ECE 6340
Intermediate EM Waves

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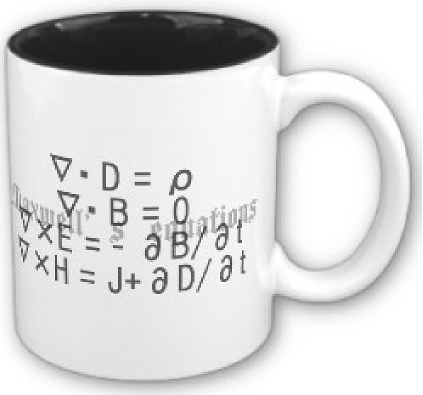
Prof. David R. Jackson
Dept. of ECE



Notes 9

Fields of a Guided Wave

Theorem on Field Representation:

Assume

$$\underline{E}(x,y,z) = \underline{E}_0(x,y)\,e^{-\gamma z}$$
$$\underline{H}(x,y,z) = \underline{H}_0(x,y)\,e^{-\gamma z}$$

Guided wave

Then

$$\underline{E}_t = \underline{E}_t(E_z, H_z)$$
$$\underline{H}_t = \underline{H}_t(E_z, H_z)$$

The "t" subscript denotes transverse (to z)

Fields of a Guided Wave (cont.)

Proof (for E_y)

$$\nabla \times \underline{H} = j\omega\varepsilon_c \underline{E}$$

so $$E_y = \frac{1}{j\omega\varepsilon_c}\left(-\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z}\right)$$

or $$E_y = \frac{1}{j\omega\varepsilon_c}\left(-\frac{\partial H_z}{\partial x} - \gamma H_x\right)$$

Need H_x

Now solve for H_x:

$$\nabla \times \underline{E} = -j\omega\mu \underline{H}$$

Fields of a Guided Wave (cont.)

$$H_x = -\frac{1}{j\omega\mu}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)$$

$$= -\frac{1}{j\omega\mu}\left(\frac{\partial E_z}{\partial y} + \gamma E_y\right)$$

Substituting this into the equation for E_y yields the following result:

$$E_y = \frac{1}{j\omega\varepsilon_c}\left(-\frac{\partial H_z}{\partial x} - \gamma H_x\right)$$

⇓

$$E_y = \frac{1}{j\omega\varepsilon_c}\left[-\frac{\partial H_z}{\partial x} - \gamma\left(-\frac{1}{j\omega\mu}\right)\left(\frac{\partial E_z}{\partial y} + \gamma E_y\right)\right]$$

Next, multiply by $-j\omega\mu\left(j\omega\varepsilon_c\right) = k^2$

Fields of a Guided Wave (cont.)

$$k^2 E_y = j\omega\mu \frac{\partial H_z}{\partial x} - \gamma \frac{\partial E_z}{\partial y} - \gamma^2 E_y$$

so

$$E_y = \left(\frac{j\omega\mu}{\gamma^2 + k^2} \right) \frac{\partial H_z}{\partial x} - \left(\frac{\gamma}{\gamma^2 + k^2} \right) \frac{\partial E_z}{\partial y}$$

The other components may be found similarly.

Fields of a Guided Wave (cont.)

Summary of Fields

$$E_x = \left(\frac{-j\omega\mu}{k^2+\gamma^2} \right) \frac{\partial H_z}{\partial y} - \left(\frac{\gamma}{k^2+\gamma^2} \right) \frac{\partial E_z}{\partial x}$$

$$E_y = \left(\frac{j\omega\mu}{\gamma^2+k^2} \right) \frac{\partial H_z}{\partial x} - \left(\frac{\gamma}{\gamma^2+k^2} \right) \frac{\partial E_z}{\partial y}$$

$$H_x = \left(\frac{j\omega\varepsilon}{k^2+\gamma^2} \right) \frac{\partial E_z}{\partial y} - \left(\frac{\gamma}{k^2+\gamma^2} \right) \frac{\partial H_z}{\partial x}$$

$$H_y = \left(\frac{-j\omega\varepsilon}{\gamma^2+k^2} \right) \frac{\partial E_z}{\partial x} - \left(\frac{\gamma}{\gamma^2+k^2} \right) \frac{\partial H_z}{\partial y}$$

Fields of a Guided Wave (cont.)

These may be written more compactly as

$$\underline{E}_t = \frac{j\omega\mu}{k^2+\gamma^2}\left(\underline{\hat{z}}\times\nabla_t H_z\right) - \frac{\gamma}{k^2+\gamma^2}\left(\nabla_t E_z\right)$$

$$\underline{H}_t = \frac{-j\omega\varepsilon}{k^2+\gamma^2}\left(\underline{\hat{z}}\times\nabla_t E_z\right) - \frac{\gamma}{k^2+\gamma^2}\left(\nabla_t H_z\right)$$

Where the 2-D gradient is defined as

$$\nabla_t \Phi \equiv \underline{\hat{x}}\frac{\partial \Phi}{\partial x} + \underline{\hat{y}}\frac{\partial \Phi}{\partial y}$$

Fields of a Guided Wave (cont.)

In cylindrical coordinates we have

$$\nabla_t \Phi = \underline{\hat{\rho}} \frac{\partial \Phi}{\partial \rho} + \underline{\hat{\phi}} \frac{1}{\rho} \frac{\partial \Phi}{\partial \phi}$$

We can thus also express the fields of a guided wave in terms of E_z and H_z in cylindrical coordinates (please see next slide).

Fields of a Guided Wave (cont.)

Summary of Fields

cylindrical coordinates

$$E_\rho = -\frac{j\omega\mu}{k^2+\gamma^2}\frac{1}{\rho}\left(\frac{\partial H_z}{\partial \phi}\right) - \frac{\gamma}{k^2+\gamma^2}\left(\frac{\partial E_z}{\partial \rho}\right)$$

$$E_\phi = \frac{j\omega\mu}{k^2+\gamma^2}\left(\frac{\partial H_z}{\partial \rho}\right) - \frac{\gamma}{k^2+\gamma^2}\frac{1}{\rho}\left(\frac{\partial E_z}{\partial \phi}\right)$$

$$H_\rho = \frac{j\omega\varepsilon}{k^2+\gamma^2}\frac{1}{\rho}\left(\frac{\partial E_z}{\partial \phi}\right) - \frac{\gamma}{k^2+\gamma^2}\left(\frac{\partial H_z}{\partial \rho}\right)$$

$$H_\phi = -\frac{j\omega\varepsilon}{k^2+\gamma^2}\left(\frac{\partial E_z}{\partial \rho}\right) - \frac{\gamma}{k^2+\gamma^2}\frac{1}{\rho}\left(\frac{\partial H_z}{\partial \phi}\right)$$

Types of Guided Waves

TEM$_z$ mode: $E_z = 0 \quad H_z = 0$ Transmission line

TM$_z$ mode: $E_z \neq 0 \quad H_z = 0$ Waveguide

TE$_z$ mode: $E_z = 0 \quad H_z \neq 0$ Waveguide

Hybrid mode: $E_z \neq 0 \quad H_z \neq 0$ Fiber-optic guide

Wavenumber Property of TEM Wave

Assume a TEM wave:

$$E_z = 0$$
$$H_z = 0$$

To avoid having a completely zero field,

$$\gamma^2 + k^2 = 0$$

$$\Rightarrow \quad \gamma = \pm jk$$

$$k = k' - jk''$$

We then have

$$\gamma = jk$$

Note: The plus sign is chosen to give a decaying wave:

$$e^{-\gamma z} = e^{-jkz} = e^{-jk'z} e^{-k''z}$$

Wavenumber Property (cont.)

