathscr{E}}$$

Note: We assume no increase in kinetic energy, so the power goes to <u>heat</u>.

Power Dissipated by Current (cont.)

Power dissipated per unit volume for ohmic current (we assume here a simple linear media that obeys Ohm's law):

$$\mathscr{P}_d = \underline{\mathscr{J}}^c \cdot \underline{\mathscr{E}} = \left(\sigma \underline{\mathscr{E}}\right) \cdot \underline{\mathscr{E}} = \sigma \left(\underline{\mathscr{E}} \cdot \underline{\mathscr{E}}\right) = \sigma \left|\underline{\mathscr{E}}\right|^2 = \sigma \mathscr{E}^2$$

Note: $\mathscr{E}^2 \equiv \left|\underline{\mathscr{E}}\right|^2 = \underline{\mathscr{E}} \cdot \underline{\mathscr{E}}$

Hence, we have

$$\boxed{\mathscr{P}_d = \sigma \mathscr{E}^2}$$

Power Dissipated by Current (cont.)

Return to the general result:

$$\mathscr{P}_d = \underline{\mathscr{J}} \cdot \underline{\mathscr{E}}$$

This is the power <u>dissipated</u> per unit volume.

From this we can also write the power generated per unit volume due to an impressed source current:

$$\boxed{\mathscr{P}_s = -\underline{\mathscr{J}}^{\,i} \cdot \underline{\mathscr{E}}}$$

Poynting Theorem: Time-Domain

$$\nabla \times \underline{\mathscr{E}} = -\underline{\mathscr{M}} - \frac{\partial \underline{\mathscr{B}}}{\partial t}$$

$$\nabla \times \underline{\mathscr{H}} = \underline{\mathscr{J}} + \frac{\partial \underline{\mathscr{D}}}{\partial t}$$

From these we obtain

$$\underline{\mathscr{H}} \cdot \left(\nabla \times \underline{\mathscr{E}}\right) = -\underline{\mathscr{M}} \cdot \underline{\mathscr{H}} - \underline{\mathscr{H}} \cdot \frac{\partial \underline{\mathscr{B}}}{\partial t}$$

$$\underline{\mathscr{E}} \cdot \left(\nabla \times \underline{\mathscr{H}}\right) = \underline{\mathscr{J}} \cdot \underline{\mathscr{E}} + \underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{D}}}{\partial t}$$

Subtract, and use the following vector identity:

$$\underline{\mathscr{H}} \cdot \left(\nabla \times \underline{\mathscr{E}}\right) - \underline{\mathscr{E}} \cdot \left(\nabla \times \underline{\mathscr{H}}\right) = \nabla \cdot \left(\underline{\mathscr{E}} \times \underline{\mathscr{H}}\right)$$

Poynting Theorem: Time-Domain (cont.)

We then have

$$\nabla\cdot\left(\underline{\mathcal{E}}\times\underline{\mathcal{H}}\right)=-\underline{\mathcal{M}}\cdot\underline{\mathcal{H}}-\underline{\mathcal{J}}\cdot\underline{\mathcal{E}}-\underline{\mathcal{H}}\cdot\frac{\partial\underline{\mathcal{B}}}{\partial t}-\underline{\mathcal{E}}\cdot\frac{\partial\underline{\mathcal{D}}}{\partial t}$$

Now let

$$\underline{\mathcal{J}}=\underline{\mathcal{J}}^{i}+\sigma\,\underline{\mathcal{E}}$$

$$\underline{\mathcal{M}}=\underline{\mathcal{M}}^{i}$$

so that

$$\nabla\cdot\left(\underline{\mathcal{E}}\times\underline{\mathcal{H}}\right)=-\underline{\mathcal{J}}^{i}\cdot\underline{\mathcal{E}}-\sigma\,\mathcal{E}^{2}-\underline{\mathcal{M}}^{i}\cdot\underline{\mathcal{H}}-\underline{\mathcal{E}}\cdot\frac{\partial\underline{\mathcal{D}}}{\partial t}-\underline{\mathcal{H}}\cdot\frac{\partial\underline{\mathcal{B}}}{\partial t}$$

Poynting Theorem: Time-Domain (cont.)

Next, use

$$\underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{D}}}{\partial t} = \varepsilon \left(\underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{E}}}{\partial t} \right)$$

Assume: $\underline{\mathscr{D}} = \varepsilon \underline{\mathscr{E}}$

(simple linear media)

and $$\frac{\partial \mathscr{E}^2}{\partial t} = \frac{\partial}{\partial t} \left(\underline{\mathscr{E}} \cdot \underline{\mathscr{E}} \right) = 2 \underline{\mathscr{E}} \cdot \left(\frac{\partial \underline{\mathscr{E}}}{\partial t} \right)$$

Hence $$\underline{\mathscr{E}} \cdot \left(\frac{\partial \underline{\mathscr{D}}}{\partial t} \right) = \varepsilon \left(\underline{\mathscr{E}} \cdot \frac{\partial \underline{\mathscr{E}}}{\partial t} \right) = \varepsilon \left(\frac{1}{2} \frac{\partial \mathscr{E}^2}{\partial t} \right)$$

And similarly, $$\underline{\mathscr{H}} \cdot \frac{\partial \underline{\mathscr{B}}}{\partial t} = \mu \left(\frac{1}{2} \frac{\partial \mathscr{H}^2}{\partial t} \right)$$

Poynting Theorem: Time-Domain (cont.)

Define $$\boxed{\underline{\mathscr{S}} \equiv \underline{\mathscr{E}} \times \underline{\mathscr{H}}}$$

We then have

$$\boxed{\nabla \cdot \underline{\mathscr{S}} = -\underline{\mathscr{J}}^{i} \cdot \underline{\mathscr{E}} - \sigma \mathscr{E}^{2} - \underline{\mathscr{M}}^{i} \cdot \underline{\mathscr{H}} - \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon \mathscr{E}^{2} + \frac{1}{2}\mu \mathscr{H}^{2}\right]}$$

Next, integrate throughout V and use the divergence theorem:

$$\boxed{\oint_{S} \underline{\mathscr{S}} \cdot \underline{\hat{n}}\, dS = \int_{V} \left(-\underline{\mathscr{J}}^{i} \cdot \underline{\mathscr{E}} - \underline{\mathscr{M}}^{i} \cdot \underline{\mathscr{H}}\right) dV - \int_{V} \sigma \mathscr{E}^{2}\, dV - \frac{\partial}{\partial t} \int_{V} \left(\frac{1}{2}\varepsilon \mathscr{E}^{2} + \frac{1}{2}\mu \mathscr{H}^{2}\right) dV}$$

Note: We are assuming the volume to be stationary (not moving) here.

Poynting Theorem: Time-Domain (cont.)

$$\oint_S \underline{\mathscr{S}} \cdot \underline{\hat{n}}\, dS = \int_V \left(-\underline{\mathscr{J}}^i \cdot \underline{\mathscr{E}} - \underline{\mathscr{M}}^i \cdot \underline{\mathscr{H}} \right) dV - \int_V \sigma\, \mathscr{E}^2\, dV - \frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \varepsilon \mathscr{E}^2 + \frac{1}{2} \mu \mathscr{H}^2 \right) dV$$

Interpretation:

$$\mathscr{W}_e = \int_V \frac{1}{2} \varepsilon\, \mathscr{E}^2\, dV = \text{stored electric energy}$$

$$\mathscr{W}_m = \int_V \frac{1}{2} \mu\, \mathscr{H}^2\, dV = \text{stored magnetic energy}$$

$$\mathscr{P}_d = \int_V \sigma\, \mathscr{E}^2\, dV = \text{dissipated power}$$

$$\mathscr{P}_s = \int_V \left(-\underline{\mathscr{J}}^i \cdot \underline{\mathscr{E}} - \underline{\mathscr{M}}^i \cdot \underline{\mathscr{H}} \right) dV = \text{source power}$$

$$\Rightarrow \quad \mathscr{P}_f = \oint_S \underline{\mathscr{S}} \cdot \underline{\hat{n}}\, dS = \text{power flowing out of } S$$

(see next figure)

Poynting Theorem: Time-Domain (cont.)

$$\oint_S \underline{\mathscr{S}} \cdot \underline{\hat{n}}\, dS = \int_V \left(-\underline{\mathscr{J}}^{i} \cdot \underline{\mathscr{E}} - \underline{\mathscr{M}}^{i} \cdot \underline{\mathscr{H}} \right) dV - \int_V \sigma\, \mathscr{E}^2\, dV - \frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \varepsilon \mathscr{E}^2 + \frac{1}{2} \mu \mathscr{H}^2 \right) dV$$

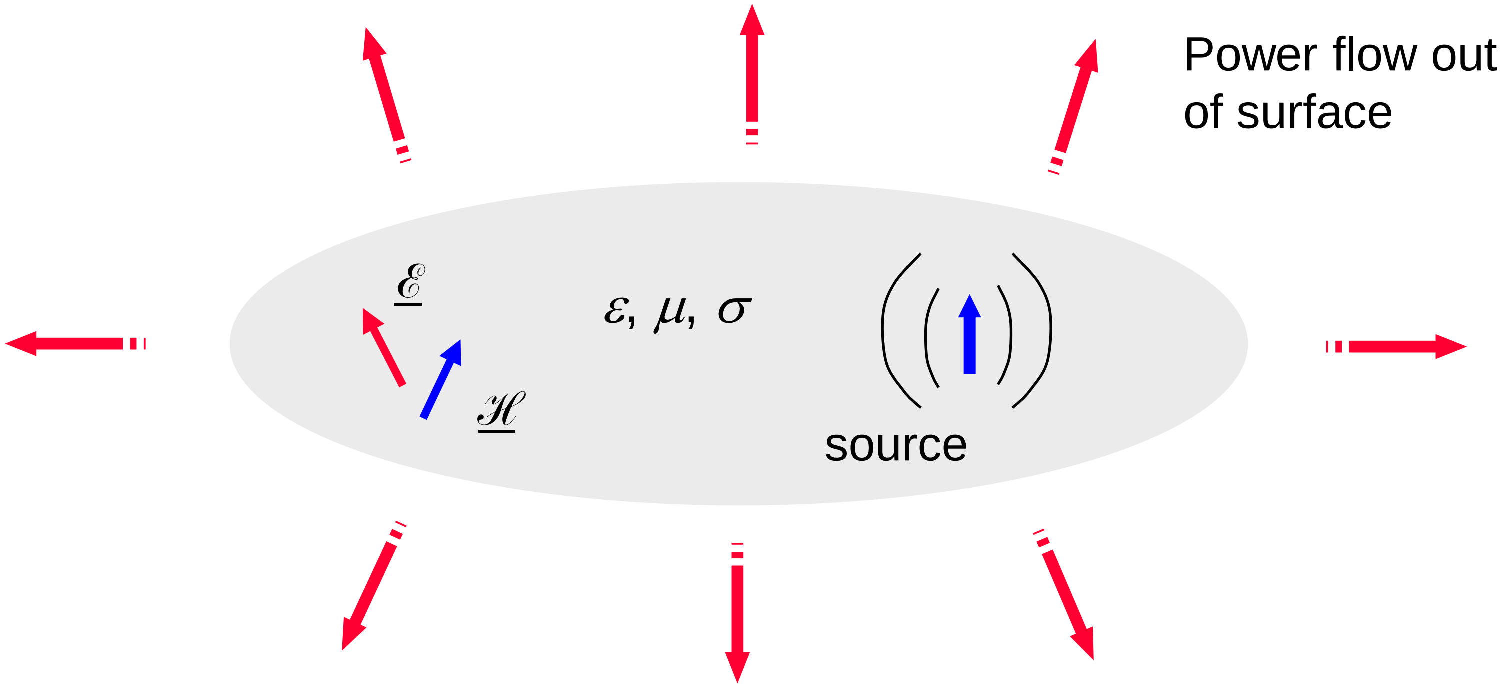
$$\mathscr{P}_f = \mathscr{P}_s - \mathscr{P}_d - \frac{\partial}{\partial t} \left(\mathscr{W}_e + \mathscr{W}_m \right)$$

or

$$\mathscr{P}_s = \mathscr{P}_d + \mathscr{P}_f + \frac{\partial}{\partial t} \left(\mathscr{W}_e + \mathscr{W}_m \right)$$

