ECE 6340
Intermediate EM Waves

Fall 2016

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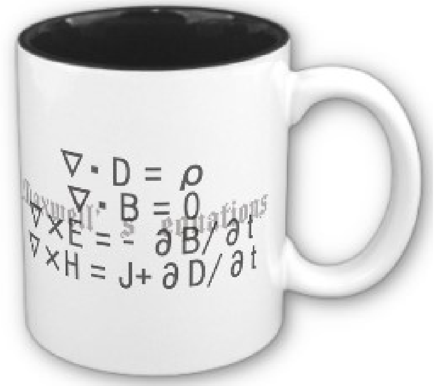
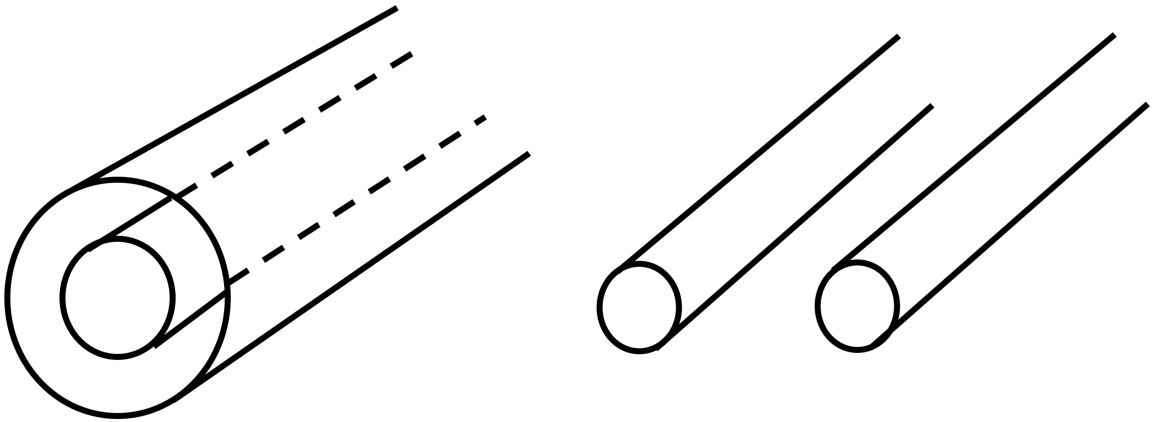
Dept. of ECE



Notes 7

TEM Transmission Line

2 conductors



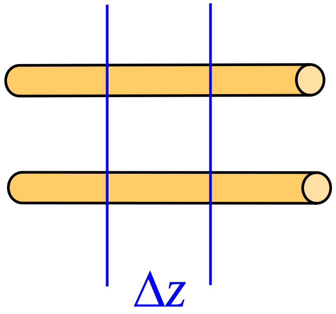
4 parameters

C = capacitance/length [F/m]

L = inductance/length [H/m]

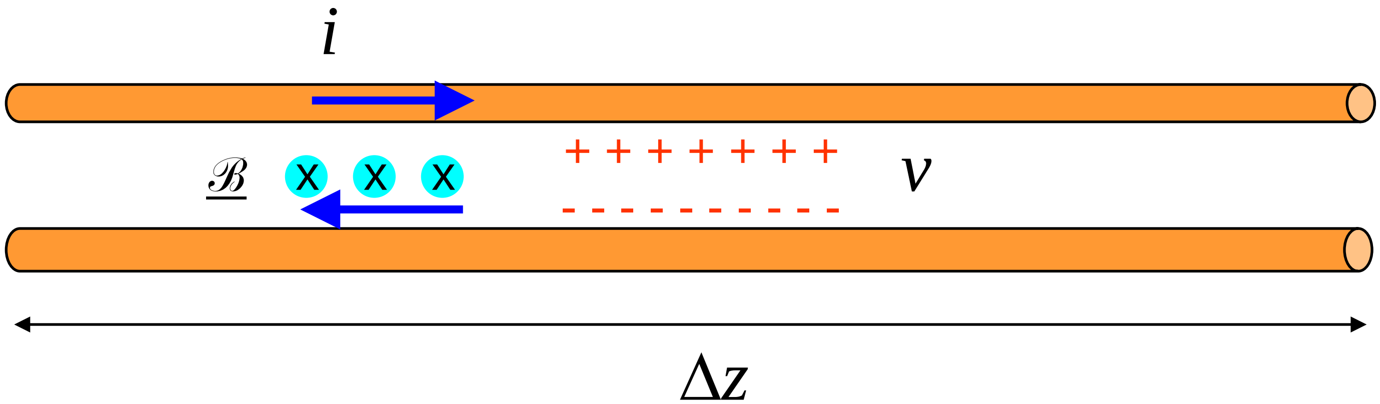
R = resistance/length [Ω/m]

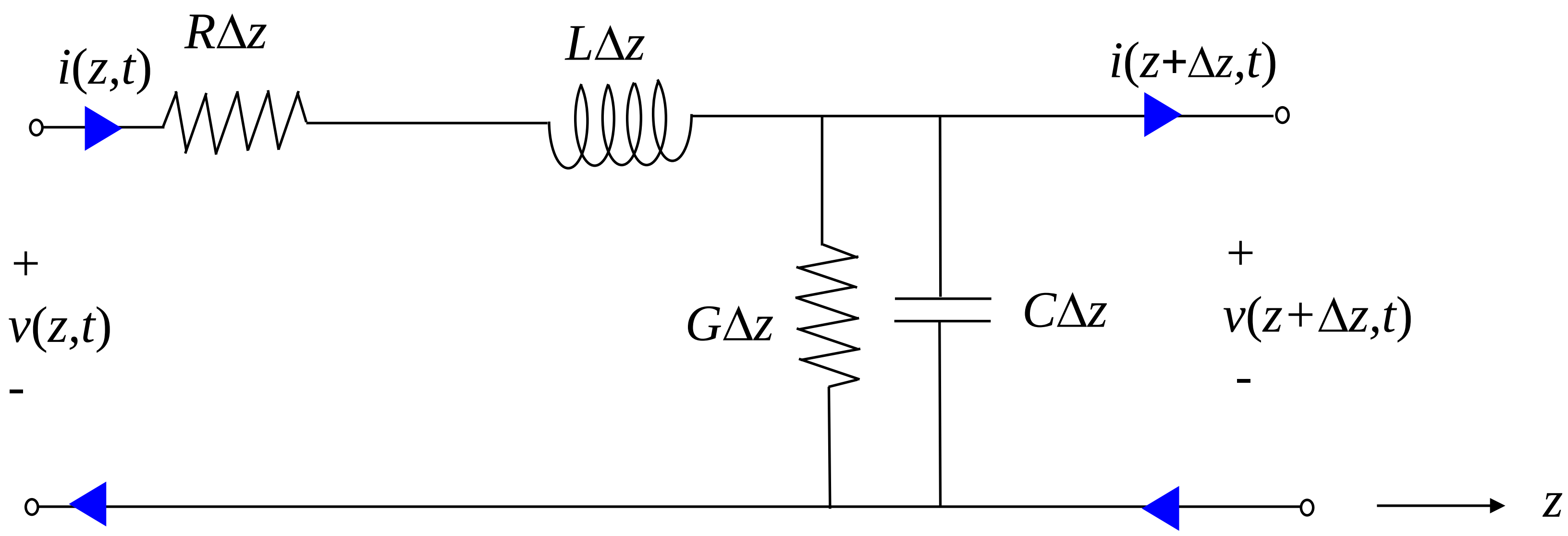
G = conductance/length [℧/m or S/m]