Propagation constant vs. wavenumber notation:

$$\gamma = \alpha + j\beta$$
$$k_z = \beta - j\alpha$$

$$k_z = -j\gamma, \; \gamma = jk_z$$

Note that k_z is called the "propagation wavenumber" of the mode.

$$e^{-\gamma z} \Rightarrow e^{-jk_z z}$$

$$e^{-\gamma z} = e^{-jk_z z} = e^{-\alpha z} \, e^{-j\beta z}$$

TEM mode: $$k_z = k$$

Note:
A TEM mode can propagate on a lossless transmission line at any frequency.

Wavenumber Property (cont.)

The field on a lossless transmission line is a TEM mode (proven later).

Lossless TL:

$$k_z = \omega\sqrt{LC} = k = \omega\sqrt{\mu\varepsilon}$$

so $$LC = \mu\varepsilon$$

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$v_p = c_d$$

The phase velocity is equal to the speed of light in the dielectric.

Wavenumber Property (cont.)

Lossy TL (dielectric but <u>no</u> conductor loss): The mode is still a TEM mode

Hence

$$k_z = -j\gamma = -j\sqrt{(\cancel{R} + j\omega L)(G + j\omega C)} = k = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu(\varepsilon_c' - j\varepsilon_c'')}$$

Note:
The TEM_z assumption requires that $R = 0$. Otherwise, $E_z \neq 0$ (from Ohm's law).

$$\Rightarrow \quad -j\sqrt{(j\omega L)(G + j\omega C)} = \omega\sqrt{\mu(\varepsilon_c' - j\varepsilon_c'')}$$

$$\Rightarrow \quad -(j\omega L)(G + j\omega C) = \omega^2\mu(\varepsilon_c' - j\varepsilon_c'')$$

Real part: $LC = \mu\varepsilon_c'$

Imaginary part: $LG = \omega\mu\varepsilon_c''$

Dividing these two equations gives us: $G = (\omega C)\dfrac{\varepsilon_c''}{\varepsilon_c'}$

Static Property of TEM Wave

The fields of a TEM mode may be written as:

$$\underline{E}(x,y,z) = \underline{E}_0(x,y)\,e^{-\gamma z}$$
$$= \underline{E}_{t0}(x,y)\,e^{-\gamma z}$$

$$\underline{H}(x,y,z) = \underline{H}_{t0}(x,y)\,e^{-\gamma z} \qquad \gamma = jk_z = jk$$

Theorem

$\underline{E}_{t0}(x,y)$ and $\underline{H}_{t0}(x,y)$ are 2D static field functions.

Static Property of TEM Wave (cont.)

Proof

$$\nabla \times \underline{E}_{t0} = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_{x0} & E_{y0} & 0 \end{vmatrix}$$

$$= \hat{\underline{z}} \left(\frac{\partial E_{y0}}{\partial x} - \frac{\partial E_{x0}}{\partial y} \right)$$

Therefore, only a z component of the curl exists.
We next prove that this must be zero.

Static Property of TEM Wave (cont.)

Use

$$\begin{aligned}\nabla \times \underline{E} &= \nabla \times \left(\underline{E}_{t0}\, e^{-\gamma z}\right) \\ &= e^{-\gamma z}\, \nabla \times \underline{E}_{t0} + \nabla\left(e^{-\gamma z}\right) \times \underline{E}_{t0} \\ &= e^{-\gamma z}\, \nabla \times \underline{E}_{t0} - \gamma e^{-\gamma z}\, \underline{\hat{z}} \times \underline{E}_{t0}\end{aligned}$$

$$\underline{\hat{z}} \cdot \left(\nabla \times \underline{E}\right) = e^{-\gamma z}\, \underline{\hat{z}} \cdot \left(\nabla \times \underline{E}_{t0}\right)$$

Also,

$$\begin{aligned}\underline{\hat{z}} \cdot \left(\nabla \times \underline{E}\right) &= \underline{\hat{z}} \cdot \left(-j\omega\mu \underline{H}\right) \\ &= -j\omega\mu \cancel{H_z} = 0\end{aligned}$$

Static Property of TEM Wave (cont.)

Hence

$$\underline{\hat{z}} \cdot (\nabla \times \underline{E}_{t0}) = 0$$

Therefore,

$$\nabla \times \underline{E}_{t0}(x, y) = \underline{0}$$

Static Property of TEM Wave (cont.)

Also,

$$\nabla \cdot \underline{E} = 0$$

(No charge density in the time-harmonic steady state, for a homogeneous medium)

Therefore, $$\nabla \cdot \left(\underline{E}_{t0}\, e^{-\gamma z} \right) = 0$$

$$\left(\nabla \cdot \underline{E}_{t0} \right) e^{-\gamma z} + \underline{E}_{t0} \cdot \nabla \left(e^{-\gamma z} \right) = 0$$

$$\left(\nabla \cdot \underline{E}_{t0} \right) e^{-\gamma z} + \underline{E}_{t0} \cdot \left(\underline{\hat{z}} \left(-\gamma e^{-\gamma z} \right) \right) = 0$$

Hence, $$\nabla \cdot \underline{E}_{t0} \left(x, y \right) = 0$$

Static Property of TEM Wave (cont.)

$$\nabla \times \underline{E}_{t0}\left(x, y\right) = \underline{0}$$

$$\nabla \cdot \underline{E}_{t0}\left(x, y\right) = 0$$

$$\nabla \times \underline{E}_{t0}\left(x, y\right) = \underline{0}$$

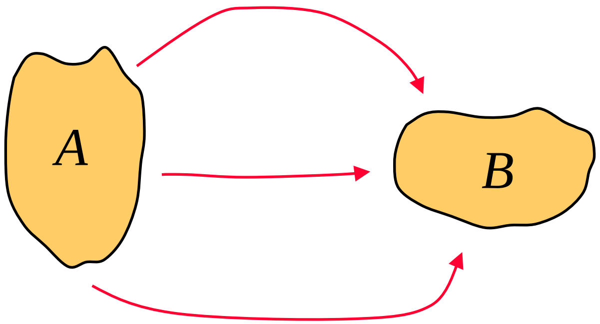
$\Rightarrow$ $$\underline{E}_{t0} = -\nabla \Phi(x, y)$$

$$\nabla \cdot \underline{E}_{t0} = 0$$

$\Rightarrow$ $$\nabla^2 \Phi = 0$$

Static Property of TEM Wave (cont.)

$$\underline{E}_{t0}\left(x, y\right) = -\nabla\Phi(x, y)$$



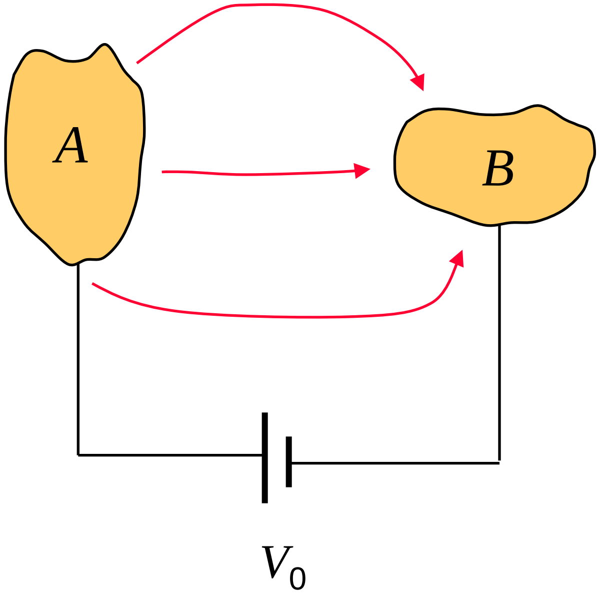
$$\nabla^2\Phi(x, y) = 0$$

$\Phi =$ constant on A or B (since $\underline{E}_{\tan} = \underline{0}$ on conductors)

The potential function is unique (because of the uniqueness theorem of statics), and hence is the same as a static potential function (which also obeys the Laplace equation and the same BCs).