Poynting Theorem: Time-Domain (cont.)

$$\mathscr{P}_s = \mathscr{P}_d + \mathscr{P}_f + \frac{\partial}{\partial t}\left(\mathscr{W}_e + \mathscr{W}_m\right)$$



Poynting Theorem: Note on Interpretation

$$\underline{\mathscr{S}} = \underline{\mathscr{E}} \times \underline{\mathscr{H}}$$

Does the Poynting vector really represent local power flow?

$$\nabla \cdot \underline{\mathscr{S}} = -\underline{\mathscr{J}}^i \cdot \underline{\mathscr{E}} - \sigma \mathscr{E}^2 - \underline{\mathscr{M}}^i \cdot \underline{\mathscr{H}} - \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon \mathscr{E}^2 + \frac{1}{2}\mu \mathscr{H}^2\right]$$

Consider: $$\underline{\mathscr{S}}' = \underline{\mathscr{S}} + \nabla \times \underline{\mathscr{A}}$$

$\underline{\mathscr{A}} =$ Arbitrary vector function

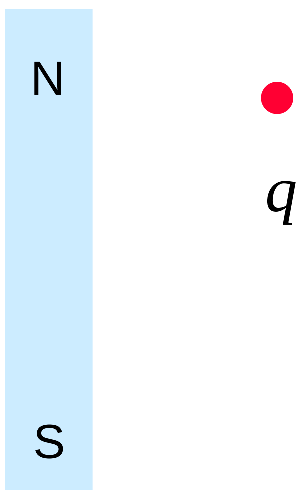
Note that $$\nabla \cdot \underline{\mathscr{S}}' = \nabla \cdot \underline{\mathscr{S}}$$

This “new Poynting vector” is equally valid! They both give same TOTAL power flowing out of the volume, but different local power flow.

Note on Interpretation (cont.)

Another “dilemma”:

A static point charge is sitting next to a bar magnet.



$$\underline{\mathscr{S}} = \underline{\mathscr{E}} \times \underline{\mathscr{H}} \neq \underline{0}$$

Is there really power flowing in space?

Note: There certainly must be zero net power out of any closed surface:

$$\cancel{\mathscr{P}_s} = \cancel{\mathscr{P}_d} + \mathscr{P}_f + \cancel{\frac{\partial}{\partial t}}\left(\mathscr{W}_e + \mathscr{W}_m\right)$$

Note on Interpretation (cont.)

Bottom line: We always get the correct result if we assume that the Poynting vector represents local power flow.

Because…

In a practical measurement, all we can ever really measure is the power flowing through a closed surface.

Comment:

At high frequency, where the power flow can be visualized as "photons" moving in space, it becomes more plausible to think of local power flow. In such situations, the Poynting vector has always given the correct result that agrees with measurements.

Complex Poynting Theorem

Frequency domain:

$$\nabla \times \underline{E} = -\underline{M}^{i} - j\omega\mu\underline{H}$$

$$\nabla \times \underline{H} = \underline{J}^{i} + j\omega\varepsilon_c\underline{E}$$

Generalized linear media:
ε may be complex.

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$

Hence

$$\underline{H}^{*} \cdot (\nabla \times \underline{E}) = -\underline{M}^{i} \cdot \underline{H}^{*} - j\omega\mu\left|\underline{H}\right|^{2}$$

$$\underline{E} \cdot \left(\nabla \times \underline{H}^{*}\right) = \underline{E} \cdot \underline{J}^{i*} - j\omega\varepsilon_c^{*}\left|\underline{E}\right|^{2}$$

Subtract and use the following vector identity:

$$\nabla \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot (\nabla \times \underline{A}) - \underline{A} \cdot (\nabla \times \underline{B})$$

$$\Rightarrow \quad \nabla \cdot \left(\underline{E} \times \underline{H}^{*}\right) = \underline{H}^{*} \cdot (\nabla \times \underline{E}) - \underline{E} \cdot \left(\nabla \times \underline{H}^{*}\right)$$

Hence

$$\nabla \cdot \left(\underline{E} \times \underline{H}^{*}\right) = -\underline{M}^{i} \cdot \underline{H}^{*} - \underline{E} \cdot \underline{J}^{i*} - j\omega\mu\left|\underline{H}\right|^{2} + j\omega\varepsilon_c^{*}\left|\underline{E}\right|^{2}$$

Complex Poynting Theorem (cont.)

Define $$\boxed{\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^*}$$ (complex Poynting vector)

Note: $$\operatorname{Re}\underline{S} = \frac{1}{2}\operatorname{Re}\left(\underline{E} \times \underline{H}^*\right) = \left\langle \underline{\mathscr{E}} \times \underline{\mathscr{H}} \right\rangle = \left\langle \underline{\mathscr{S}} \right\rangle$$

Then

$$\nabla \cdot \underline{S} = -\frac{1}{2}\left(\underline{E} \cdot \underline{J}^{i*} + \underline{M}^{i} \cdot \underline{H}^*\right) + \frac{1}{2} j\omega\left(\varepsilon_c^* \left|\underline{E}\right|^2 - \mu\left|\underline{H}\right|^2\right)$$

Next use

$$\varepsilon_c = \varepsilon_c' - j\varepsilon_c'' \quad \rightarrow \quad \varepsilon_c^* = \varepsilon_c' + j\varepsilon_c''$$

$$\mu = \mu' - j\,\mu''$$

Next, collect real and imaginary parts in the last term on the RHS.

Complex Poynting Theorem (cont.)

$$\nabla \cdot \underline{S} = -\frac{1}{2}\left(\underline{E} \cdot \underline{J}^{i*} + \underline{M}^{i} \cdot \underline{H}^{*}\right) + \frac{1}{2} j\,\omega\left(\varepsilon_c' \left|\underline{E}\right|^2 - \mu' \left|\underline{H}\right|^2\right) - \frac{1}{2}\omega\left(\varepsilon_c'' \left|\underline{E}\right|^2 + \mu'' \left|\underline{H}\right|^2\right)$$

or

$$\nabla \cdot \underline{S} = -\frac{1}{2}\left(\underline{E} \cdot \underline{J}^{i*} + \underline{M}^{i} \cdot \underline{H}^{*}\right) + 2 j\,\omega\left(\frac{1}{4}\varepsilon_c' \left|\underline{E}\right|^2 - \frac{1}{4}\mu' \left|\underline{H}\right|^2\right) - \frac{1}{2}\omega\left(\varepsilon_c'' \left|\underline{E}\right|^2 + \mu'' \left|\underline{H}\right|^2\right)$$

Next, integrate over a volume V and apply the divergence theorem:

$$\oint_S \underline{S} \cdot \underline{\hat{n}}\, dS = \int_V -\frac{1}{2}\left(\underline{E} \cdot \underline{J}^{i*} + \underline{M}^{i} \cdot \underline{H}^{*}\right) dV + 2 j\,\omega \int_V \left(\frac{1}{4}\varepsilon_c' \left|\underline{E}\right|^2 - \frac{1}{4}\mu' \left|\underline{H}\right|^2\right) dV$$
$$-\int_V \left(\frac{1}{2}\omega\varepsilon_c'' \left|\underline{E}\right|^2 + \frac{1}{2}\omega\mu'' \left|\underline{H}\right|^2\right) dV$$

Complex Poynting Theorem (cont.)

Final form of complex Poynting theorem:

$$\int_V -\frac{1}{2}\left(\underline{E}\cdot\underline{J}^{i*}+\underline{M}^{i}\cdot\underline{H}^{*}\right)dV=\oint_S \underline{S}\cdot\underline{\hat{n}}\,dS$$
$$+2j\omega\int_V\left(\frac{1}{4}\mu'\left|\underline{H}\right|^2-\frac{1}{4}\varepsilon_c'\left|\underline{E}\right|^2\right)dV$$
$$+\int_V\left(\frac{1}{2}\omega\varepsilon_c''\left|\underline{E}\right|^2+\frac{1}{2}\omega\mu''\left|\underline{H}\right|^2\right)dV$$

Complex Poynting Theorem (cont.)

Interpretation of P_s :

$$P_s = \int_V -\frac{1}{2}\left(\underline{E} \cdot \underline{J}^{i*} + \underline{M}^{i} \cdot \underline{H}^{*}\right) dV$$

$$\operatorname{Re} P_s = \int_V \left(\left\langle -\underline{\mathscr{E}} \cdot \underline{\mathscr{J}}^{i} \right\rangle + \left\langle -\underline{\mathscr{M}}^{i} \cdot \underline{\mathscr{H}} \right\rangle\right) dV \; = \left\langle \mathscr{P}_s \right\rangle$$

We therefore identify that

$P_s \equiv$ complex source power $\left[\mathrm{VA}\right]$

$\operatorname{Re} P_s \equiv$ real power (watts) from the sources $\left[\mathrm{W}\right]$

$\operatorname{Im} P_s \equiv$ imaginary power (vars) from the sources $\left[\mathrm{VAR}\right]$

Complex Poynting Theorem (cont.)

Interpretation of P_f:

$$P_f = \oint_S \frac{1}{2}\left(\underline{E} \times \underline{H}^*\right) \cdot \underline{\hat{n}}\, dS$$

$$\operatorname{Re} P_f = \oint_S \left(\left\langle \underline{\mathscr{E}} \times \underline{\mathscr{H}} \right\rangle\right) \cdot \underline{\hat{n}}\, dS = \left\langle \mathscr{P}_f \right\rangle$$

We therefore identify that

$P_f \equiv$ complex power flowing out of S

$\operatorname{Re} P_f \equiv$ real power (watts) flowing out of S $[\mathrm{W}]$

$\operatorname{Im} P_f \equiv$ imaginary power (vars) flowing out of S $[\mathrm{VAR}]$

Complex Poynting Theorem (cont.)