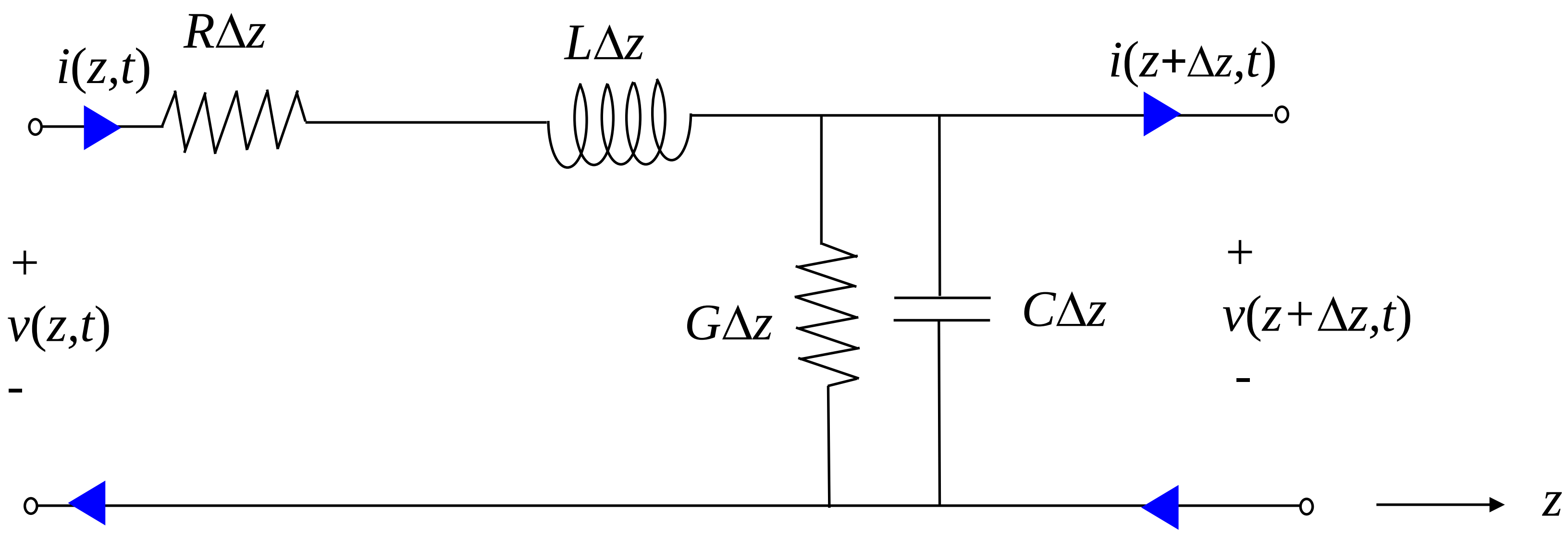
TEM Transmission Line (cont.)

Physical Transmission Line





TEM Transmission Line (cont.)



$$v(z,t) = v(z+\Delta z,t) + i(z,t)R\Delta z + L\Delta z \frac{\partial i(z,t)}{\partial t}$$

$$i(z,t) = i(z+\Delta z,t) + v(z+\Delta z,t)G\Delta z + C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t}$$

TEM Transmission Line (cont.)

Hence

$$\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z} = -Gv(z+\Delta z,t) - C\frac{\partial v(z+\Delta z,t)}{\partial t}$$

Now let $\Delta z \longrightarrow 0$:

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t}$$

"Telegrapher's Equations"

TEM Transmission Line (cont.)

To combine these, take the derivative of the first one with respect to z:

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$

$$\frac{\partial^2 v}{\partial z^2} = -R\frac{\partial i}{\partial z} - L\frac{\partial}{\partial z}\left(\frac{\partial i}{\partial t}\right)$$

$$= -R\frac{\partial i}{\partial z} - L\frac{\partial}{\partial t}\left(\frac{\partial i}{\partial z}\right)$$

Switch the order of the derivatives.

$$= -R\left[-Gv - C\frac{\partial v}{\partial t}\right]$$

$$- L\left[-G\frac{\partial v}{\partial t} - C\frac{\partial^2 v}{\partial t^2}\right]$$

$$\frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t}$$

Substitute from the second Telegrapher's equation.

TEM Transmission Line (cont.)

$$\frac{\partial^2 v}{\partial z^2} = -R\left[-Gv - C\frac{\partial v}{\partial t}\right] - L\left[-G\frac{\partial v}{\partial t} - C\frac{\partial^2 v}{\partial t^2}\right]$$

Hence, collecting terms, we have:

$$\frac{\partial^2 v}{\partial z^2} - \left(RG\right)v - (RC + LG)\frac{\partial v}{\partial t} - LC\left(\frac{\partial^2 v}{\partial t^2}\right) = 0$$

The same equation also holds for i.

TEM Transmission Line (cont.)

Time-Harmonic Waves

$$\frac{\partial^2 v}{\partial z^2} - (RG)v - (RC + LG)\frac{\partial v}{\partial t} - LC\left(\frac{\partial^2 v}{\partial t^2}\right) = 0$$

$$\frac{d^2 V}{dz^2} - (RG)V - (RC + LG)\, j\omega V - LC(-\omega^2)V = 0$$

TEM Transmission Line (cont.)

$$\frac{d^2V}{dz^2} = \left(RG\right)V + j\omega(RC + LG)V - \left(\omega^2 LC\right)V$$

Note that

$$RG + j\omega(RC + LG) - \omega^2 LC = (R + j\omega L)(G + j\omega C)$$

$Z = R + j\omega L$ = series impedance/length

$Y = G + j\omega C$ = parallel admittance/length

Then we can write:

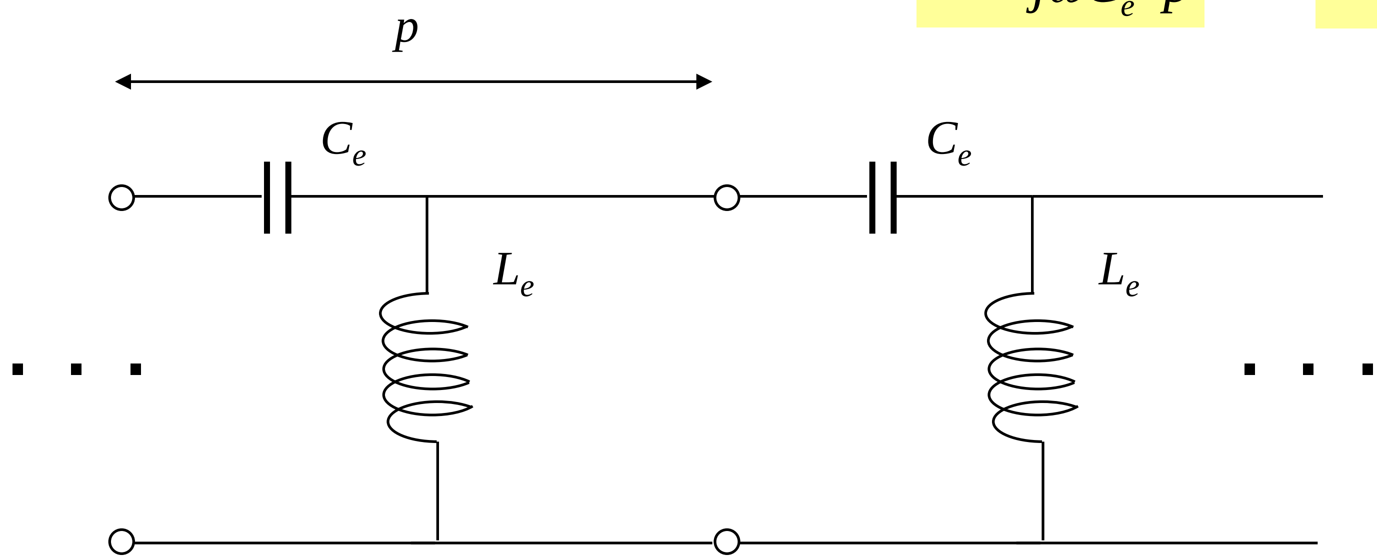
$$\frac{d^2V}{dz^2} = (ZY)V$$

Artificial Transmission Line (ATL)