Static Property of TEM Wave (cont.)

The static property shows us why a TEM_z wave can exist on a transmission line (two parallel conductors).



Transmission line

A nonzero field can exist at DC.

Static Property of TEM Wave (cont.)

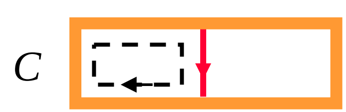
The static property also tells us why a TEM_z wave cannot exist inside of a waveguide (hollow conducting pipe).

Waveguide

No field can exist inside at DC.

(This would violate Faraday's law:

at DC the voltage drop around a closed path must be zero.)



Static Property of TEM Wave (cont.)

Similarly,

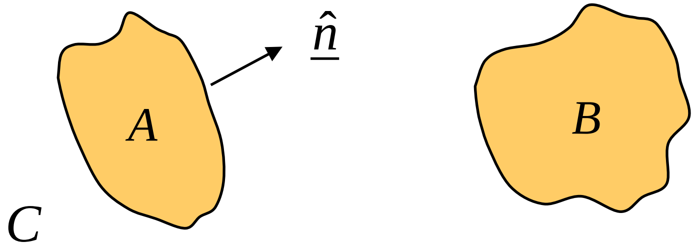
$$\nabla \times \underline{H}_{t0} = \underline{0}$$

$$\nabla \cdot \underline{H}_{t0} = 0$$

so

$$\underline{H}_{t0} = -\nabla \Phi_m (x, y)$$

$$\nabla^2 \Phi_m = 0$$



$$\begin{cases} \nabla^2 \Phi_m = 0 \\ \dfrac{\partial \Phi_m}{\partial n} = 0 \qquad C_1 \text{ and } C_2 \end{cases}$$

$$\left(\underline{H}_{t0} \cdot \hat{\underline{n}} = 0 \right)$$

TEM Mode: Magnetic Field

$$\nabla \times \underline{H} = j\,\omega\,\varepsilon_c\,\underline{E}$$

$$\underline{E} = \frac{1}{j\,\omega\,\varepsilon_c}\begin{vmatrix} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & -\gamma \\ H_x & H_y & 0 \end{vmatrix}$$

so

$$E_x = \frac{\gamma}{j\omega\varepsilon_c} H_y \qquad E_y = -\frac{\gamma}{j\omega\varepsilon_c} H_x$$

TEM Magnetic Field (cont.)

Also,

$$\frac{\gamma}{j\omega\varepsilon_c} = \frac{jk_z}{j\omega\varepsilon_c} = \frac{jk}{j\omega\varepsilon_c} = \frac{\omega\sqrt{\mu\varepsilon_c}}{\omega\varepsilon_c} = \sqrt{\frac{\mu}{\varepsilon_c}} = \eta$$

so

$$E_x = \eta H_y$$

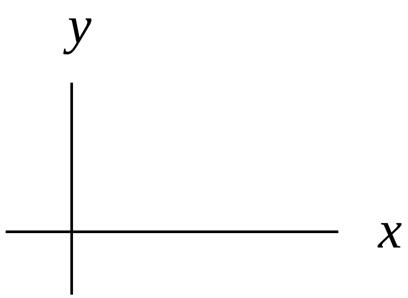
$$E_y = -\eta H_x$$

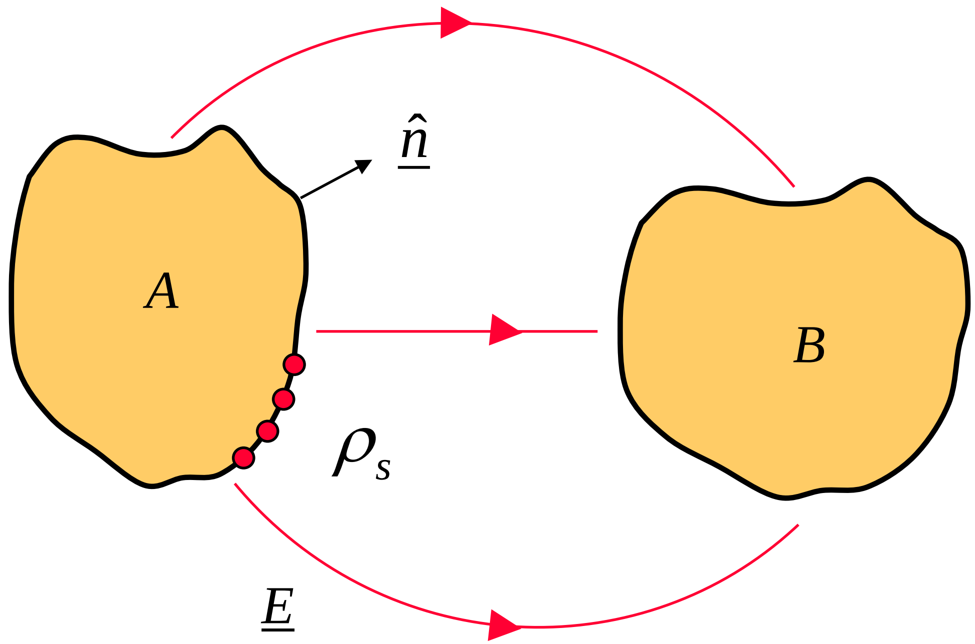
This can be written as

$$\underline{H} = \frac{1}{\eta}\left(\underline{\hat{z}} \times \underline{E}\right)$$

TEM Mode: Current and Charge Density

TEM mode





$$\rho_l = \oint_{C_A} \rho_s(l)\, dl$$

$$\rho_l = CV$$

$C =$ capacitance / length

$$V = \int_{A}^{B} \underline{E} \cdot d\underline{r}$$

$$\underline{J}_s = \underline{\hat{n}} \times \underline{H} = \underline{\hat{n}} \times \left[\frac{1}{\eta} \underline{\hat{z}} \times \underline{E} \right]$$

TEM Mode: Current and Charge Density (cont.)

$$\underline{A}\times\left(\underline{B}\times\underline{C}\right)=\underline{B}\left(\underline{A}\cdot\underline{C}\right)-\underline{C}\left(\underline{A}\cdot\underline{B}\right)$$

$$\underline{\hat{n}}\times\left(\underline{\hat{z}}\times\underline{E}\right)=\underline{\hat{z}}\left(\underline{\hat{n}}\cdot\underline{E}\right)-\underline{E}\left(\underline{\hat{n}}\cdot\underline{\hat{z}}\right)$$

so

$$\underline{J}_s=\underline{\hat{z}}\frac{1}{\eta}\left(\underline{\hat{n}}\cdot\underline{E}\right)=\underline{\hat{z}}\frac{1}{\eta\,\varepsilon}\left(\underline{\hat{n}}\cdot\underline{D}\right)$$