Interpretation of energy terms:

$$\frac{1}{4}\varepsilon_c'\left|\underline{E}\right|^2=\frac{1}{2}\varepsilon_c'\left(\frac{1}{2}\left(\underline{E}\cdot\underline{E}^*\right)\right)$$

$$\frac{1}{2}\varepsilon_c'\left(\frac{1}{2}\mathrm{Re}\left(\underline{E}\cdot\underline{E}^*\right)\right)$$

Note: The real-part operator may be added here since it has no effect.

$$=\frac{1}{2}\varepsilon_c'\left\langle\underline{\mathscr{E}}\cdot\underline{\mathscr{E}}\right\rangle=\left\langle\frac{1}{2}\varepsilon_c'\left|\underline{\mathscr{E}}\right|^2\right\rangle$$

Note: We know this result represents stored energy for simple linear media ($\varepsilon_c' = \varepsilon$), so we assume it is true for generalized linear media.

Hence $$\int_V\frac{1}{4}\varepsilon_c'\left|\underline{E}\right|^2dV=\left\langle\int_V\frac{1}{2}\varepsilon_c'\left|\underline{\mathscr{E}}\right|^2dV\right\rangle=\left\langle\mathscr{W}_e\right\rangle$$

Similarly, $$\int_V\frac{1}{4}\mu'\left|\underline{H}\right|^2dV=\left\langle\int_V\frac{1}{2}\mu'\left|\underline{\mathscr{H}}\right|^2dV\right\rangle=\left\langle\mathscr{W}_m\right\rangle$$

Note:
The formulas for stored energy can be improved for *dispersive* media (discussed later).

Complex Poynting Theorem (cont.)

Interpretation of dissipation terms:

$$\frac{1}{2}\omega\varepsilon_c''\left|\underline{E}\right|^2 = \omega\varepsilon_c''\left(\frac{1}{2}\left(\underline{E}\cdot\underline{E}^*\right)\right) = \omega\varepsilon_c''\left(\frac{1}{2}\mathrm{Re}\left(\underline{E}\cdot\underline{E}^*\right)\right)$$

$$= \omega\varepsilon_c''\left\langle\underline{\mathscr{E}}\cdot\underline{\mathscr{E}}\right\rangle = \left\langle\omega\varepsilon_c''\left|\underline{\mathscr{E}}\right|^2\right\rangle$$

Recall:

$$\varepsilon_c = \varepsilon - j\left(\frac{\sigma}{\omega}\right)$$

ε is real for simple linear media

For simple linear media: $\omega\varepsilon_c'' = \omega\left(\dfrac{\sigma}{\omega}\right) = \sigma$

Hence

$$\int_V \frac{1}{2}\omega\varepsilon_c''\left|\underline{E}\right|^2 dV = \int_V \left\langle\sigma\left|\underline{\mathscr{E}}\right|^2\right\rangle dV = \left\langle\mathscr{P}_d^e\right\rangle$$

Note:
This formula gives the correct time-average power dissipated due to electric losses for simple linear media.

We assume the same interpretation holds for *generalized linear media* (ε is complex).

Complex Poynting Theorem (cont.)

Interpretation of dissipation terms (cont.):

Similarly,

$$\int_V \frac{1}{2}\omega\mu''\left|\underline{H}\right|^2 dV = \left\langle \mathscr{P}_d^m \right\rangle$$

This is the time-average power dissipated due to magnetic losses.

Note: There is no magnetic conductivity, and hence no magnetic conduction loss, but there can be magnetic <u>polarization</u> loss.

Complex Poynting Theorem (cont.)

Summary of Final Form

$$\int_V -\frac{1}{2}\left(\underline{E}\cdot\underline{J}^{i*}+\underline{M}^{i}\cdot\underline{H}^{*}\right)dV=\oint_S \underline{S}\cdot\underline{\hat{n}}\,dS$$

$$+\int_V\left(\frac{1}{2}\omega\varepsilon_c''\left|\underline{E}\right|^2+\frac{1}{2}\omega\mu''\left|\underline{H}\right|^2\right)dV$$

$$+2j\omega\int_V\left(\frac{1}{4}\mu'\left|\underline{H}\right|^2-\frac{1}{4}\varepsilon_c'\left|\underline{E}\right|^2\right)dV$$

$$\Downarrow$$

$$P_s=P_f+\langle\mathcal{P}_d\rangle+j(2\omega)\left(\langle\mathcal{W}_m\rangle-\langle\mathcal{W}_e\rangle\right)$$

Complex Poynting Theorem (cont.)

$$\underbrace{P_s = P_f + \langle \mathscr{P}_d \rangle + j(2\omega)\left(\langle \mathscr{W}_m \rangle - \langle \mathscr{W}_e \rangle\right)}$$

$$\underbrace{\langle \mathscr{P}_d \rangle + j(2\omega)\left(\langle \mathscr{W}_m \rangle - \langle \mathscr{W}_e \rangle\right)}_{P_{abs}}$$

We can write this as $P_s = P_f + P_{abs}$

where we have defined a complex power absorbed P_{abs} :

$$\mathrm{Re}\left(P_{abs}\right) = \langle \mathscr{P}_d \rangle$$

$$\mathrm{Im}\left(P_{abs}\right) = (2\omega)\left(\langle \mathscr{W}_m \rangle - \langle \mathscr{W}_e \rangle\right)$$

Complex Poynting Theorem (cont.)

$$P_s = P_f + P_{abs}$$

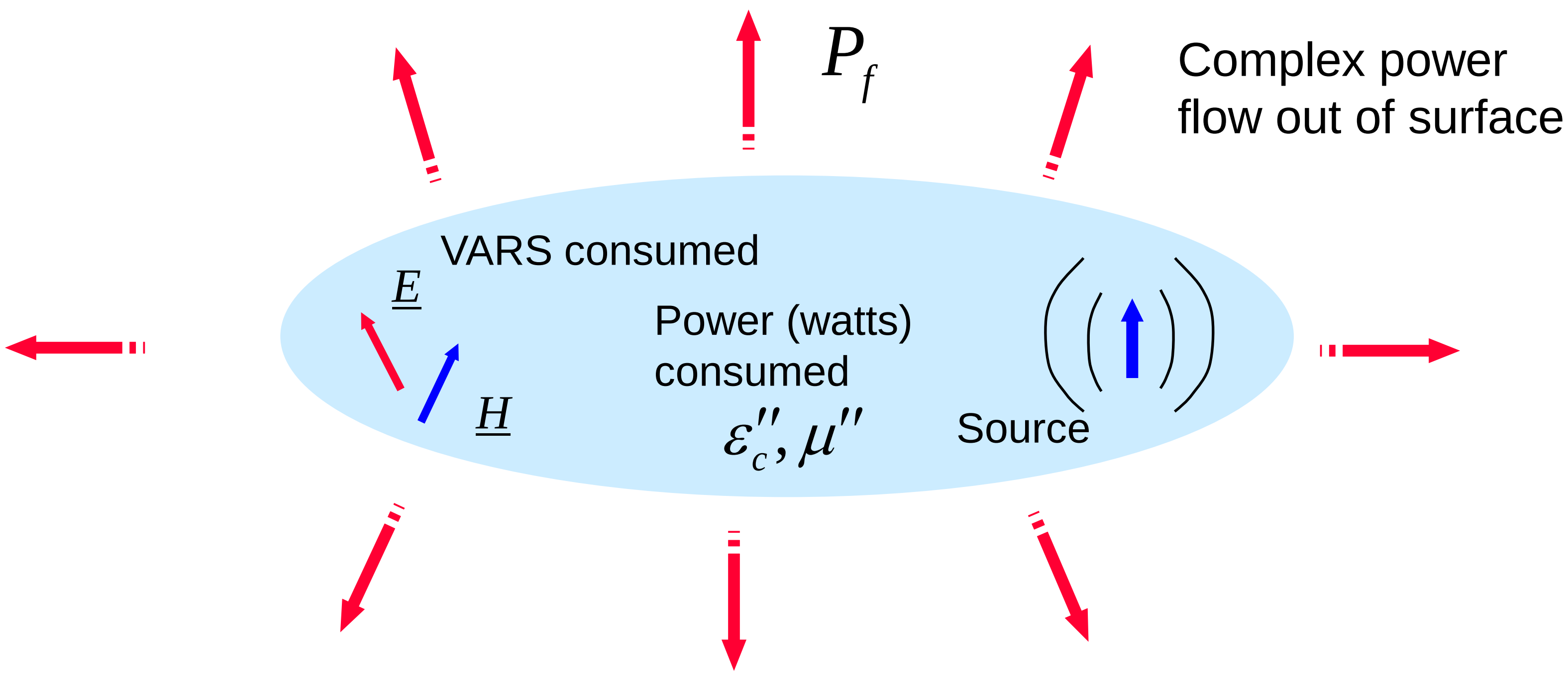
This is a conservation statement for complex power.

$$\mathrm{Re}\left(P_{abs}\right) = \left\langle \mathscr{P}_d \right\rangle$$

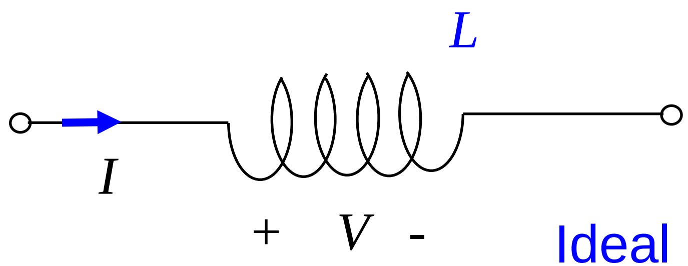
Watts

$$\mathrm{Im}\left(P_{abs}\right) = \left(2\omega\right)\left(\left\langle \mathscr{W}_m \right\rangle - \left\langle \mathscr{W}_e \right\rangle\right)$$

VARs



Example



Ideal (lossless) inductor

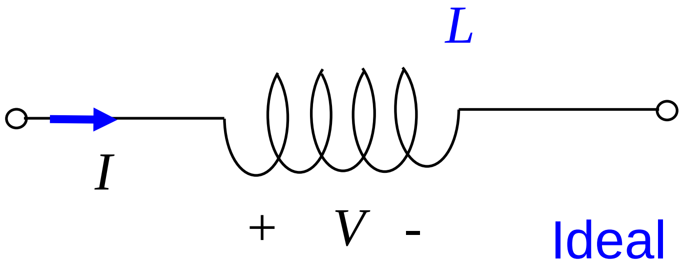
Denote:

$$P_{abs} = \text{complex power absorbed} = W_{abs} + jQ_{abs}$$

Calculate P_{abs} using circuit theory, and verify that the result is consistent with the complex Poynting theorem.

Note: $W_{abs} = 0$ (lossless element)

Example (cont.)



Ideal inductor

$$P_{abs} = \frac{1}{2} V\, I^* = \frac{1}{2}(Z\, I)\, I^*$$

$$= \frac{1}{2}(j\omega L\, I)\, I^*$$

$$= \frac{1}{2} j\omega L |I|^2$$

Note:

$$P_{abs} = jQ_{abs}$$

(lossless element)

Example (cont.)

$$
\begin{aligned}
Q_{abs} &= \operatorname{Im} P_{abs} = \frac{1}{2}\omega L \left|I\right|^2 \\
&= \omega L \left(\operatorname{Re}\left(\frac{1}{2} I\, I^* \right) \right) \\
&= \omega L \left(\left\langle i^2 \right\rangle \right) = 2\omega \left(\left\langle \frac{1}{2} L\, i^2 \right\rangle \right) \\
&= 2\,\omega \left(\left\langle \mathscr{W}_m \right\rangle \right)
\end{aligned}
$$

Example (cont.)