This is one that is made by artificially cascading periodic sections of elements, which are arranged differently from what an actual (TEM) transmission line would have.

$$p \ll \lambda_0$$

$$Z = \frac{1}{j\omega C_e}\frac{1}{p} \qquad Y = \frac{1}{j\omega L_e}\frac{1}{p}$$



We still have: $$\frac{d^2V}{dz^2} = (ZY)V$$

$$Z \neq R + j\omega L$$

$$Y \neq G + j\omega C$$

Propagation Constant

Let $\gamma^2 = ZY$ Then $\frac{d^2V}{dz^2} = (\gamma^2)V$

Solution: $V(z) = Ae^{-\gamma z} + Be^{+\gamma z}$

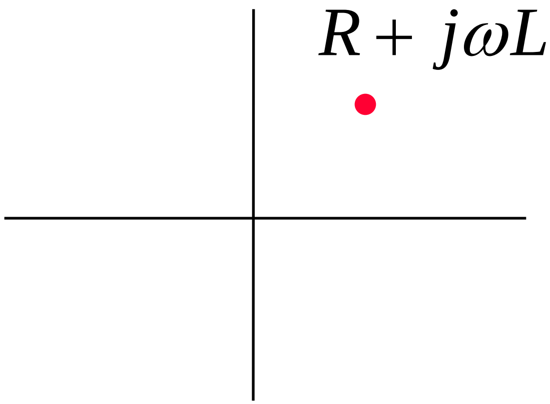
γ is called the "*propagation constant*."

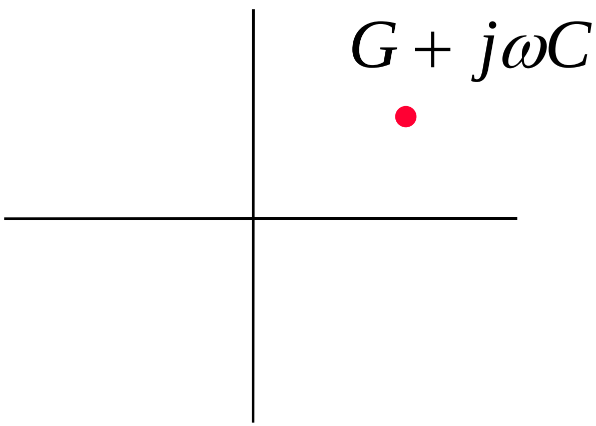
Convention: $+z$ wave is $V^+(z) = V_0^+ e^{-\gamma z}$

To see what this implies, use

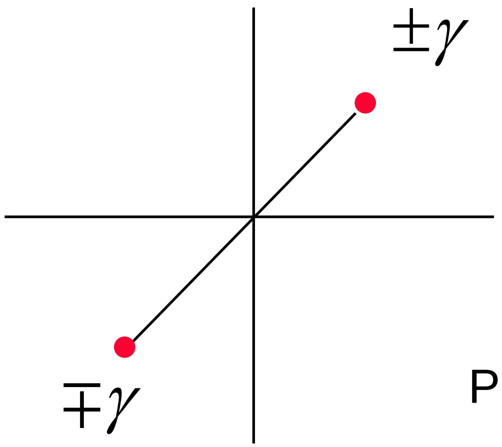
$$\gamma = \left[(R + j\omega L)(G + j\omega C)\right]^{1/2} = \left(ZY\right)^{1/2}$$

Propagation Constant (cont.)





There are two possible locations for γ:



Write:

$$\gamma = \alpha + j\beta$$

Propagation constant

Attenuation constant

Phase constant

Propagation Constant (cont.)

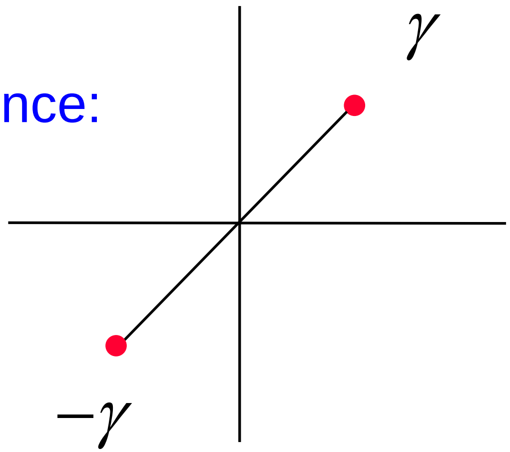
Consider a wave going in the $+z$ direction:

$$V^+(z) = V_0^+ e^{-\gamma z} = V_0^+ e^{-\alpha z} e^{-j\beta z} = \left|V_0^+\right| e^{j\phi} e^{-\alpha z} e^{-j\beta z}$$

We require $\alpha \geq 0$ ⇨

The propagation constant γ is always in the first quadrant.

Hence:



Propagation Constant (cont.)

The *principal branch* of the square root function:

$$z = r\,e^{j\theta}$$

$$\sqrt{z} \equiv \sqrt{r}\,e^{j\theta/2}$$

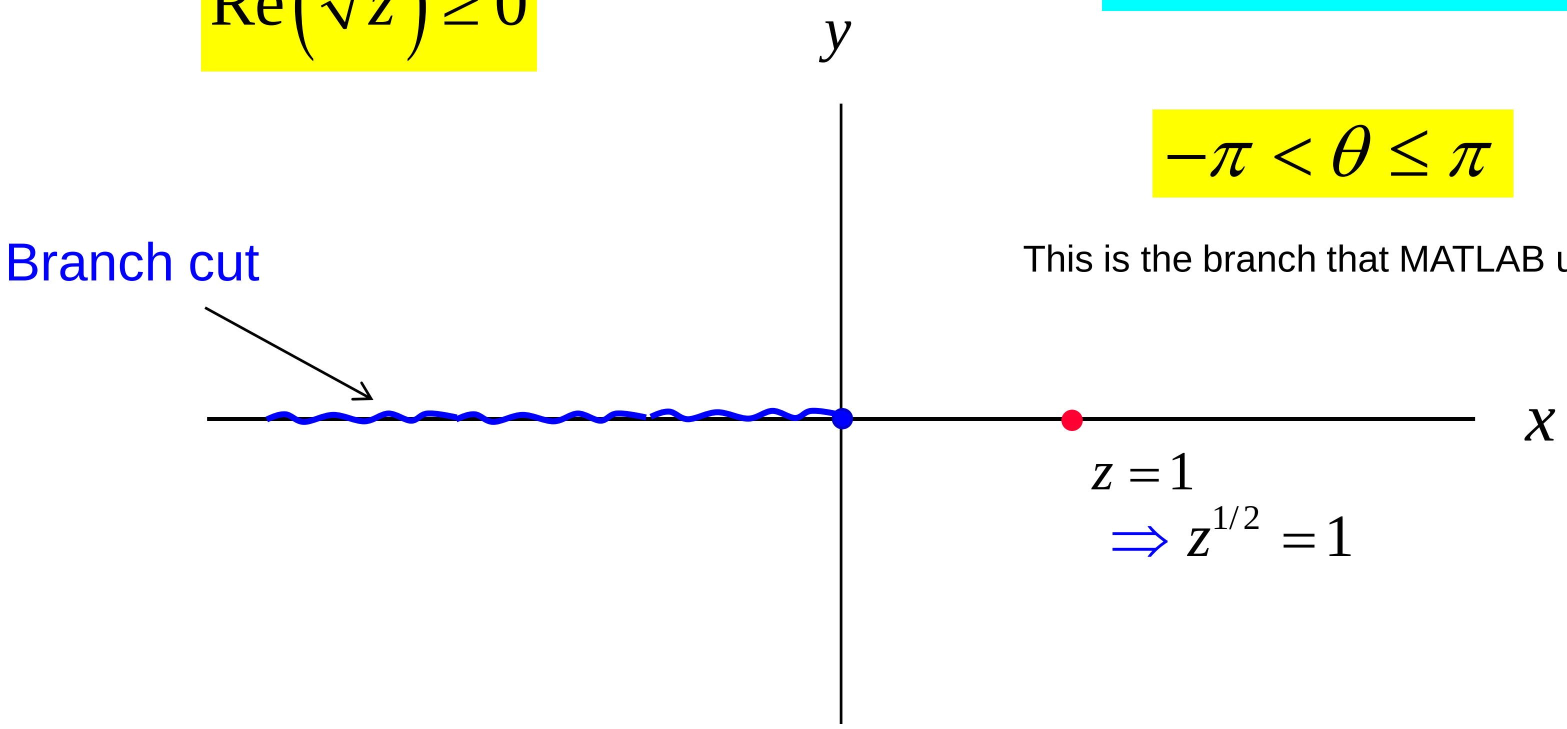
$$-\pi < \theta \leq \pi$$

We have the property that

$$\mathrm{Re}\left(\sqrt{z}\right) \geq 0$$

$$-\pi < \theta \leq \pi$$

This is the branch that MATLAB uses.



Propagation Constant (cont.)

Hence:

$$\gamma = \sqrt{\left(R + j\omega L\right)\left(G + j\omega C\right)}$$

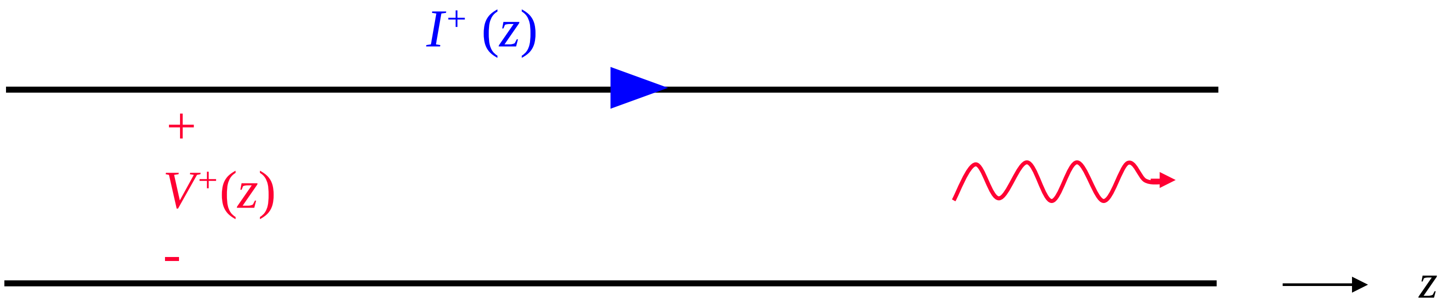
The principal branch ensures that $\alpha \geq 0$

To be more general,

$$\gamma = \sqrt{ZY}$$

(This also holds for an ATL.)

Characteristic Impedance Z_0



A wave is traveling in the positive z direction.

$$Z_0 \equiv \frac{V^+(z)}{I^+(z)}$$

$$V^+(z) = V_0^+ e^{-\gamma z}$$

$$I^+(z) = I_0^+ e^{-\gamma z}$$

so $$Z_0 = \frac{V_0^+}{I_0^+}$$

(Z_0 is a number, not a function of z.)

Characteristic Impedance Z_0 (cont.)

Use the first Telegrapher's Equation:

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$

so $$\frac{dV}{dz} = -RI - j\omega LI$$

$$= -ZI$$ (This form also holds for an ATL.)

Hence $$-\gamma V_0^{+} e^{-\gamma z} = -Z I_0^{+} e^{-\gamma z}$$

$$Z_0 = \frac{V_0^{+}}{I_0^{+}} = \frac{Z e^{-\gamma z}}{\gamma e^{-\gamma z}}$$

Recall:

$$V^{+}(z) = V_0^{+} e^{-\gamma z}$$

$$I^{+}(z) = I_0^{+} e^{-\gamma z}$$

Characteristic Impedance Z_0 (cont.)

From this we have:

$$Z_0 = \frac{Z}{\gamma}$$

(This also holds for an artificial TL.)

We can also write

$$Z_0 = \frac{Z}{\gamma} = \frac{Z}{\sqrt{ZY}} = \pm\sqrt{\frac{Z}{Y}}$$

Which sign is correct?

Positive power flow in the z direction ⇒ $\operatorname{Re} Z_0 \geq 0$

Hence

$$Z_0 = \sqrt{\frac{Z}{Y}}$$

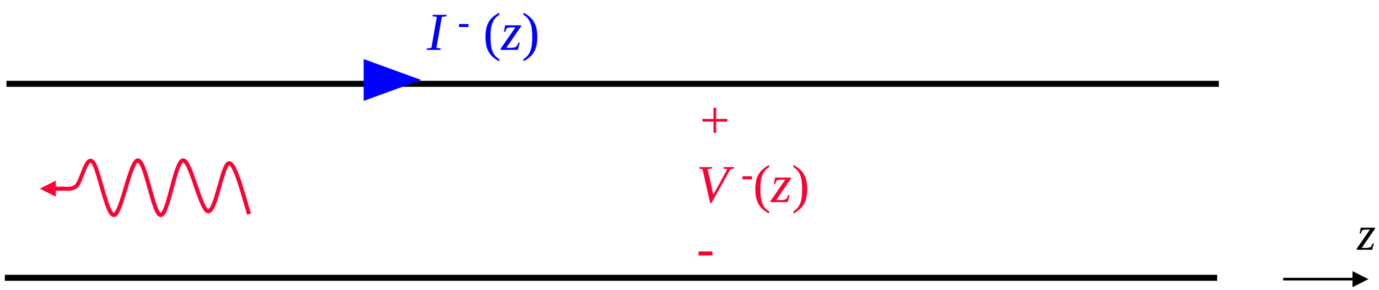
(This also holds for an artificial TL.)

Characteristic Impedance Z_0 (cont.)