Hence

$$\underline{J}_s=\underline{\hat{z}}\left(\frac{\rho_s}{\varepsilon\eta}\right)$$

$$\eta=\sqrt{\frac{\mu}{\varepsilon_c}}$$

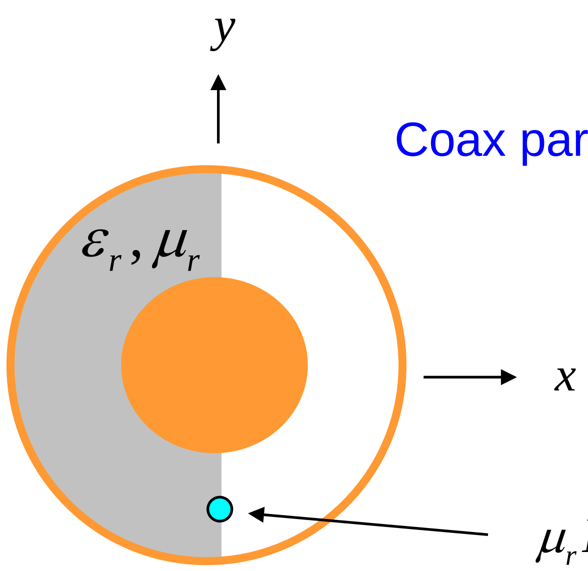
Note:

In general

$$\varepsilon\neq\varepsilon_c$$

TEM Mode: Homogeneous Substrate

A TEM$_z$ mode requires a homogeneous substrate.



Coax partially filled with dielectric

<u>Assume</u> a TEM mode

$$\mu_r H_x^- = H_x^+ \qquad E_y^- = E_y^+$$

$$x = 0^+ : \quad \frac{E_y^+}{H_x^+} = -\eta_0$$

$$x = 0^- : \quad \frac{E_y^-}{H_x^-} = -\eta_1$$

$$\frac{E_y^+}{E_y^-}\left(\frac{H_x^-}{H_x^+}\right) = \frac{\eta_0}{\eta_1} = \sqrt{\frac{\varepsilon_r}{\mu_r}}$$

Take the ratio

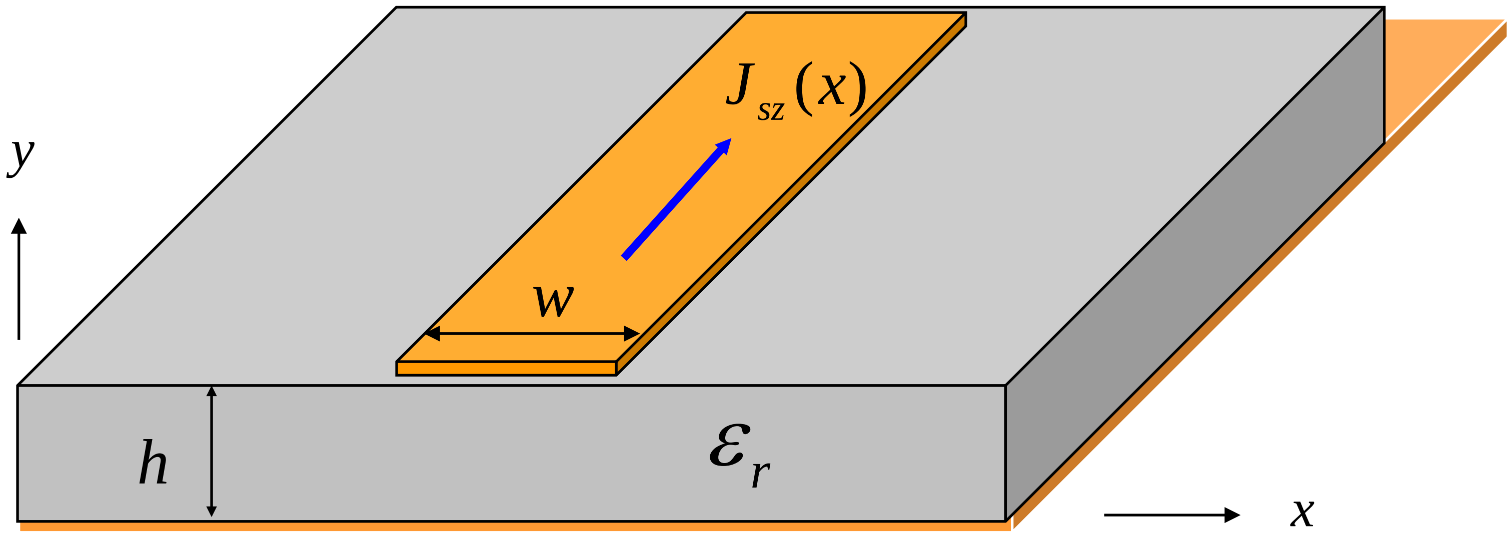
BCs

$$\left(\frac{1}{\mu_r}\right) = \sqrt{\frac{\varepsilon_r}{\mu_r}}$$

$$\sqrt{\mu_r \varepsilon_r} = 1$$

Contradiction!

Example: Microstrip Line

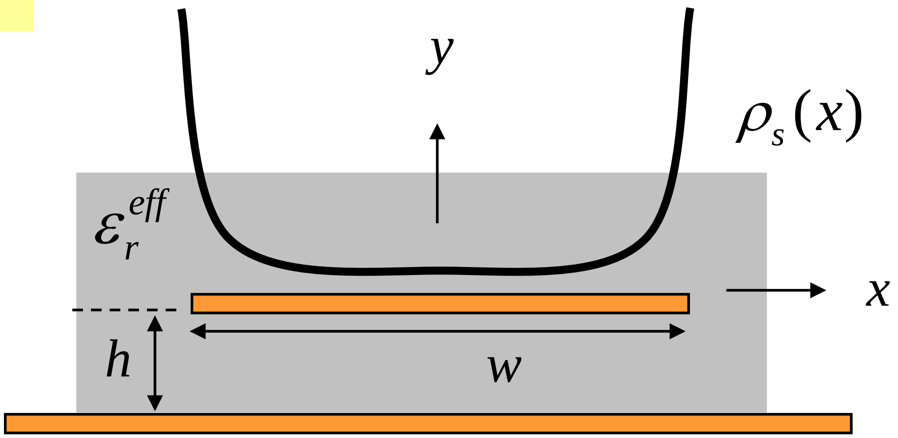


$$k_z = k_0\sqrt{\varepsilon_r^{eff}}$$

Assume a TEM mode:

(requires a homogeneous space of material)

$$J_{sz} = \frac{1}{\varepsilon_0 \varepsilon_r \eta^{eff}} \rho_s(x)$$

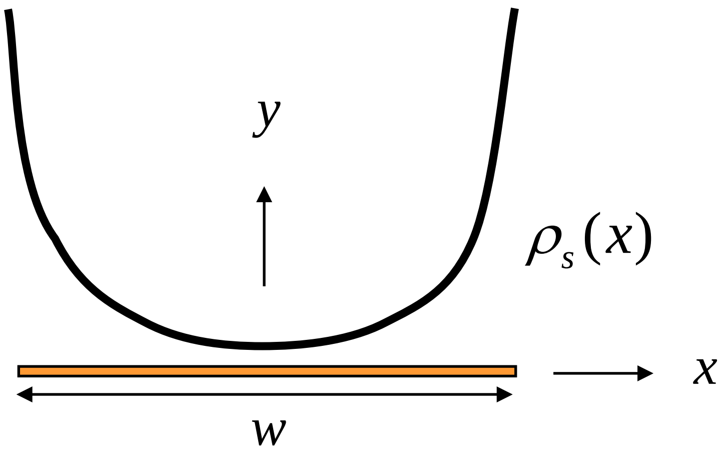


Homogeneous model

Example (cont.)