Since there is no stored electric energy in the inductor, we can write

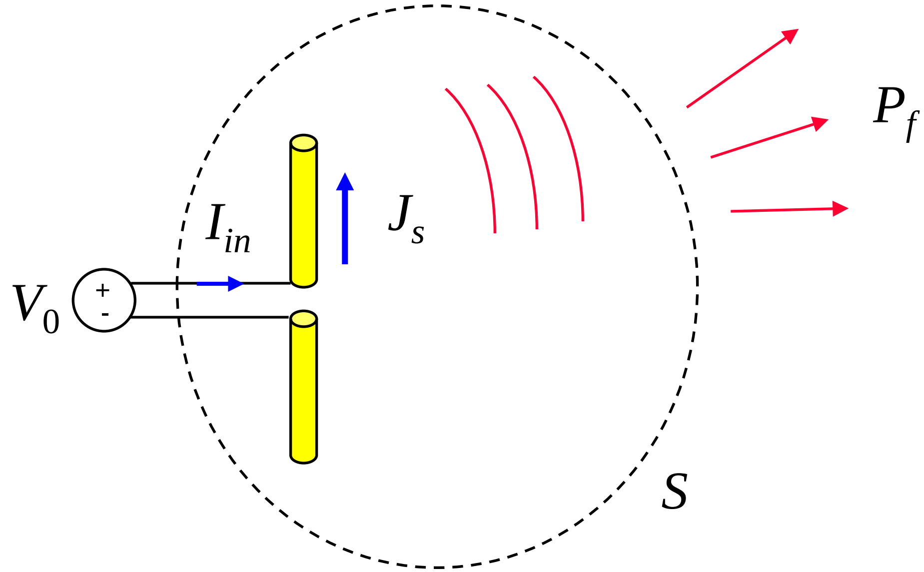
$$Q_{abs} = 2\,\omega\left(\left\langle \mathscr{W}_m \right\rangle - \left\langle \mathscr{W}_e \right\rangle\right)$$

Hence, the circuit-theory result is consistent with the complex Poynting theorem.

Note: The inductor absorbs <u>positive</u> VARS.

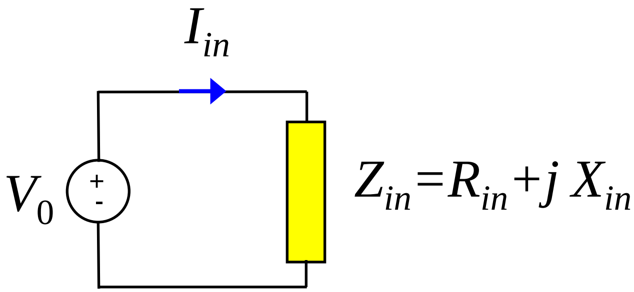
Example

We use the complex Poynting theorem to examine the input impedance of an antenna.



Antenna

Model:



$$P_{in} = \frac{1}{2} V_0\, I_{in}^* = \frac{1}{2}\left(Z_{in}\, I_{in}\right) I_{in}^*$$
$$= \frac{1}{2} Z_{in} \left| I_{in} \right|^2$$

Example (cont.)

so $\boxed{Z_{in} = \dfrac{2P_{in}}{\left|I_{in}\right|^2}}$

Real part: $R_{in} = \dfrac{2\,\mathrm{Re}\left(P_{in}\right)}{\left|I_{in}\right|^2}$

$$\begin{aligned}\mathrm{Re}\left(P_{in}\right) &= \left\langle \mathscr{P}_{in} \right\rangle \\ &= \left\langle \mathscr{P}_{rad}^{\infty} \right\rangle \\ &= \mathrm{Re}\,P_{rad}^{\infty}\end{aligned}$$

(no losses in vacuum surrounding antenna)

Note:
The far-field Poynting vector is much easier to calculate than the near-field Poynting vector.

Hence $R_{in} = \dfrac{2}{\left|I_{in}\right|^2} \oint_{S_\infty} \mathrm{Re}\left(\underline{S} \cdot \underline{\hat{n}}\right) dS$

Example (cont.)

$$R_{in} = \frac{2}{\left|I_{in}\right|^2} \oint_{S_\infty} \mathrm{Re}\left(\underline{S} \cdot \underline{\hat{n}}\right) dS$$

In the far field ($r \to \infty$)

$$\mathrm{Im}\, \underline{S} = \underline{0}$$

(This follows from plane-wave properties in the far field.)

Hence

$$\boxed{R_{in} = \frac{2}{\left|I_{in}\right|^2} \oint_{S_\infty} \left(\underline{S} \cdot \underline{\hat{n}}\right) dS}$$

Example (cont.)

Imaginary part: $$X_{in} = \frac{2}{\left|I_{in}\right|^2} \operatorname{Im}\left(P_{in}\right)$$

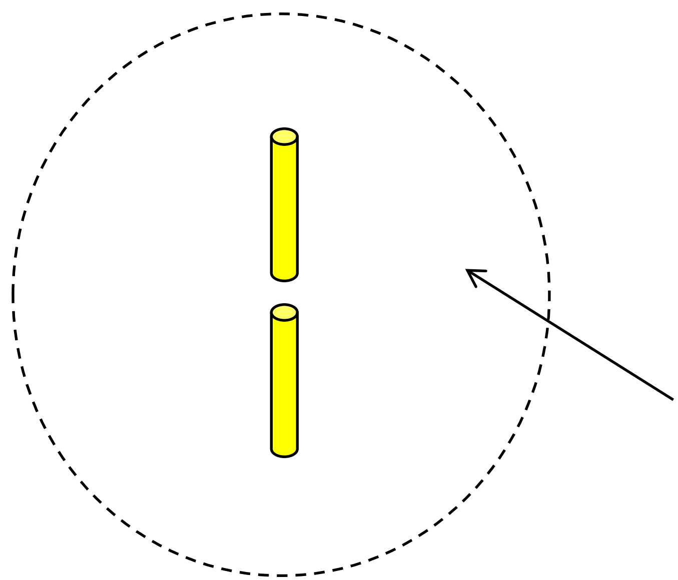
where

$$\operatorname{Im}\left(P_{in}\right) = \cancel{\operatorname{Im}\left(P_{rad}^{\infty}\right)} + 2\omega\left[\left\langle \mathscr{W}_m \right\rangle - \left\langle \mathscr{W}_e \right\rangle\right]$$

Hence

$$X_{in} \neq \frac{2}{\left|I_{in}\right|^2} \oint_{S_\infty} \operatorname{Im} \underline{S} \cdot \underline{\hat{n}}\, dS$$

The RHS is zero!



$$\left\langle \mathscr{W}_m \right\rangle, \left\langle \mathscr{W}_e \right\rangle$$

Example (cont.)

We can say that

$$X_{in} = \frac{2}{\left|I_{in}\right|^2}\left(2\omega\left[\left\langle \mathscr{W}_m \right\rangle - \left\langle \mathscr{W}_e \right\rangle\right]\right)$$

Hence

$$\boxed{X_{in} = \frac{4\omega}{\left|I_{in}\right|^2}\int_V \left(\frac{1}{4}\mu_0\left|\underline{H}\right|^2 - \frac{1}{4}\varepsilon_0\left|\underline{E}\right|^2\right)dV}$$

However, it would be <u>very difficult</u> to calculate the input impedance using this formula!

Dispersive Material

The permittivity and permeability are now functions of frequency:

$$\varepsilon = \varepsilon(\omega)$$

$$\mu = \mu(\omega)$$

(A lossy dispersive media is assumed here. It is assumed that the fields are defined over a fairly narrow band of frequencies.)

The formulas for stored electric and magnetic energy now become:

$$\langle \mathscr{W}_e \rangle = \frac{1}{4} \left| \underline{E} \right|^2 \frac{\partial}{\partial \omega} (\omega \varepsilon')$$

$$\langle \mathscr{W}_m \rangle = \frac{1}{4} \left| \underline{H} \right|^2 \frac{\partial}{\partial \omega} (\omega \mu')$$

Note:
The stored energy should always be positive, even if the permittivity or permeability become negative.

Reference:

J. D. Jackson, *Classical Electrodynamics*, Wiley, 1998 (p. 263).

Momentum Density Vector

The electromagnetic field has a momentum density (momentum per volume):

$$\underline{\mathscr{p}} = \underline{\mathscr{D}} \times \underline{\mathscr{B}}$$

In free space:

$$\underline{\mathscr{p}} = \left(\mu_0 \varepsilon_0\right) \underline{\mathscr{E}} \times \underline{\mathscr{H}}$$

or $$\boxed{\underline{\mathscr{p}} = \frac{1}{c^2} \underline{\mathscr{S}}}$$ $[(\mathrm{kg\ m/s})/\mathrm{m}^3] = [\mathrm{kg}/(\mathrm{s\ m}^2)]$

Reference:

J. D. Jackson, *Classical Electrodynamics*, Wiley, 1998 (p. 262).

Momentum Density Vector (cont.)

Photon:

From physics, we have a relation between the energy E and the momentum p of a single photon.

$$\boxed{E = pc} \qquad \boxed{E = hf} \qquad h = 6.626068\times 10^{-34}\ [\mathrm{J\,s}]$$

(Planck's constant)

Calculation of power flow:

$$\mathscr{S}_z = \frac{Energy}{A\,\Delta t} = \frac{E\left(A\left(c\,\Delta t\right)\right)N_p}{A\,\Delta t}$$

$$= E c\, N_p$$

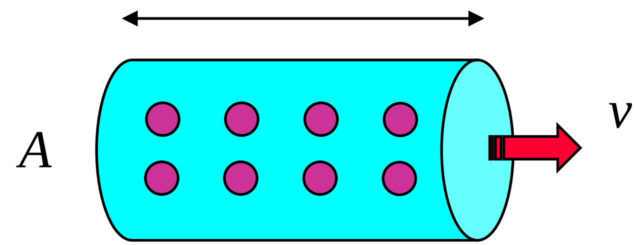
$$= p c^2 N_p = \left(p N_p\right)c^2$$

$$= \wp_z c^2$$

Hence $\boxed{\wp_z = \dfrac{1}{c^2}\mathscr{S}_z}$

Photons moving:

$$\Delta\ell = c\Delta t$$



$$\underline{\mathscr{S}} = \mathscr{S}_z\,\underline{\hat{z}}$$

N_p photons per unit volume

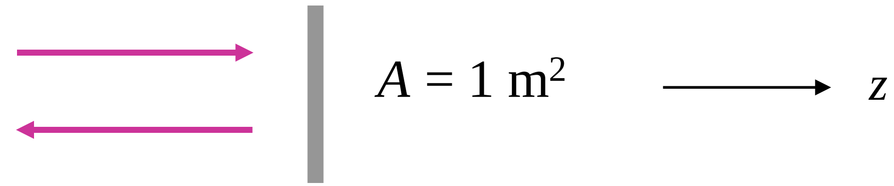
This is consistent with the previous momentum formula.

Momentum Density Vector (cont.)

Example:

Find the time-average force on a $1\ [\mathrm{m}^2]$ mirror illuminated by normally incident sunlight, having a power density of $1\ [\mathrm{kW/m}^2]$.