For a physical transmission line we have:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Backward-Traveling Wave



A wave is traveling in the negative z direction.

$$\frac{V^{-}(z)}{-I^{-}(z)} = Z_0 \qquad \text{so} \qquad \frac{V^{-}(z)}{I^{-}(z)} = -Z_0$$

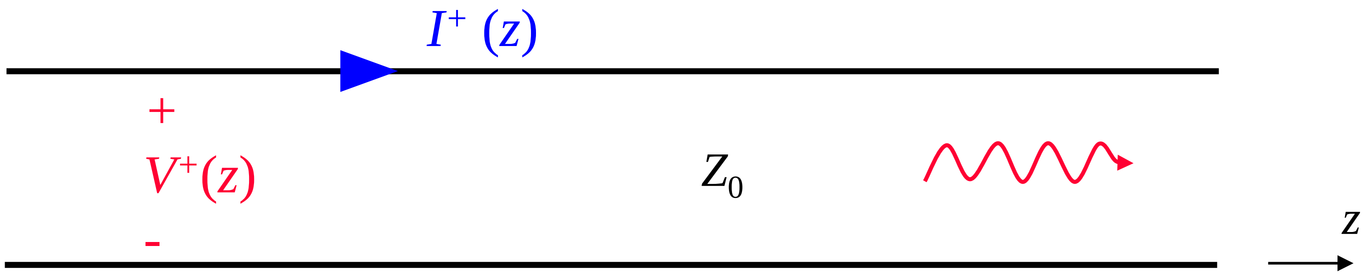
Most general case:

A general superposition of forward and backward traveling waves:

$$V(z) = V_0^{+} e^{-\gamma z} + V_0^{-} e^{+\gamma z}$$

$$I(z) = \frac{1}{Z_0}\left[V_0^{+} e^{-\gamma z} - V_0^{-} e^{+\gamma z}\right]$$

Power Flow



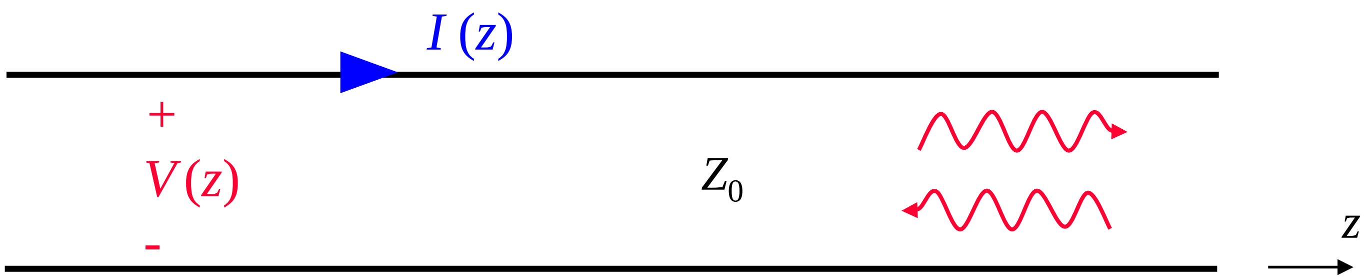
A wave is traveling in the positive z direction.

$$P_z = \frac{1}{2}V^{+}(z)I^{+*}(z) = \frac{1}{2}\left(Z_0 I^{+}(z)I^{+*}(z)\right) = \frac{1}{2}Z_0\left|I^{+}(z)\right|^2$$

$$< \mathscr{P}_z > = \operatorname{Re} P_z$$

Hence $$< \mathscr{P}_z > = \frac{1}{2}\operatorname{Re} Z_0\left|I^{+}(z)\right|^2$$

Power Flow (cont.)



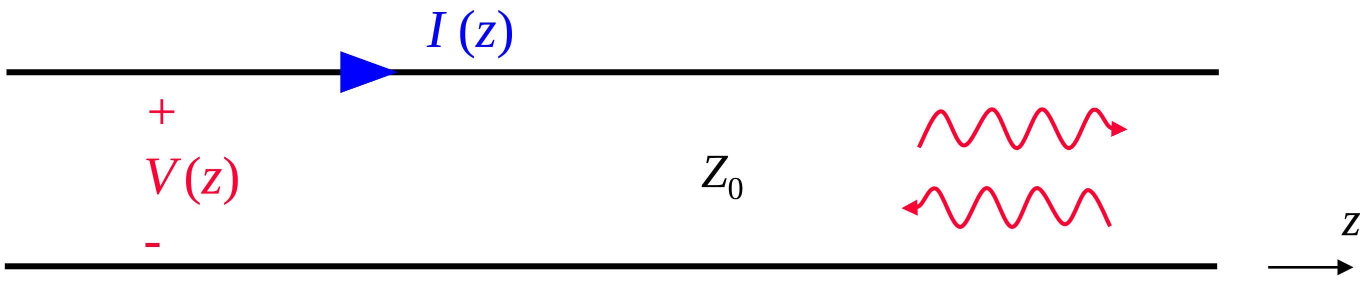
Allow for waves in both directions:

$$P_z = \frac{1}{2} V(z) I^*(z)$$

$$= \frac{1}{2}\left(Z_0 I^+(z) - Z_0 I^-(z)\right)\left(I^{+*}(z) + I^{-*}(z)\right)$$

$$= \frac{1}{2} Z_0 \left|I^+(z)\right|^2 - \frac{1}{2} Z_0 \left|I^-(z)\right|^2 + \left[\frac{1}{2} Z_0 I^+(z) I^{-*}(z) - \frac{1}{2} Z_0 I^-(z) I^{+*}(z)\right]$$

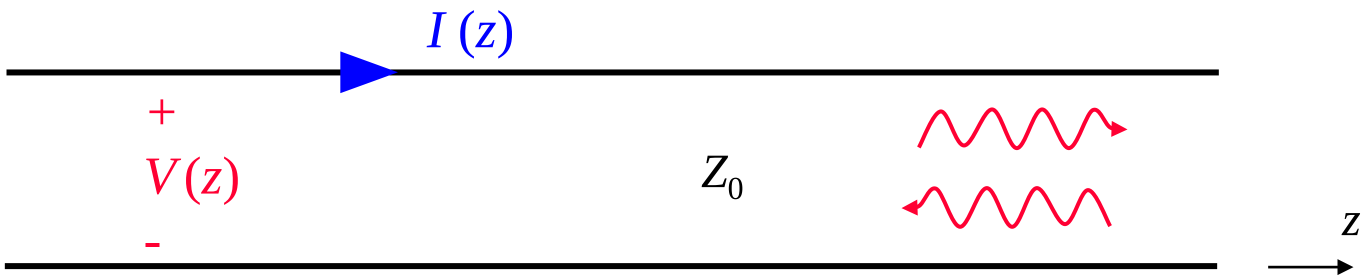
Power Flow (cont.)



$$P_z = \frac{1}{2} Z_0 \left| I^+(z) \right|^2 - \frac{1}{2} Z_0 \left| I^-(z) \right|^2 + \left[\frac{1}{2} Z_0 I^+(z) I^{-*}(z) - \frac{1}{2} Z_0 I^-(z) I^{+*}(z) \right]$$

$$P_z = P_z^+ + P_z^- + \frac{1}{2} Z_0 \underbrace{\left[I^+(z) I^{-*}(z) - I^-(z) I^{+*}(z) \right]}_{\text{pure imaginary}}$$

Power Flow (cont.)



$$P_z = P_z^+ + P_z^- + \frac{1}{2} Z_0 \underbrace{\left[I^+(z) I^{-*}(z) - I^-(z) I^{+*}(z) \right]}_{\text{pure imaginary}}$$

For a lossless line, Z_0 is pure real: $Z_0 = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}} \Rightarrow \sqrt{\dfrac{L}{C}}$

In this case, $\operatorname{Re} P_z = \operatorname{Re} P_z^+ + \operatorname{Re} P_z^-$

so $\left\langle \mathscr{P}_z \right\rangle = \left\langle \mathscr{P}_z^+ \right\rangle + \left\langle \mathscr{P}_z^- \right\rangle$ (orthogonality)