Strip in free space (or homogeneous space) with a static charge density (no ground plane):

$$\rho_s(x) = \left(\frac{1/\pi}{\sqrt{(w/2)^2 - x^2}} \right) \rho_l$$



$$\rho_l = \int_{-w/2}^{w/2} \rho_s(x)\,dx$$

(This was first derived by Maxwell using conformal mapping.)

Hence:

$$J_{sz}(x) \approx \left(\frac{1/\pi}{\sqrt{(w/2)^2 - x^2}} \right) I_0$$

In this result, I_0 is the total current [Amps] on the strip at $z = 0$.

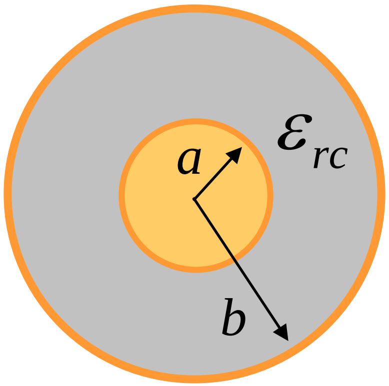
This is accurate for a narrow strip (since we ignored the ground plane).

Example: Coaxial Cable

Find $\underline{E}$, $\underline{H}$

We first find $\underline{E}_{t0}$ and $\underline{H}_{t0}$

$$\underline{E}_{t0}\left(x,y\right) = -\nabla\Phi(x,y)$$



$$\nabla^2\Phi(\rho) = 0$$
$$\Phi(a) = V_0$$
$$\Phi(b) = 0$$

$$\Phi = \Phi(\rho)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) = 0$$

$$\rho\frac{\partial\Phi}{\partial\rho} = c_1$$

$$\frac{\partial\Phi}{\partial\rho} = \frac{c_1}{\rho}$$

$$\Rightarrow \quad \Phi = c_1 \ln\rho + c_2$$

Example (cont.)

Boundary conditions:

$$c_1 \ln a + c_2 = V_0$$

$$c_1 \ln b + c_2 = 0$$

so

$$c_1 (\ln a - \ln b) = V_0$$

Hence

$$c_1 = \frac{V_0}{\ln\left(\frac{a}{b}\right)} \qquad c_2 = -c_1 \ln b = -\frac{V_0 \ln b}{\ln\left(\frac{a}{b}\right)}$$

Therefore

$$\Phi = \left(\frac{V_0}{\ln\left(\frac{a}{b}\right)} \right) \ln(\rho) + \left(-\frac{V_0 \ln b}{\ln\left(\frac{a}{b}\right)} \right) = \frac{V_0}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{\rho}{b}\right)$$

Example (cont.)

$$\Phi = \frac{V_0}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{\rho}{b}\right)$$

$$\underline{E}_{t0}(x,y) = -\nabla\Phi(x,y) = -\underline{\hat{\rho}}\frac{\partial\Phi}{\partial\rho}$$

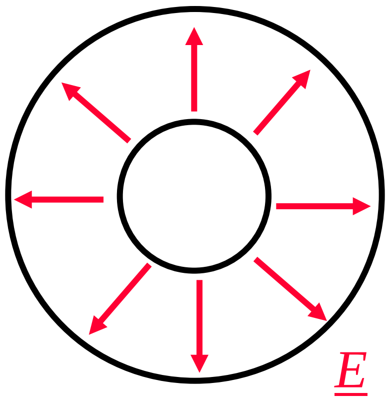
Hence

$$\underline{E}_{t0} = -\underline{\hat{\rho}}\left(\frac{1}{\rho}\right)\frac{V_0}{\ln\left(\frac{a}{b}\right)}$$

or

$$\underline{E}_{t0} = \underline{\hat{\rho}}\left(\frac{1}{\rho}\right)\frac{V_0}{\ln\left(\frac{b}{a}\right)}$$

Example (cont.)



$$\underline{H} = \frac{1}{\eta}\underline{\hat{z}} \times \underline{E} = \frac{1}{\eta}\underline{\hat{z}} \times (\underline{\hat{\rho}}\, E_\rho)$$

$$\Rightarrow \underline{H} = \underline{\hat{\phi}}\left(\frac{E_\rho}{\eta}\right) \qquad \eta = \sqrt{\frac{\mu}{\varepsilon_c}}$$

The three-dimensional fields are then as follows:

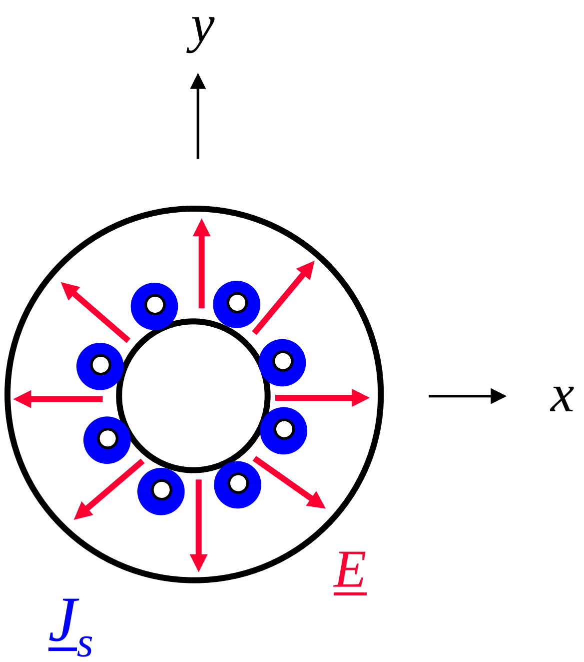
$$\underline{E} = \underline{\hat{\rho}}\left(\frac{1}{\rho}\right)\frac{V_0}{\ln\left(\dfrac{b}{a}\right)}e^{-jkz}$$

$$\underline{H} = \underline{\hat{\phi}}\left(\frac{1}{\eta\rho}\right)\frac{V_0}{\ln\left(\dfrac{b}{a}\right)}e^{-jkz}$$

This result is valid at <u>any</u> frequency.

$$k = \omega\sqrt{\mu\varepsilon_c}$$

Example (cont.)



Find the characteristic impedance.

$$V^{+} = V_0 \, e^{-jkz}$$

$$I^{+} = 2\pi a J_{sz} = 2\pi a H_{\phi}(a)$$

$$= 2\pi a \left(\frac{1}{\eta a}\right) \frac{V_0}{\ln\left(\frac{b}{a}\right)} e^{-jkz}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}}$$

$$Z_0 = \frac{V^{+}}{I^{+}} \quad \Rightarrow \quad Z_0 = \frac{\eta}{2\pi} \ln\left(\frac{b}{a}\right) \quad [\Omega]$$

Example (cont.)

Find (L, C) for lossless coax.

$$\sqrt{LC} = \sqrt{\mu\varepsilon}$$ (assume $\mu = \mu_0$)

$$\sqrt{\frac{L}{C}} = Z_0$$ ← Use

$$Z_0 = \frac{\eta}{2\pi}\ln\left(\frac{b}{a}\right)$$

Solve for L and C (multiply and divide the above two equations):

$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \quad [\text{H/m}]$$

Example (cont.)