Note: the fields vary sinusoidally, but the frequency is arbitrary.



$$\left\langle \mathscr{p}_z^{inc} \right\rangle = \frac{1}{c^2}\left\langle \mathscr{S}_z^{inc} \right\rangle = \frac{1}{c^2}(1000)$$

$$\left\langle \mathscr{F}_z \right\rangle = \frac{2\left(\left\langle \mathscr{p}_z^{inc} \right\rangle\right)\left(Ac\,\Delta t\right)}{\Delta t} = 2\left(\left\langle \mathscr{p}_z^{inc} \right\rangle\right)\left(Ac\right) = 2\left[\frac{1}{c^2}(1000)\right]\left[(1)c\right] = \frac{2000}{c}$$

$$\boxed{\left\langle \mathscr{F}_z \right\rangle = 6.671\times 10^{-6}\ [\mathrm{N}]}$$

Solar Sail

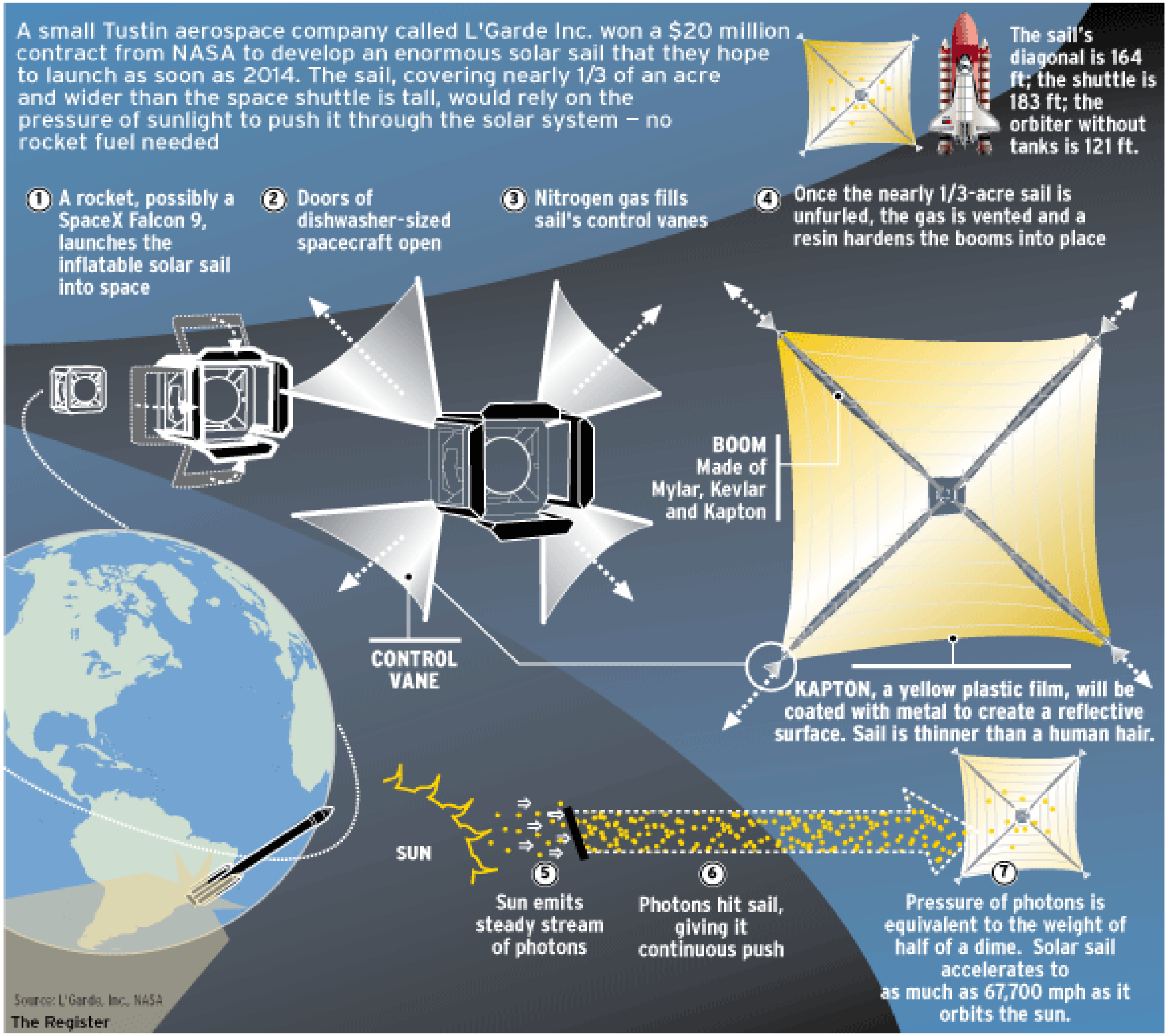
Sunjammer Project

NASA has awarded $20 million to L’Garde, Inc. (Tustin, CA), a maker of “inflatable space structures,” to develop a solar sail, which will rely on the pressure of sunlight to move through space when it takes its first flight as soon as 2014.

A smaller version of L'Garde's solar sail unfurled in a vacuum chamber in Ohio in 2005. This one was about 3,400 square feet, a quarter the size of the sail the company plans to loft into space as soon as 2014. Photo courtesy NASA and L'Garde.

http://www.lgarde.com

Solar Sail (cont.)



http://www.lgarde.com

Maxwell Stress Tensor

This gives us the stress (vector force per unit area) on an object, from knowledge of the fields on the surface of the object.

References:

D. J. Griffiths, Introduction to Electrodynamics, Prentice-Hall, 1989.

J. D. Jackson, *Classical Electrodynamics*, Wiley, 1998.

J. Schwinger, L. L. DeRaad, Jr., K. A. Milton, and W.-Y. Tsai, *Classical Electrodynamics*, Perseus, 1998.

Maxwell Stress Tensor

$$\underline{\underline{T}} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$\boxed{T_{ij} = \varepsilon_0\, \mathscr{E}_i \mathscr{E}_j + \mu_0\, \mathscr{H}_i \mathscr{H}_j - \delta_{ij}\left(\frac{1}{2}\varepsilon_0 \left|\underline{\mathscr{E}}\right|^2 + \frac{1}{2}\mu_0 \left|\underline{\mathscr{H}}\right|^2\right)} \qquad i, j = x, y, z$$

(for vacuum)

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$ (Kronecker delta)

Maxwell Stress Tensor (cont.)

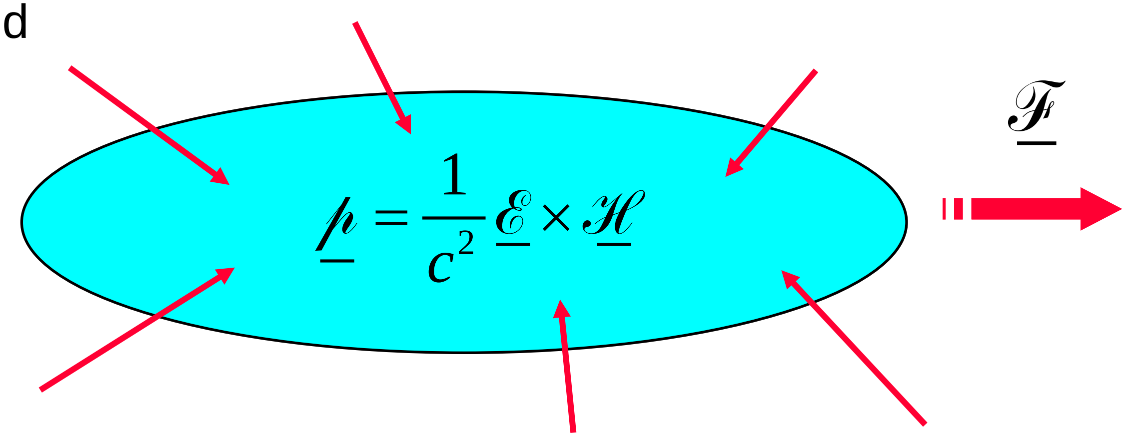
Momentum equation:

$$\oint_S \underline{\underline{T}} \cdot \underline{\hat{n}}\; dS = \underline{\mathscr{F}} + \frac{d}{dt}\int_V \underline{\wp}\; dV$$

Total flow rate of momentum given to object from EM field

Total force on object (rate of change of mechanical momentum)

Rate of change of electromagnetic momentum inside of region



Maxwell Stress Tensor (cont.)

$$\oint_S \underline{\underline{T}} \cdot \underline{\hat{n}}\, dS = \underline{\mathscr{F}} + \frac{d}{dt}\int_V \underline{\wp}\, dV$$

In many practical cases the <u>time-average</u> of the last term (the rate of change of electromagnetic momentum inside of region) is zero:

- No fields inside the body
- Sinusoidal steady-state fields

Example :

$$\frac{d}{dt}\left(\sin^2(\omega t)\right) = 2\omega \sin(\omega t)\cos(\omega t) = \omega \sin(2\omega t)$$

$$\left\langle \frac{d}{dt}\left(\sin^2(\omega t)\right)\right\rangle = \omega \left\langle \sin(2\omega t)\right\rangle = 0$$

In this case we have

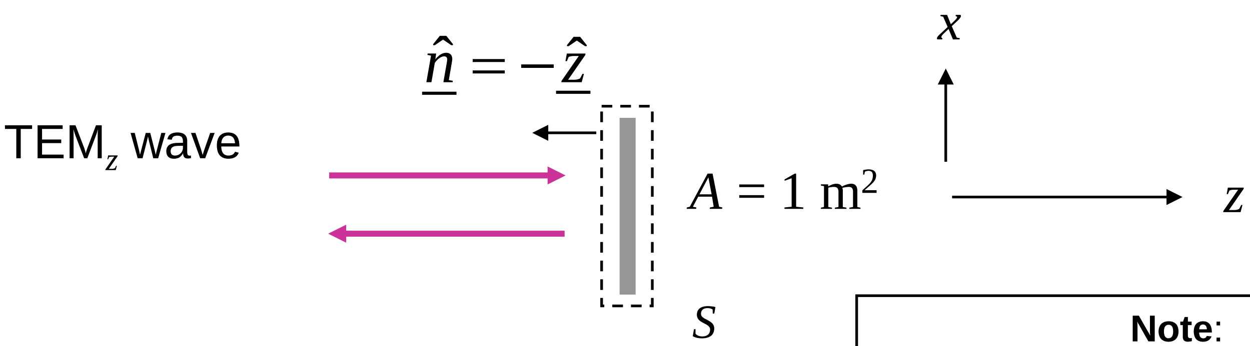
$$\boxed{\left\langle \underline{\mathscr{F}} \right\rangle = \oint_S \left\langle \underline{\underline{T}} \right\rangle \cdot \underline{\hat{n}}\, dS}$$

The Maxwell stress tensor (matrix) is then interpreted as the stress (vector force per unit area) on the surface of the body.

Maxwell Stress Tensor (cont.)

Example:

Find the time-average force on a $1\ \mathrm{m}^2$ mirror illuminated by normally incident sunlight, having a power density of $1\ [\mathrm{kW/m^2}]$.