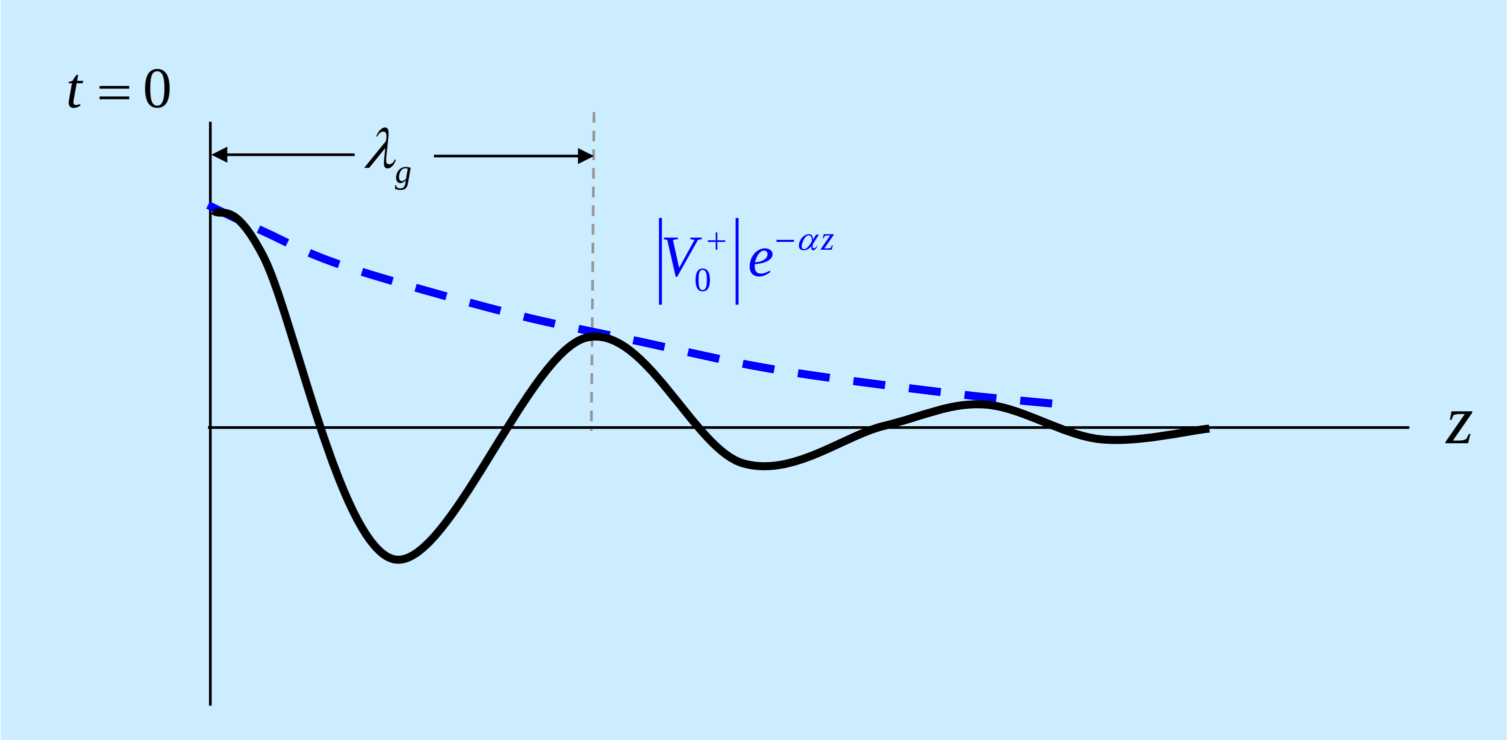
Traveling Wave

Let's look at the traveling wave in the time domain.

$$
\begin{aligned}
v^{+}(z,t) &= \mathrm{Re}\left\{V^{+}(z)\,e^{j\omega t}\right\} \\
&= \mathrm{Re}\left\{V_0^{\;+}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right\} \\
&= \mathrm{Re}\left\{\left|V_0^{\;+}\right|e^{j\phi}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right\} \\
&= \left|V_0^{\;+}\right|e^{-\alpha z}\cos(\omega t-\beta z+\phi)
\end{aligned}
$$

Wavelength

$$v^{+}(z,t) = \left|V_0^{+}\right| e^{-\alpha z} \cos(\omega t - \beta z + \phi)$$



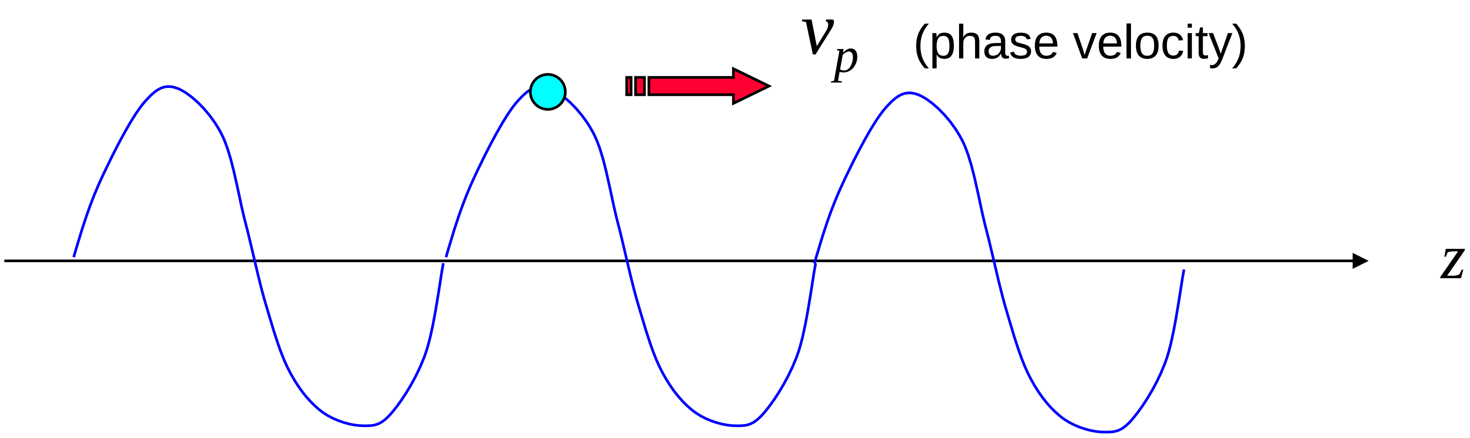
Wavelength (cont.)

The wave "repeats" when: $\beta\lambda_g = 2\pi$

Hence: $$\beta = \frac{2\pi}{\lambda_g}$$

Phase Velocity

Track the velocity of a fixed point on the wave (a point of constant phase), e.g., the crest.



$$v^{+}(z,t) = \left|V_0^{+}\right| e^{-\alpha z} \cos(\omega t - \beta z + \phi)$$

Phase Velocity (cont.)

Set $\omega t - \beta z = \text{constant}$

$$\omega - \beta \frac{dz}{dt} = 0$$

$$\frac{dz}{dt} = \frac{\omega}{\beta}$$

Hence $$v_p = \frac{\omega}{\beta}$$

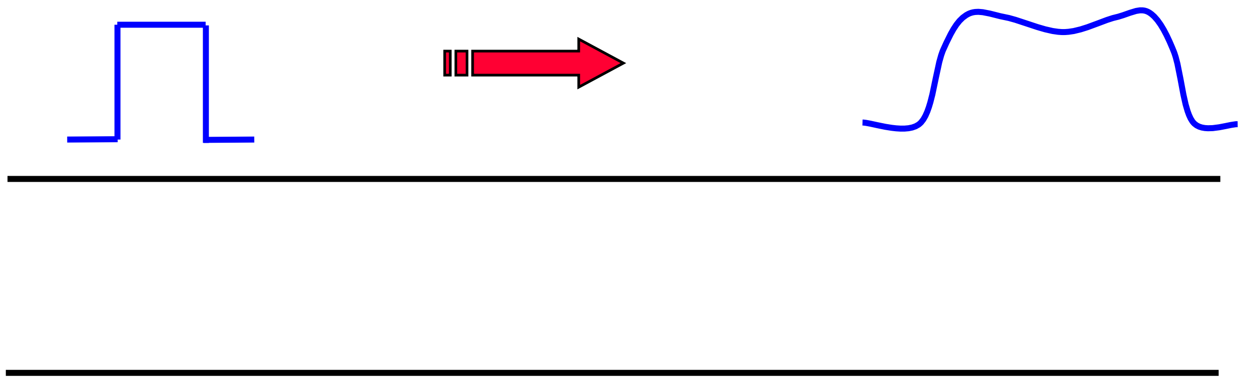
Phase Velocity (cont.)