Find (L, C, G) for lossy coax.

$$LC = \mu \varepsilon_c'$$

$$\sqrt{\frac{L}{C}} = Z_0^{lossless}$$

Use

$$Z_0^{lossless} = \sqrt{\frac{\mu_0}{\varepsilon_c'}} \frac{1}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$G = (\omega C) \frac{\varepsilon_c''}{\varepsilon_c'}$$

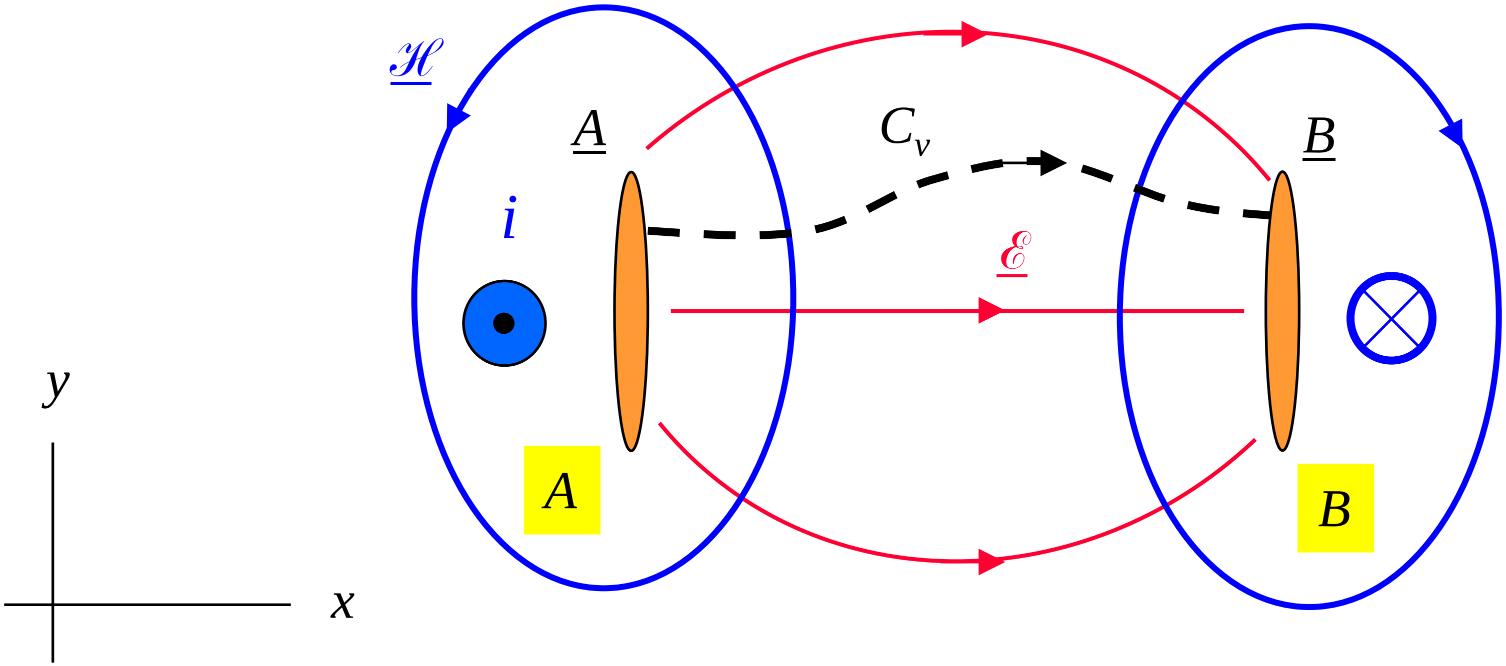
Result:

$$C = \frac{2\pi \varepsilon_0 \varepsilon_{rc}'}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

$$L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \quad [\text{H/m}]$$

$$G = (\omega C) \frac{\varepsilon_{rc}''}{\varepsilon_{rc}'} \quad [\text{S/m}]$$

TEM Mode: Telegrapher's Eqs.



TEM mode (lossless conductors)

$$v = \int_{\underline{A}}^{\underline{B}} \underline{\mathscr{E}} \cdot \underline{dr}$$

Telegrapher's Eqs. (cont.)

$$v = \int_{\underline{A}}^{\underline{B}} \left(\mathscr{E}_x dx + \mathscr{E}_y dy \right)$$

$$\frac{\partial v}{\partial z} = \int_{\underline{A}}^{\underline{B}} \left(\frac{\partial \mathscr{E}_x}{\partial z} dx + \frac{\partial \mathscr{E}_y}{\partial z} dy \right)$$

Note:
The voltage v is path independent in the (x,y) plane.

$$\oint_C \underline{\mathscr{E}} \cdot \underline{dr} = \int_S \left(\nabla \times \underline{\mathscr{E}} \right) \cdot \underline{\hat{z}}\, ds$$

$$= \int_S -\frac{\partial \mathscr{B}_z}{\partial t} ds$$

$$= 0$$

Telegrapher's Eqs. (cont.)

Use $$\nabla \times \underline{\mathscr{E}} = -\frac{\partial \underline{\mathscr{B}}}{\partial t}$$

Take x and y components:

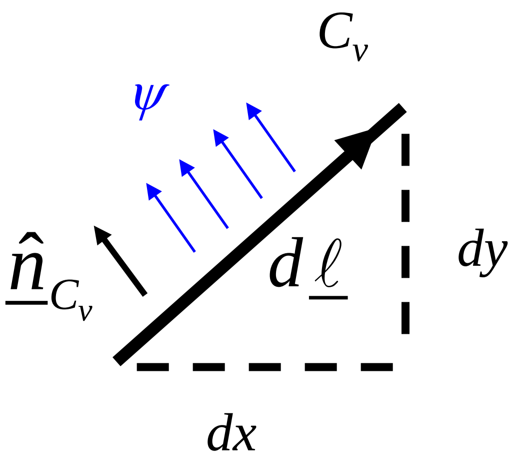
$$-\frac{\partial \mathscr{B}_x}{\partial t} = \cancel{\frac{\partial \mathscr{E}_z}{\partial y}} - \frac{\partial \mathscr{E}_y}{\partial z} \qquad -\frac{\partial \mathscr{B}_y}{\partial t} = -\cancel{\frac{\partial \mathscr{E}_z}{\partial x}} + \frac{\partial \mathscr{E}_x}{\partial z}$$

Hence, we have

$$\frac{\partial v}{\partial z} = \int_{\underline{A}}^{\underline{B}} \left(-\frac{\partial \mathscr{B}_y}{\partial t} dx + \frac{\partial \mathscr{B}_x}{\partial t} dy \right)$$

$$= \frac{\partial}{\partial t} \int_{\underline{A}}^{\underline{B}} \left(\mathscr{B}_x dy - \mathscr{B}_y dx \right)$$

Telegrapher's Eqs. (cont.)



$$\underline{\hat{n}}_{C_v}\, d\underline{\ell} = \underline{\hat{z}} \times d\underline{\ell} = \underline{\hat{z}} \times \left(\underline{\hat{x}}dx + \underline{\hat{y}}dy \right)$$
$$= \underline{\hat{y}}dx - \underline{\hat{x}}dy$$

$$\int_{\underline{A}}^{\underline{B}} \left(\mathscr{B}_x dy - \mathscr{B}_y dx \right) = -\int_{\underline{A}}^{\underline{B}} \left(\underline{\hat{x}}\mathscr{B}_x + \underline{\hat{y}}\mathscr{B}_y \right) \cdot \left(\underline{\hat{y}}dx - \underline{\hat{x}}dy \right)$$
$$= -\int_{\underline{A}}^{\underline{B}} \underline{\mathscr{B}} \cdot \underline{\hat{n}}_{C_v}\, dl = -\psi_l$$

(flux per meter)

But

$$L \equiv \frac{\psi_l}{i}$$

$$\psi_l = Li$$

so

$$\frac{\partial v}{\partial z} = \frac{\partial}{\partial t}\left(-Li \right)$$

Hence

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

Note:
L is the magnetostatic (DC) value (a fixed number).