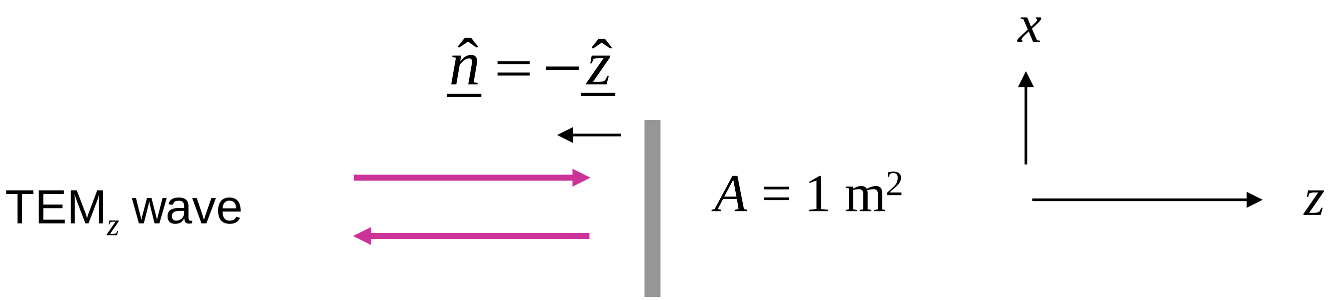
Note:
The fields are assumed to be zero on the back side of the mirror, and no fields are inside the mirror.

$$\mathcal{F}_z = \underline{\hat{z}} \cdot \underline{\mathcal{F}}$$

$$\underline{\mathcal{F}} = \underline{\underline{T}} \cdot \underline{\hat{n}} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -T_{xz} \\ -T_{yz} \\ -T_{zz} \end{bmatrix}$$

$$\mathcal{F}_z = -T_{zz}$$

Maxwell Stress Tensor (cont.)



$$T_{zz} = \cancel{\varepsilon_0\, \mathcal{E}_z \mathcal{E}_z} + \cancel{\mu_0\, \mathcal{H}_z \mathcal{H}_z} - \left(\frac{1}{2}\varepsilon_0 \left|\underline{\mathcal{E}}\right|^2 + \frac{1}{2}\mu_0 \left|\underline{\mathcal{H}}\right|^2 \right)$$

$$= -\left(\frac{1}{2}\varepsilon_0 \left|\underline{\mathcal{E}}\right|^2 + \frac{1}{2}\mu_0 \left|\underline{\mathcal{H}}\right|^2 \right)_{\text{front side}}$$

Assume: $\underline{\mathcal{E}} = \mathcal{E}_x\, \underline{\hat{x}}\ ,\ \ \underline{\mathcal{H}} = \mathcal{H}_y\, \underline{\hat{y}}$ $\qquad \mathcal{E}_x^{inc} \,/\, \mathcal{H}_y^{inc} = \eta_0$

Maxwell Stress Tensor (cont.)

$$\mathscr{F}_z = \left(\frac{1}{2}\varepsilon_0 \cancel{\mathscr{E}_x^2} + \frac{1}{2}\mu_0 \mathscr{H}_y^2 \right)_{\text{front side}}$$

(PEC mirror)

$$= \frac{1}{2}\mu_0 \mathscr{H}_y^2$$

$$\langle \mathscr{F}_z \rangle = \frac{1}{2}\mu_0 \langle \mathscr{H}_y^2 \rangle$$

$$= \frac{1}{2}\mu_0 \left(\frac{1}{2}\operatorname{Re}\left(H_y H_y^* \right) \right)$$

$$= \frac{1}{2}\mu_0 \left(\frac{1}{2}\operatorname{Re}\left(\left| H_y \right|^2 \right) \right)$$

$$= \frac{1}{4}\mu_0 \left| H_y \right|^2$$

We then have:

$$\langle \mathscr{F}_z \rangle = \frac{1}{4}\mu_0 \left| H_y \right|^2$$

$$= \frac{1}{4}\mu_0 \left| 2H_y^{inc} \right|^2$$

(The tangential magnetic field doubles at the shorting plate.)

Maxwell Stress Tensor (cont.)

$$\left\langle \mathscr{F}_z \right\rangle = \frac{1}{4}\mu_0 \left|2H_y^{inc}\right|^2 = \mu_0 \left|H_y^{inc}\right|^2$$

Next, use

$$\left\langle S_z^{inc} \right\rangle = \left\langle \mathscr{E}_x^{inc} \mathscr{H}_y^{inc} \right\rangle = \frac{1}{2} E_x^{inc} H_y^{inc*} = \frac{1}{2}\left|H_y^{inc}\right|^2 \eta_0$$

$$\left(E_x^{inc} / H_y^{inc} = \eta_0 \right)$$

$$\begin{aligned} \left\langle \mathscr{F}_z \right\rangle &= \mu_0 \left|H_y^{inc}\right|^2 \\ &= \mu_0 \left(2\left\langle S_z^{inc} \right\rangle / \eta_0 \right) \\ &= 4\pi \times 10^{-7} \left(2(1000) / 376.7303 \right) \end{aligned}$$

so

$$\boxed{\left\langle \mathscr{F}_z \right\rangle = 6.671 \times 10^{-6} \, [\mathrm{N}]}$$

Force on Dielectric Object

Here we calculate the electrostatic force on a dielectric object in an electric field using dipole moments.

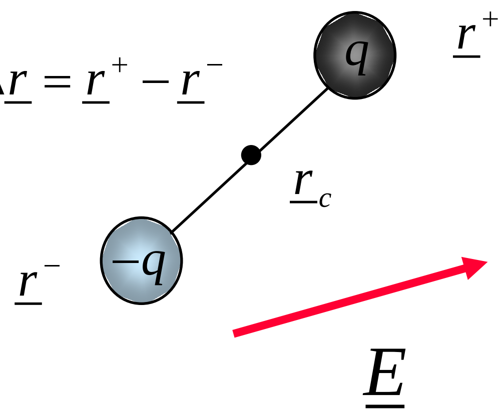
This is an alternative to using the Maxwell stress tensor.

Consider first the force on a single electrostatic dipole:

$$\underline{F} = q\left(\underline{E}\left(\underline{r}^{+}\right) - \underline{E}\left(\underline{r}^{-}\right)\right)$$

$$\Delta\underline{r} = \underline{r}^{+} - \underline{r}^{-}$$

$$\underline{p} = q\,\Delta\underline{r}$$



For small dipoles:

$$\underline{F} \approx q\left(\Delta\underline{r} \cdot \nabla\underline{E}\left(\underline{r}_c\right)\right)$$

or

$$\underline{F} \approx \underline{p} \cdot \nabla\underline{E}\left(\underline{r}_c\right)$$

Note that we have a gradient of a vector here. This is called a “dyad.”

Force on Dielectric Object (cont.)

Dyadic representation:

$$\nabla \underline{E} = \underline{\hat{x}} \frac{\partial}{\partial x}\left(\underline{\hat{x}} E_x + \underline{\hat{y}} E_y + \underline{\hat{z}} E_z\right) + \underline{\hat{y}} \frac{\partial}{\partial y}\left(\underline{\hat{x}} E_x + \underline{\hat{y}} E_y + \underline{\hat{z}} E_z\right) + \underline{\hat{z}} \frac{\partial}{\partial z}\left(\underline{\hat{x}} E_x + \underline{\hat{y}} E_y + \underline{\hat{z}} E_z\right)$$

or

$$\nabla \underline{E} = \left(\underline{\hat{x}}\,\underline{\hat{x}} \frac{\partial E_x}{\partial x} + \underline{\hat{x}}\,\underline{\hat{y}} \frac{\partial E_y}{\partial x} + \underline{\hat{x}}\,\underline{\hat{z}} \frac{\partial E_z}{\partial x}\right) + \left(\underline{\hat{y}}\,\underline{\hat{x}} \frac{\partial E_x}{\partial y} + \underline{\hat{y}}\,\underline{\hat{y}} \frac{\partial E_y}{\partial y} + \underline{\hat{y}}\,\underline{\hat{z}} \frac{\partial E_z}{\partial y}\right) + \left(\underline{\hat{z}}\,\underline{\hat{x}} \frac{\partial E_x}{\partial z} + \underline{\hat{z}}\,\underline{\hat{y}} \frac{\partial E_y}{\partial z} + \underline{\hat{z}}\,\underline{\hat{z}} \frac{\partial E_z}{\partial z}\right)$$

Hence we have:

$$\Delta \underline{r} \cdot \nabla \underline{E} = \left(\underline{\hat{x}} \Delta_x + \underline{\hat{y}} \Delta_y + \underline{\hat{z}} \Delta_z\right) \cdot \nabla \underline{E} =$$

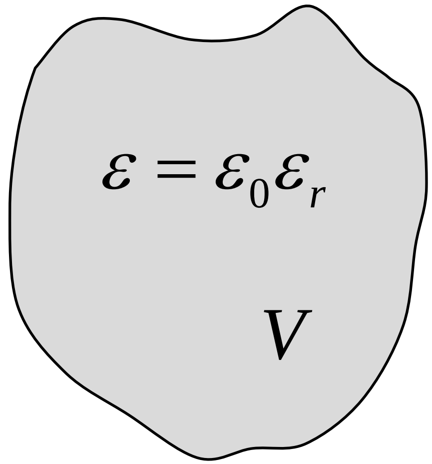
$$\Delta_x \left(\underline{\hat{x}} \frac{\partial E_x}{\partial x} + \underline{\hat{y}} \frac{\partial E_y}{\partial x} + \underline{\hat{z}} \frac{\partial E_z}{\partial x}\right) + \Delta_y \left(\underline{\hat{x}} \frac{\partial E_x}{\partial y} + \underline{\hat{y}} \frac{\partial E_y}{\partial y} + \underline{\hat{z}} \frac{\partial E_z}{\partial y}\right) + \Delta_z \left(\underline{\hat{x}} \frac{\partial E_x}{\partial z} + \underline{\hat{y}} \frac{\partial E_y}{\partial z} + \underline{\hat{z}} \frac{\partial E_z}{\partial z}\right)$$

Force on Dielectric Object (cont.)

For a dielectric object:

$$\boxed{\underline{P} = \varepsilon_0 \chi_e \, \underline{E}}$$

$$\varepsilon_r = (1 + \chi_e)$$



$$\underline{F} = \underline{p} \cdot \nabla \underline{E}(\underline{r}_c) \quad \Rightarrow \quad d\underline{F}(\underline{r}) = \underline{P}(\underline{r}) \cdot \nabla \underline{E}(\underline{r}) \, dV$$

$$\underline{F} = \int_V \underline{P}(\underline{r}) \cdot \nabla \underline{E}(\underline{r}) \, dV$$

Since the body does not exert a force on itself, we can use the <u>external</u> electric field $\underline{E}_0$.

$$\underline{F} = \int_V \underline{P}(\underline{r}) \cdot \nabla \underline{E}_0(\underline{r}) \, dV$$

Force on Dielectric Object (cont.)