In general,

$$v_p = \frac{\omega}{\operatorname{Im}\left[\sqrt{(R + j\omega L)(G + j\omega C)}\right]}$$

$$v_p = f(\omega) \ \text{(function of frequency)}$$

This is <u>dispersion</u>, resulting in waveform <u>distortion</u>

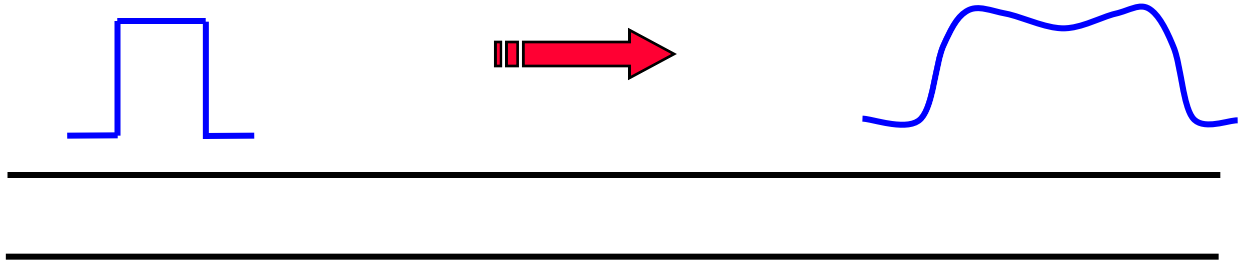


Distortion

In general, waveform distortion is caused by either of two things:

1) Phase velocity v_p is a function of frequency (dispersion)

2) Attenuation α is a function of frequency



In general, both effects arise when loss is present on a transmission line.

Lossless Case

$$R = 0,\ G = 0$$

$$\gamma = \alpha + j\beta = \sqrt{(\cancel{R} + j\omega L)(\cancel{G} + j\omega C)}$$

$$= j\omega\sqrt{LC}$$

so $\alpha = 0$

$$\beta = \omega\sqrt{LC}$$

$$v_p(\omega) = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \text{constant}$$

no dispersion + no attenuation

⇩

no distortion

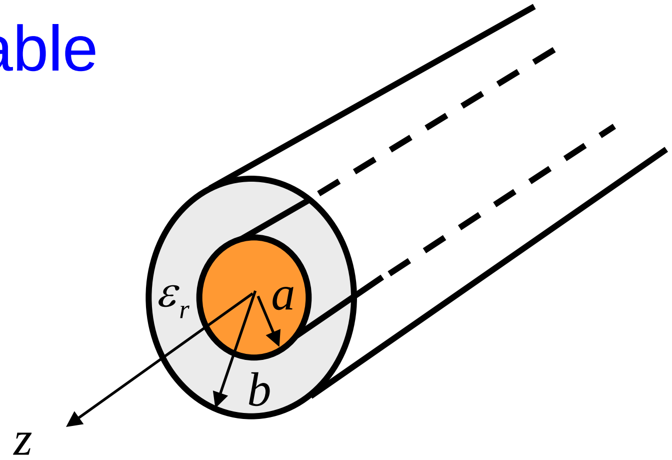
Lossless Case (cont.)

$$Z_0 = \sqrt{\frac{\cancel{R} + j\omega L}{\cancel{G} + j\omega C}}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

Example

Lossless coaxial cable



$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \qquad [\mathrm{F/m}]$$

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \qquad [\mathrm{H/m}]$$

Example (cont.)

Using

$$v_p = \frac{1}{\sqrt{LC}} \qquad Z_0 = \sqrt{\frac{L}{C}}$$

We have

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} \quad [\text{m/s}]$$

$$Z_0 = \frac{\eta_0}{2\pi}\frac{1}{\sqrt{\varepsilon_r}}\ln\left(\frac{b}{a}\right) \; [\Omega]$$

$$c = 2.99792458 \times 10^8 \, [\text{m/s}] \qquad \eta_0 = \sqrt{\mu_0 / \varepsilon_0} \doteq 376.7303 [\Omega]$$

Generalization

Generalization to general lossless two-conductor transmission line with a homogeneous non-magnetic material filling:

$$C = \varepsilon_0 \varepsilon_r \text{GF}$$

$$L = \frac{\mu_0}{\text{GF}}$$

GF = geometrical factor

Note:
A proof of this independence is given in Notes 9.

Justification:

1) We have $\rho_s = \underline{D} \cdot \underline{\hat{n}}$ and $\underline{D} = \varepsilon \underline{E}$, with $\underline{E}$ independent of frequency and material.

2) The relation for L follows from the requirement that the phase velocity be equal to the speed of light in the filling material – this is valid for any TEM mode in a lossless material, as discussed later in Notes 9.

Hence: $$\sqrt{LC} = \sqrt{\mu \varepsilon_0 \varepsilon_r}$$

Generalization (cont.)

Consider next a lossy dielectric, but lossless conductors:

$$C = \varepsilon_0 \varepsilon'_{rc} \, \text{GF}$$
$$G = \omega \, \varepsilon_0 \varepsilon''_{rc} \, \text{GF}$$
$$L = \frac{\mu_0}{\text{GF}}$$

Justification of C and G formulas:

$$Y = G + j\omega C \propto j\omega\varepsilon_c$$ (principle of effective permittivity)

$$j\omega\varepsilon_c = j\omega\left(\varepsilon'_c - j\varepsilon''_c\right) = \omega\left(\varepsilon''_c + j\varepsilon'_c\right)$$

$$\Rightarrow \left(G + j\omega C\right) \propto \omega\left(\varepsilon''_c + j\varepsilon'_c\right)$$

$$\Rightarrow \quad G \propto \omega\varepsilon''_c \qquad C \propto \varepsilon'_c$$

Also, we have

$$\frac{G}{\omega C} = \frac{\varepsilon''_c}{\varepsilon'_c} = \tan\delta$$

Generalization (cont.)

Justification of L formula:

$$\sqrt{LC} = \sqrt{\mu \varepsilon_c'}$$

$$\varepsilon_c = \varepsilon_c' - j\varepsilon_c''$$

This is proven in notes 9.

Determination of (L, G, C) Parameters

Consider the general case of a lossy (dielectric loss only) transmission line:

$$\varepsilon_c = \varepsilon_c' - j\varepsilon_c''$$

We wish to calculate the parameters (G, L, C) in terms of the characteristic impedance of the lossless line and the complex permittivity of the filling material.

$$Z_0^{lossless} \equiv \sqrt{\frac{L}{C}} \qquad \sqrt{LC} = \sqrt{\mu\varepsilon_c'} \qquad \frac{G}{\omega C} = \frac{\varepsilon_c''}{\varepsilon_c'} = \tan\delta$$

For the calculation of the lossless Z_0, we set $\varepsilon_c = \varepsilon_c'$ (ignore ε_c'').