Telegrapher's Eqs. (cont.)

If we add R into the equation:

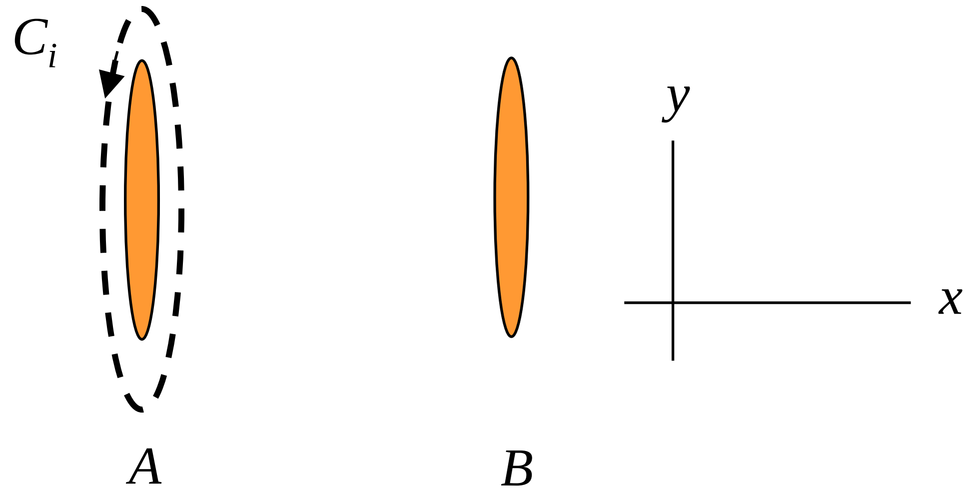
$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$

This is justifiable if the mode is approximately a TEM mode (small conductor loss).

Please see the derivation in the Appendix.

Telegrapher's Eqs. (cont.)

Now use this path:



The contour C_i hugs the A conductor.

Ampere's law: $$i = \oint_{C_i} \underline{\mathscr{H}} \cdot d\underline{r} = \oint_{C_i} \left(\mathscr{H}_x dx + \mathscr{H}_y dy \right)$$

Note:
There is no displacement current through the surface, since $E_z = 0$.

so $$\frac{\partial i}{\partial z} = \oint_{C_i} \left(\frac{\partial \mathscr{H}_x}{\partial z} dx + \frac{\partial \mathscr{H}_y}{\partial z} dy \right)$$

Telegrapher's Eqs. (cont.)

Now use

$$\nabla \times \underline{\mathscr{H}} = \frac{\partial \underline{\mathscr{D}}}{\partial t} + \underline{\mathscr{J}}$$

Take x and y components:

$$\frac{\partial \mathscr{H}_z}{\partial y} - \frac{\partial \mathscr{H}_y}{\partial z} = \frac{\partial \mathscr{D}_x}{\partial t} + \mathscr{J}_x$$

$$-\frac{\partial \mathscr{H}_z}{\partial x} + \frac{\partial \mathscr{H}_x}{\partial z} = \frac{\partial \mathscr{D}_y}{\partial t} + \mathscr{J}_y$$

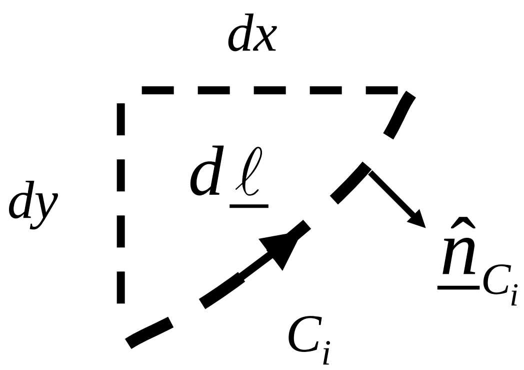
Telegrapher's Eqs. (cont.)

Hence

$$\frac{\partial i}{\partial z} = \oint_{C_i} \left(\frac{\partial \mathscr{D}_y}{\partial t} dx - \frac{\partial \mathscr{D}_x}{\partial t} dy \right) + \oint_{C_i} \left(\mathscr{J}_y dx - \mathscr{J}_x dy \right)$$

$$= \frac{\partial}{\partial t} \oint_{C_i} \left(\mathscr{D}_y dx - \mathscr{D}_x dy \right) + \oint_{C_i} \left(\mathscr{J}_y dx - \mathscr{J}_x dy \right)$$

Telegrapher's Eqs. (cont.)



$$\underline{\hat{n}}_{C_i} dl = -\underline{\hat{z}} \times d\underline{\ell} = -\underline{\hat{z}} \times \left(\underline{\hat{x}} dx + \underline{\hat{y}} dy \right)$$

$$= -\underline{\hat{y}} dx + \underline{\hat{x}} dy$$

$$\underline{\mathscr{D}} \cdot \underline{\hat{n}}_{C_i} = \rho_s$$

$$\oint_{C_i} \mathscr{D}_y dx - \mathscr{D}_x dy = -\oint_{C_i} \left(\underline{\hat{x}} \mathscr{D}_x + \underline{\hat{y}} \mathscr{D}_y \right) \cdot \left(-\underline{\hat{y}} dx + \underline{\hat{x}} dy \right) - \oint_{C_i} \underline{\mathscr{D}} \cdot \underline{\hat{n}}_{C_i} dl = -\oint_{C_i} \rho_s \, dl = -\rho_l^A = -\rho_l$$

$$\oint_{C_i} \mathscr{J}_y dx - \mathscr{J}_x dy = -\oint_{C_i} \underline{\mathscr{J}} \cdot \underline{\hat{n}}_{C_i} dl = -i_{leak}$$

But

$$C \equiv \rho_l / v \qquad\qquad \rho_l = Cv$$

$$G \equiv i_{leak} / v \qquad \Rightarrow \qquad i_{leak} = Gv$$

Note:
C and *G* are the static (DC) values.

Telegrapher's Eqs. (cont.)