We then have:

$$\boxed{\underline{F} = \varepsilon_0 \chi_e \int_V \underline{E}(\underline{r}) \cdot \nabla \underline{E}_0(\underline{r})\, dV}$$

We can also write:

$$\underline{F} = \varepsilon_0 \chi_e \int_V \underline{E}(\underline{r}) \cdot \nabla \underline{E}(\underline{r})\, dV$$

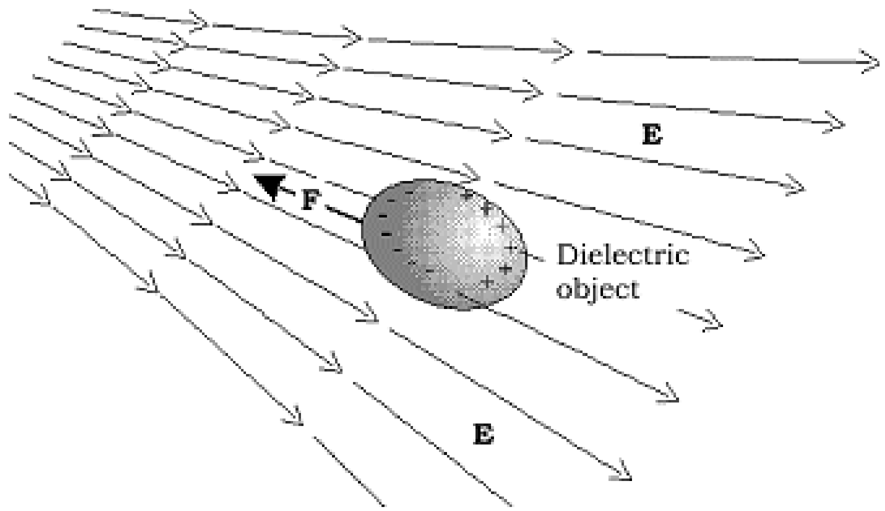
or

$$\underline{F} = \varepsilon_0 \chi_e \int_V \frac{1}{2} \nabla \left(\underline{E}(\underline{r}) \cdot \underline{E}(\underline{r}) \right) dV$$

or

$$\boxed{\underline{F} = \varepsilon_0 \chi_e \int_V \frac{1}{2} \nabla \left(\left| \underline{E}(\underline{r}) \right|^2 \right) dV}$$

Force on Dielectric Object (cont.)



A non-uniform electrical field will generate a net <u>attractive</u> force on a neutral piece of matter. The force is directed toward the region of higher field strength.

http://electrogravityphysics.com/html/sec_4.html

Force on Magnetic Object

We have (derivation omitted) the magnetostatic force as:

$$\underline{F} = \int_V \underline{M}(\underline{r}) \cdot \nabla \underline{B}_0(\underline{r})\, dV$$

Recall: $\underline{M} = \chi_m \underline{H}$

Hence, we have

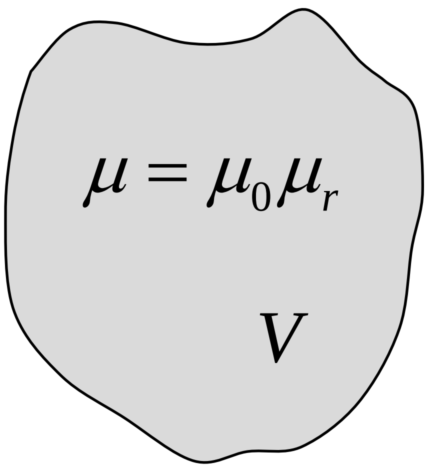
$$\underline{F} = \chi_m \int_V \underline{H}(\underline{r}) \cdot \nabla \underline{B}_0(\underline{r})\, dV$$

or

$$\underline{F} = \frac{\chi_m}{\mu} \int_V \underline{B}(\underline{r}) \cdot \nabla \underline{B}_0(\underline{r})\, dV$$

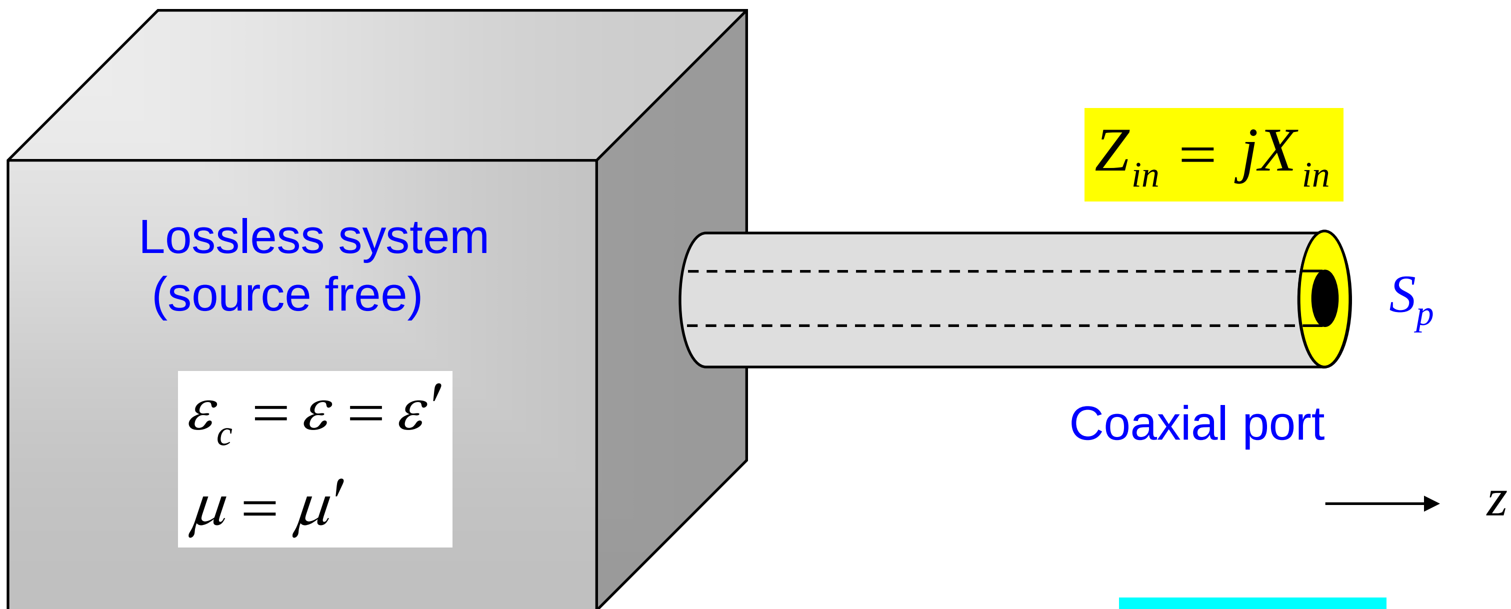
or

$$\underline{F} = \frac{\chi_m}{\mu} \int_V \frac{1}{2} \nabla \left(\left| \underline{B}(\underline{r}) \right|^2 \right) dV$$



Foster's Theorem

Consider a lossless system with a port that leads into it:



PEC enclosure

Foster's theorem: $$\frac{dX_{in}}{d\omega} > 0$$

R. E. Collin, *Field Theory of Guided Waves*, IEEE Press, Piscataway, NJ, 1991.

Foster's Theorem (cont.)

The same holds for the input susceptance:

$$\boxed{Z_{in} = \frac{1}{Y_{in}}}$$

$$\Rightarrow \quad jX_{in} = \frac{1}{jB_{in}}$$

$$\Rightarrow \quad X_{in} = -\frac{1}{B_{in}}$$

$$\Rightarrow \quad \frac{dX_{in}}{d\omega} = \frac{1}{B_{in}^2}\frac{dB_{in}}{d\omega} \quad \Rightarrow \quad \boxed{\frac{dB_{in}}{d\omega} > 0}$$

Foster's Theorem (cont.)

Start with the following vector identity:

$$\nabla \cdot \left(\underline{A} \times \underline{B} \right) = \underline{B} \cdot \left(\nabla \times \underline{A} \right) - \underline{A} \cdot \left(\nabla \times \underline{B} \right)$$

Hence we have (applying twice, for two different choices of the (vectors)

$$\nabla \cdot \left(\underline{E} \times \frac{\partial \underline{H}^*}{\partial \omega} \right) = \left(\frac{\partial \underline{H}^*}{\partial \omega} \right) \cdot \left(\nabla \times \underline{E} \right) - \underline{E} \cdot \left(\nabla \times \frac{\partial \underline{H}^*}{\partial \omega} \right)$$

$$\nabla \cdot \left(\frac{\partial \underline{E}^*}{\partial \omega} \times \underline{H} \right) = \underline{H} \cdot \left(\nabla \times \frac{\partial \underline{E}^*}{\partial \omega} \right) - \left(\frac{\partial \underline{E}^*}{\partial \omega} \right) \cdot \left(\nabla \times \underline{H} \right)$$

Add these last two equations together:

Foster's Theorem (cont.)

$$\nabla \cdot \left(\underline{E} \times \frac{\partial \underline{H}^*}{\partial \omega} + \frac{\partial \underline{E}^*}{\partial \omega} \times \underline{H} \right) = \left(\frac{\partial \underline{H}^*}{\partial \omega} \right) \cdot \left(\nabla \times \underline{E} \right) - \underline{E} \cdot \left(\nabla \times \frac{\partial \underline{H}^*}{\partial \omega} \right)$$
$$+ \underline{H} \cdot \left(\nabla \times \frac{\partial \underline{E}^*}{\partial \omega} \right) - \left(\frac{\partial \underline{E}^*}{\partial \omega} \right) \cdot \left(\nabla \times \underline{H} \right)$$

Using Maxwell's equations for a source-free region,

$$\nabla \cdot \left(\underline{E} \times \frac{\partial \underline{H}^*}{\partial \omega} + \frac{\partial \underline{E}^*}{\partial \omega} \times \underline{H} \right) = \left(\frac{\partial \underline{H}^*}{\partial \omega} \right) \cdot \left(-j\omega\mu \underline{H} \right) - \underline{E} \cdot \left(\nabla \times \frac{\partial \underline{H}^*}{\partial \omega} \right)$$
$$+ \underline{H} \cdot \left(\nabla \times \frac{\partial \underline{E}^*}{\partial \omega} \right) - \left(\frac{\partial \underline{E}^*}{\partial \omega} \right) \cdot \left(j\omega\varepsilon \underline{E} \right)$$

Foster's Theorem (cont.)

$$\nabla\cdot\left(\underline{E}\times\frac{\partial\underline{H}^*}{\partial\omega}+\frac{\partial\underline{E}^*}{\partial\omega}\times\underline{H}\right)=\left(\frac{\partial\underline{H}^*}{\partial\omega}\right)\cdot\left(-j\omega\mu\underline{H}\right)-\underline{E}\cdot\boxed{\left(\nabla\times\frac{\partial\underline{H}^*}{\partial\omega}\right)}$$

$$+\underline{H}\cdot\boxed{\left(\nabla\times\frac{\partial\underline{E}^*}{\partial\omega}\right)}-\left(\frac{\partial\underline{E}^*}{\partial\omega}\right)\cdot\left(j\omega\varepsilon\underline{E}\right)$$