From the first two equations we have (multiplying and dividing the two equations):

$$L = Z_0^{lossless}\sqrt{\mu\varepsilon_c'} \qquad C = \frac{\sqrt{\mu\varepsilon_c'}}{Z_0^{lossless}}$$

Determination of Parameters (cont.)

Summary

$$L = \left(\frac{Z_0^{lossless}}{c} \right) \sqrt{\varepsilon'_{rc}}$$

$$C = \frac{\sqrt{\varepsilon'_{rc}}}{c Z_0^{lossless}}$$

$$\frac{G}{\omega C} = \frac{\varepsilon''_{rc}}{\varepsilon'_{rc}} = \tan\delta$$

$$c = 2.99792458 \times 10^8 \text{ [m/s]}$$

These results tells us how to calculate the (L, G, C) line parameters from the characteristic impedance of the lossless line and the filling material.

This information is what we would typically know about a line (e.g., from a vendor).

Note: Later we will see how to calculate R. (This involves the concept of the surface resistance of the conductors.)

Distortionless Case

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

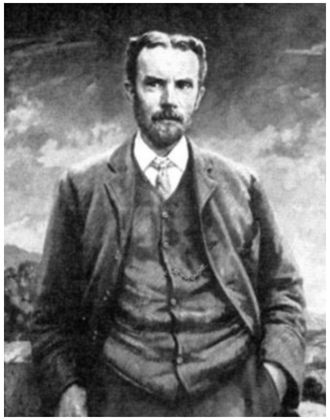
$$= \sqrt{L\left(\frac{R}{L} + j\omega\right)C\left(\frac{G}{C} + j\omega\right)}$$

Assume that the following condition holds:

$$\frac{R}{L} = \frac{G}{C}$$

This is the "Heaviside condition," discovered by Oliver Heaviside.

Then we have: $\gamma = \sqrt{LC}\left(\frac{R}{L} + j\omega\right)$



Distortionless Case (cont.)

$$\gamma = \sqrt{LC}\left(\frac{R}{L} + j\omega\right)$$

$$\alpha = \frac{R}{L}\sqrt{LC} \qquad \beta = \omega\sqrt{LC}$$

$$v_p(\omega) = \frac{1}{\sqrt{LC}} = \text{constant (no dispersion)}$$

There is then *attenuation* but no dispersion.

Note: There will be some distortion in practice, since the Heaviside condition cannot be satisfied for all frequencies:

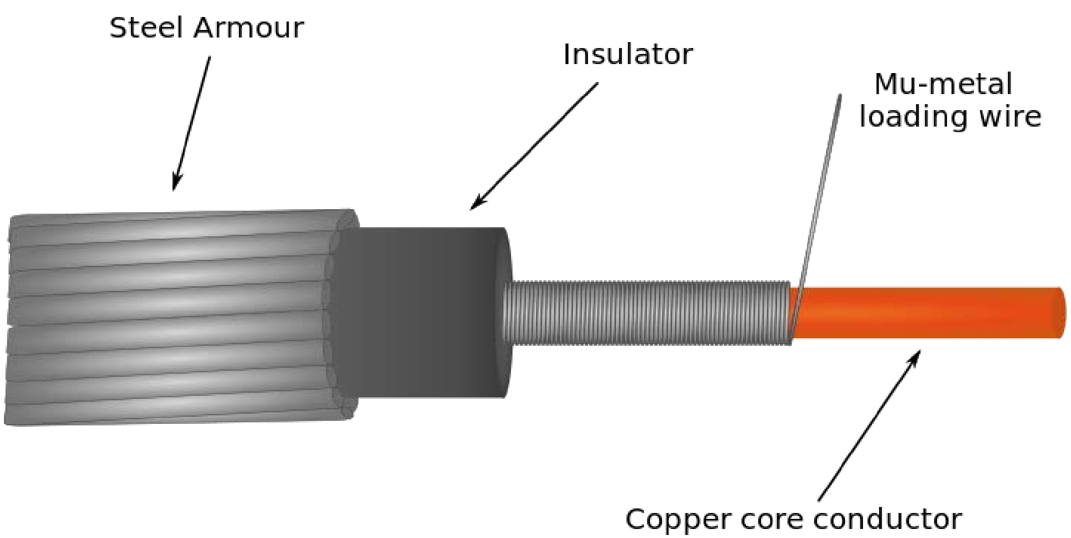
$$R = G\left(\frac{L}{C}\right) \qquad G \propto \omega,\ R \propto \sqrt{\omega} \qquad \text{(assuming a fixed loss tangent)}$$

Also, there will be distortion since α is a function of frequency.

Distortionless Case (cont.)

An example of loading a coaxial cable to achieve the Heaviside condition.

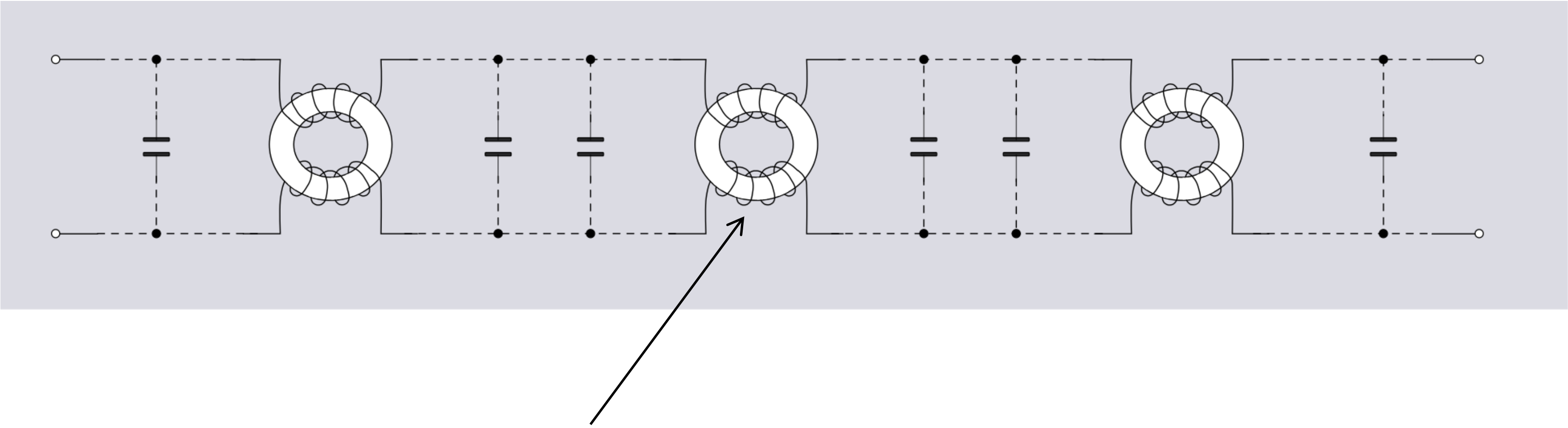
For regular coax: $\dfrac{R}{L} > \dfrac{G}{C}$ ⇒ We can increase L to achieve the Heaviside condition.



https://en.wikipedia.org/wiki/Heaviside_condition

Distortionless Case (cont.)

Loading coils can also be placed periodically along the line.



Loading coil

https://en.wikipedia.org/wiki/Heaviside_condition

Distortionless Case (cont.)

For an interesting history:

https://en.wikipedia.org/wiki/Loading_coil

- Loading cables to improve performance was popular in the early 1900's, but declined after the 1940s.
- The technology has been superseded by using digital repeaters on transmission lines.
- For long distances, transmission lines are usually replaced by fiber-optic cables (or wireless systems).