Hence

$$\frac{\partial i}{\partial z} = \frac{\partial}{\partial t}(-Cv) - Gv$$

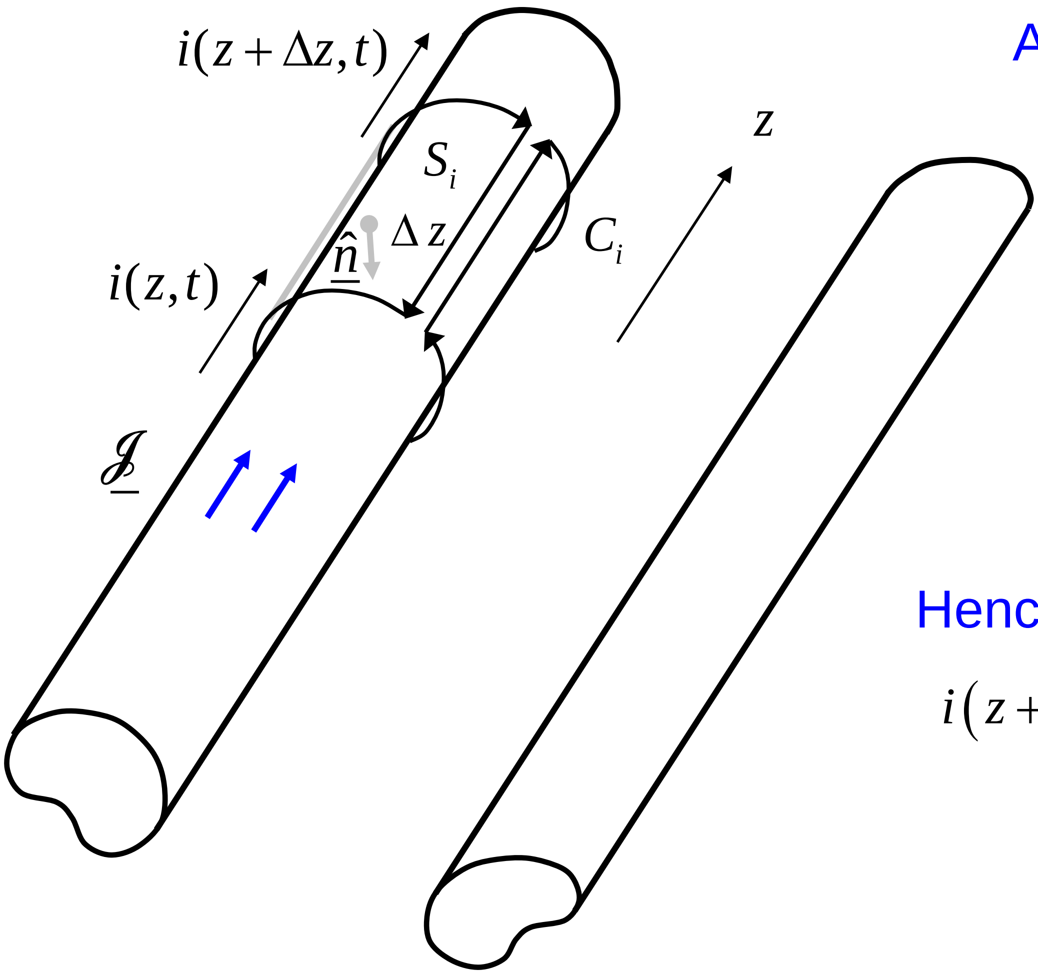
or

$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} - Gv$$

Appendix

Alternate derivation of second Telegrapher's equation

$$\oint_{C_i} \underline{\mathscr{H}} \cdot d\underline{r} = i(z+\Delta z, t) - i(z,t)$$



Ampere's law:

$$\oint_{C_i} \underline{\mathscr{H}} \cdot d\underline{r} = \int_{S_i} \underline{\mathscr{J}} \cdot \hat{\underline{n}}\, dS + \frac{d}{dt}\int_{S_i} \underline{\mathscr{D}} \cdot \hat{\underline{n}}\, dS$$

$$= -i_{leak} - \frac{d}{dt}(\rho_\ell \Delta z)$$

$$= -\Delta z\, Gv - \Delta z \frac{d}{dt}(Cv)$$

Hence

$$i(z+\Delta z,t) - i(z,t) = -\Delta z\, Gv - \Delta z \frac{d}{dt}(Cv)$$

$$\Rightarrow \quad \frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t}$$

Appendix (cont.)

Include R

Assume that current still flows in the z direction only, and R is unique in the time domain.

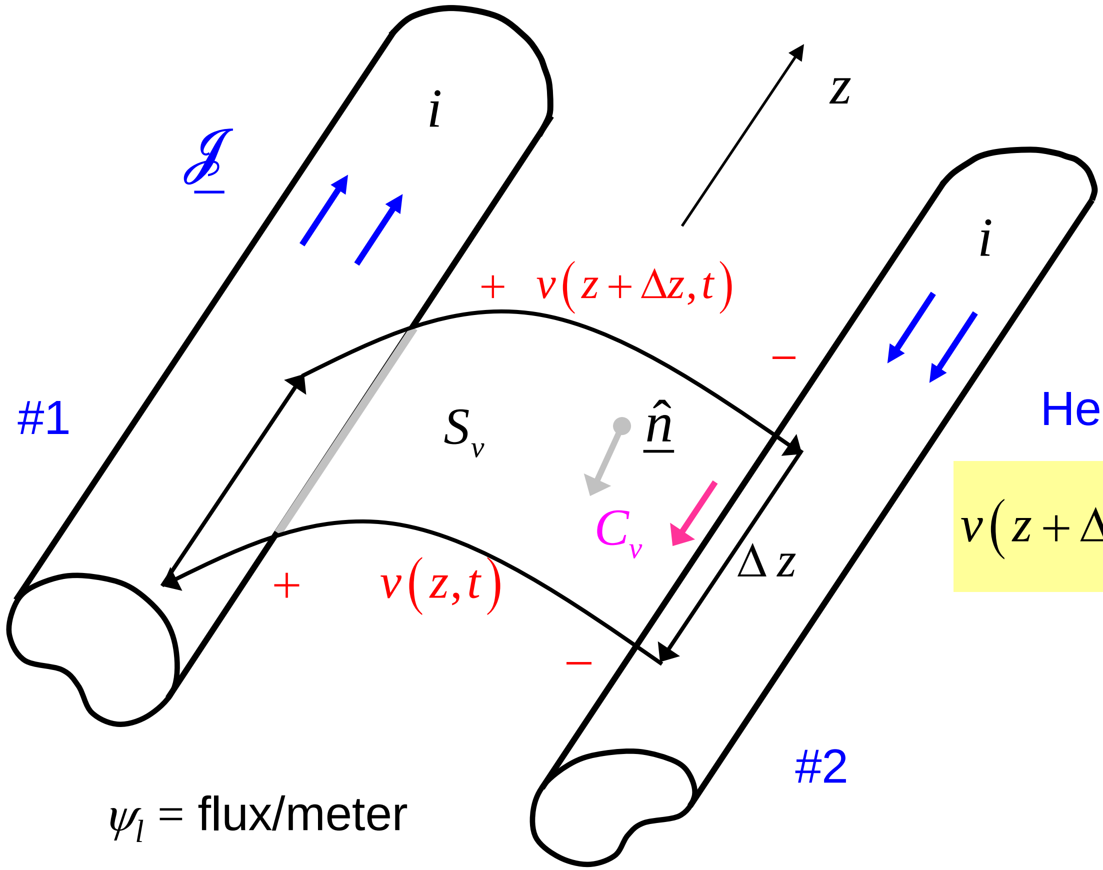
$$\oint_{C_v} \underline{\mathscr{E}} \cdot d\underline{r} = v(z+\Delta z,t) - v(z,t) + \underbrace{(R_1 + R_2)}_{R} \Delta z\, i$$

Faraday's law:

$$\oint_{C_v} \underline{\mathscr{E}} \cdot d\underline{r} = -\frac{d}{dt} \int_{S_v} \underline{\mathscr{B}} \cdot \underline{\hat{n}}\, dS$$

$$= -\frac{d}{dt}(\psi_l \Delta z)$$

$$= -\Delta z \frac{d}{dt}(Li)$$



ψ_l = flux/meter

Hence:

$$v(z+\Delta z,t) - v(z,t) + R\Delta z\, i = -\Delta z \frac{d}{dt}(Li)$$

$$\Rightarrow \quad \frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$