Using Maxwell's equations again, and the chain rule, we have:

$$\nabla\times\frac{\partial\underline{H}^*}{\partial\omega}=\frac{\partial}{\partial\omega}\left(\nabla\times\underline{H}\right)^*=\frac{\partial}{\partial\omega}\left(j\omega\varepsilon\underline{E}\right)^*=-j\frac{\partial}{\partial\omega}\left(\omega\varepsilon\right)\underline{E}^*-j\omega\varepsilon\frac{\partial\underline{E}^*}{\partial\omega}$$

$$\nabla\times\frac{\partial\underline{E}^*}{\partial\omega}=\frac{\partial}{\partial\omega}\left(\nabla\times\underline{E}\right)^*=-\frac{\partial}{\partial\omega}\left(j\omega\mu\underline{H}\right)^*=j\frac{\partial}{\partial\omega}\left(\omega\mu\right)\underline{H}^*+j\omega\mu\frac{\partial\underline{H}^*}{\partial\omega}$$

We then have

Foster's Theorem (cont.)

$$\nabla\cdot\left(\underline{E}\times\frac{\partial\underline{H}^*}{\partial\omega}+\frac{\partial\underline{E}^*}{\partial\omega}\times\underline{H}\right)=\left(\frac{\partial\underline{H}^*}{\partial\omega}\right)\cdot\left(-j\omega\mu\underline{H}\right)-\underline{E}\cdot\left(-j\frac{\partial}{\partial\omega}(\omega\varepsilon)\underline{E}^*-j\omega\varepsilon\frac{\partial\underline{E}^*}{\partial\omega}\right)$$

$$+\underline{H}\cdot\left(j\frac{\partial}{\partial\omega}(\omega\mu)\underline{H}^*+j\omega\mu\frac{\partial\underline{H}^*}{\partial\omega}\right)-\left(\frac{\partial\underline{E}^*}{\partial\omega}\right)\cdot\left(j\omega\varepsilon\underline{E}\right)$$

cancels

Simplifying, we have

$$\nabla\cdot\left(\underline{E}\times\frac{\partial\underline{H}^*}{\partial\omega}+\frac{\partial\underline{E}^*}{\partial\omega}\times\underline{H}\right)=-\underline{E}\cdot\left(-j\frac{\partial}{\partial\omega}(\omega\varepsilon)\underline{E}^*\right)+\underline{H}\cdot\left(j\frac{\partial}{\partial\omega}(\omega\mu)\underline{H}^*\right)$$

or

$$\nabla\cdot\left(\underline{E}\times\frac{\partial\underline{H}^*}{\partial\omega}+\frac{\partial\underline{E}^*}{\partial\omega}\times\underline{H}\right)=j\left[\left(\frac{\partial}{\partial\omega}(\omega\varepsilon)\left|\underline{E}\right|^2\right)+\left(\frac{\partial}{\partial\omega}(\omega\mu)\left|\underline{H}\right|^2\right)\right]$$

Foster's Theorem (cont.)

$$\nabla \cdot \left(\underline{E} \times \frac{\partial \underline{H}^*}{\partial \omega} + \frac{\partial \underline{E}^*}{\partial \omega} \times \underline{H} \right) = j \left[\left(\frac{\partial}{\partial \omega} (\omega \varepsilon) \left| \underline{E} \right|^2 \right) + \left(\frac{\partial}{\partial \omega} (\omega \mu) \left| \underline{H} \right|^2 \right) \right]$$

Applying the divergence theorem,

$$\oint_S \left(\underline{E} \times \frac{\partial \underline{H}^*}{\partial \omega} + \frac{\partial \underline{E}^*}{\partial \omega} \times \underline{H} \right) \cdot \underline{\hat{n}}\, dS = j4 \left[\langle \mathscr{W}_e \rangle + \langle \mathscr{W}_m \rangle \right]$$

Hypothesis: the total stored energy must be positive.

Therefore,

$$\operatorname{Im} \oint_S \left(\underline{E} \times \frac{\partial \underline{H}^*}{\partial \omega} + \frac{\partial \underline{E}^*}{\partial \omega} \times \underline{H} \right) \cdot \underline{\hat{n}}\, dS > 0$$

Foster's Theorem (cont.)

$$\operatorname{Im} \oint_S \left(\underline{E} \times \frac{\partial \underline{H}^*}{\partial \omega} + \frac{\partial \underline{E}^*}{\partial \omega} \times \underline{H} \right) \cdot \underline{\hat{n}} \; dS > 0$$

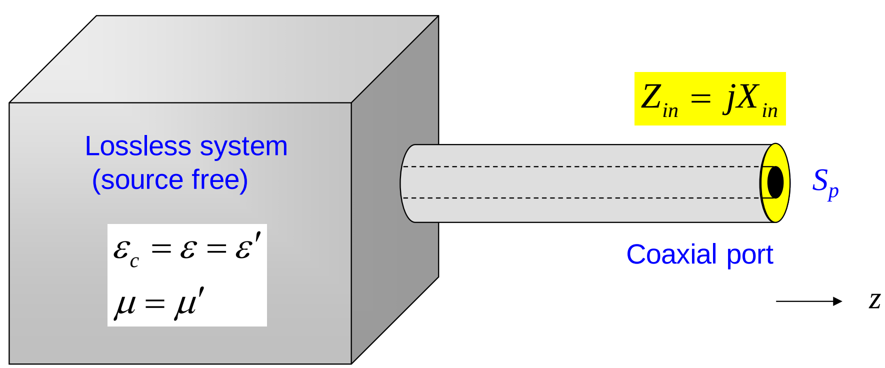
The tangential electric field is only nonzero at the port. Hence we have:

$$\operatorname{Im} \int_{S_p} \left(\underline{E} \times \frac{\partial \underline{H}^*}{\partial \omega} + \frac{\partial \underline{E}^*}{\partial \omega} \times \underline{H} \right) \cdot \underline{\hat{n}} \, dS > 0$$

Assume that the electric field (voltage) at the port is fixed (not changing with frequency).

Then we have

$$\frac{\partial \underline{E}^*}{\partial \omega} = 0$$

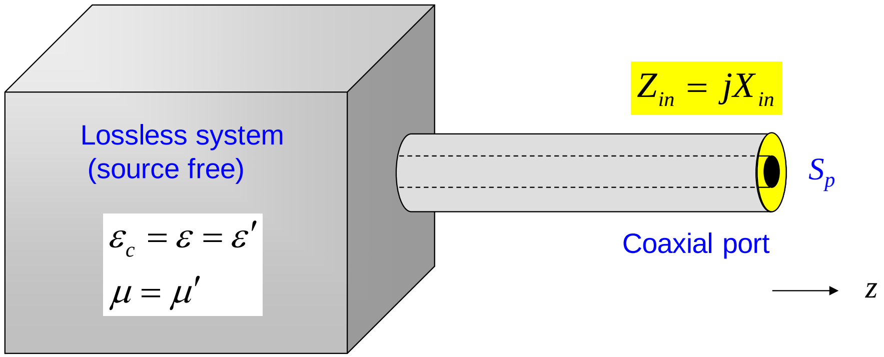


Foster's Theorem (cont.)

Then $$\operatorname{Im} \int_{S_p} \left(\underline{E} \times \frac{\partial \underline{H}^*}{\partial \omega} \right) \cdot \underline{\hat{n}}\, dS > 0$$

At the coaxial port:

$$\underline{E} = \underline{\hat{\rho}}\, E_\rho$$
$$\underline{H} = \underline{\hat{\phi}}\, H_\phi$$



Hence:

$$\operatorname{Im} \int_{S_p} \left(E_\rho \frac{\partial H_\phi^*}{\partial \omega} \right) dS > 0$$

or $$\frac{\partial}{\partial \omega} \operatorname{Im} \int_{S_p} \left(E_\rho H_\phi^* \right) dS > 0$$

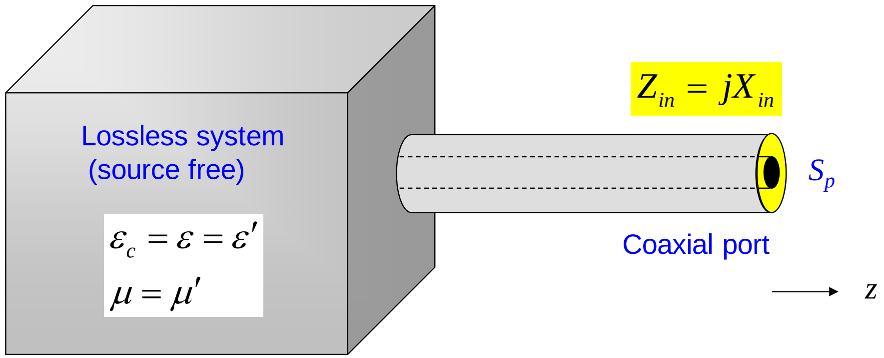
(since the electric field is fixed and not changing with frequency)

Foster's Theorem (cont.)

$$\frac{\partial}{\partial\omega}\operatorname{Im}\int_{S_p}\left(E_\rho H_\phi^{\,*}\right)dS > 0$$

$$\Rightarrow \quad \frac{\partial}{\partial\omega}\operatorname{Im}\left(2P_z\right) > 0$$

$$\Rightarrow \quad \frac{\partial}{\partial\omega}\operatorname{Im}\left[V I_z^{*}\right] > 0$$



$$\Rightarrow \quad \frac{\partial}{\partial\omega}\operatorname{Im}\left[-\left|V\right|^2 Y_{in}^{*}\right] > 0 \qquad \left(-I_z = V\,Y_{in}\right)$$

$$\Rightarrow \quad \frac{\partial}{\partial\omega}\operatorname{Im}\left[Y_{in}^{*}\right] < 0$$

Foster's Theorem (cont.)

$$\frac{\partial}{\partial \omega} \operatorname{Im}\left[Y_{in}^{*}\right] < 0$$

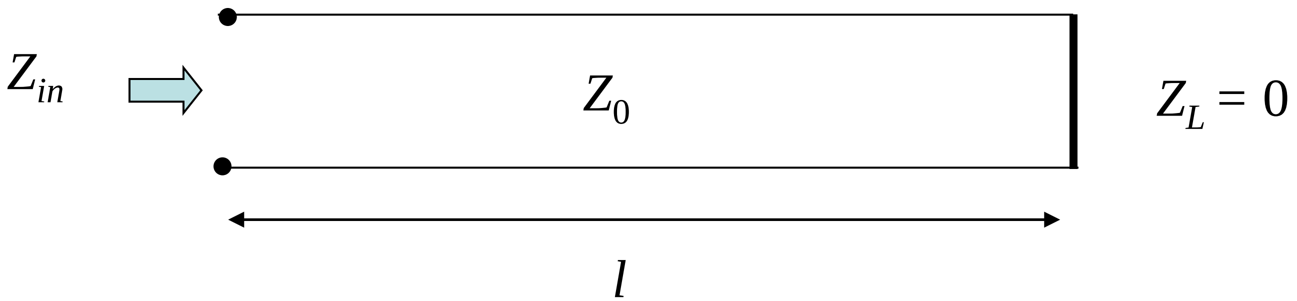
$$\Longrightarrow \quad \frac{\partial}{\partial \omega} \operatorname{Im}\left[Y_{in}\right] > 0$$

$$\Longrightarrow \quad \frac{\partial}{\partial \omega} B_{in} > 0$$

$$\Longrightarrow \quad \boxed{\frac{\partial}{\partial \omega} X_{in} > 0}$$

Foster's Theorem (cont.)

Example: Short-circuited transmission line



$$Z_{in} = jZ_0 \tan\left(\beta l\right)$$

$$\beta = \omega\sqrt{LC}$$